Product-Sum-Gravity Method
= Fuzzy Singleton-type Reasoning Method
= Simplified Fuzzy Reasoning Method

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Abstract

Three fuzzy reasoning methods of product-sum-gravity method, fuzzy singleton-type reasoning method and simplified reasoning method which are all widely used as fuzzy control methods are shown to be equivalent to each other.

1. Introduction

This paper shows that product-sum-gravity method [1,2,3] whose consequent parts consist of fuzzy sets is reduced to fuzzy singleton-type reasoning method [4] by regarding the area of fuzzy set of the consequent part as the weight of fuzzy singleton-type reasoning method. Moreover, it is shown that the fuzzy singleton-type reasoning method is reduced to a simplified reasoning method [5,6] by multiplying fuzzy sets in the antecedent part of simplified reasoning method by the weight of fuzzy singleton-type reasoning method. Therefore, these three fuzzy reasoning methods are equivalent to each other.

2. Three Fuzzy Reasoning Methods

We shall consider the following multiple fuzzy reasoning form:

Rule 1: A_1 and B_1 => C_1
Rule 2: A_2 and B_2 => C_2
.... (1)
Rule n: A_n and B_n => C_n
Fact: x_0 and y_0
Cons: C'

where A_i is a fuzzy set in X; B_i in Y; and C_i in Z and $x_0 \in X$, $y_0 \in Y$.

At first, we shall explain a fuzzy reasoning method called *product-sum-gravity method* [1,2,3] for the fuzzy reasoning form (see Fig.1).

Each inference result C_i' which is inferred from the fact 「x_0 and y_0」 and the fuzzy rule 「A_i and B_i => C_i」 is given as follows:

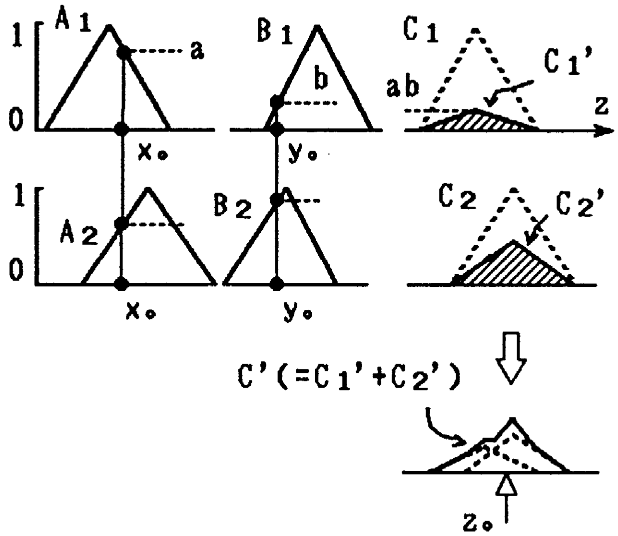

Fig.1 Product-sum-gravity method

The degree of fitness of the fact 「x_0 and y_0」 to the antecedent part 「A_i and B_i」 is given as

$$h_i = \mu_{A_i}(x_0) \cdot \mu_{B_i}(y_0) \quad (2)$$

where · stands for algebraic product. Thus the inference result C_i' is given as

$$\mu_{C_i'}(z) = \mu_{A_i}(x_0) \cdot \mu_{B_i}(y_0) \cdot \mu_{C_i}(z) \quad (3)$$

$$= h_i \cdot \mu_{C_i}(z) \quad (4)$$

The final consequence C' of (1) is aggregated from C_1', C_2', ···, C_n' by using the sum (+). Namely,

$$\mu_{C'}(z) = \mu_{C_1'}(z) + \mu_{C_2'}(z) + \cdots + \mu_{C_n'}(z) \quad (5)$$

The representative point z_0 for the resulting fuzzy set C' is obtained as the center of gravity of C':

$$z_0 = \frac{\int z \cdot \mu_{C'}(z)dz}{\int \mu_{C'}(z)dz} \quad (6)$$

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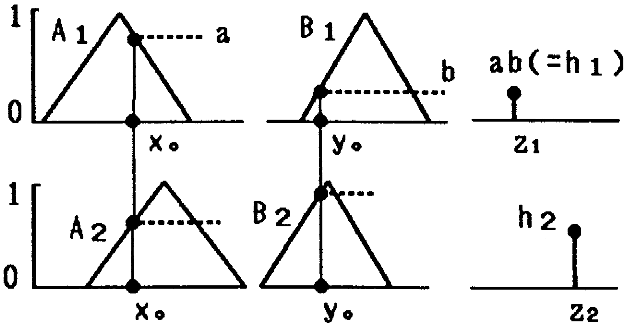


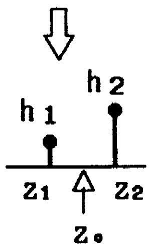


Fig.2 Simplified fuzzy reasoning method

Note that it is possible to define ***min-max-gravity method*** known as Mamdani's fuzzy reasoning method [7] by replacing product with min in (3) and sum with max in (5).

As a special case of the product-sum-gravity method, we can give a ***simplified fuzzy reasoning method*** [5,6] for the following fuzzy reasoning form:

$$
\begin{array}{ll}
\text{Rule 1:} & A_1 \text{ and } B_1 \Rightarrow z_1 \\
\text{Rule 2:} & A_2 \text{ and } B_2 \Rightarrow z_2 \\
 & \cdots\cdots\cdots \\
\text{Rule n:} & A_n \text{ and } B_n \Rightarrow z_n \\
\text{Fact:} & x_\circ \text{ and } y_\circ \\
\hline
\text{Cons:} & \qquad z_\circ
\end{array}
\qquad (7)
$$

where $z_1, z_2, \cdots, z_n, z_\circ$ are not fuzzy sets but real numbers in Z.

The consequence $z_\circ$ by the simplified fuzzy reasoning method is obtained as follows (see Fig.2). The degree of fitness, h_i, of the fact 「$x_\circ$ and $y_\circ$」 to the antecedent part 「A_i and B_i」 is given as (2). The final consequence $z_\circ$ of (7) is obtained as the weighted average of z_i by the degree h_i. Namely,

$$z_\circ = \frac{h_1 \cdot z_1 + h_2 \cdot z_2 + \cdots + h_n \cdot z_n}{h_1 + h_2 + \cdots + h_n} \qquad (8)$$

Note that the simplified reasoning method is regarded as a special case of product-sum-gravity method, but not a special case of min-max-gravity method.

Usually, an identical fuzzy rule is not used two or many times at a time in the execution of fuzzy controls. However, an identical fuzzy rule may be used twice or more in the simplified fuzzy reasoning method.

For example, suppose that a fuzzy rule A and B => z of a simplified fuzzy reasoning method is used twice, that is,

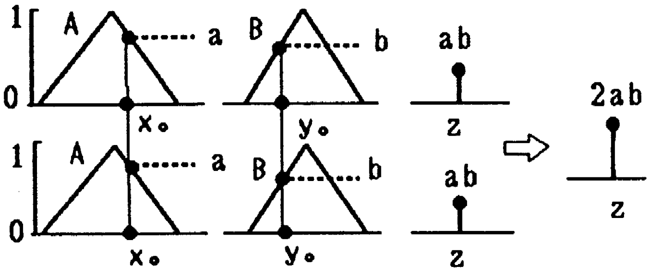


Fig.3 Emphatic effect by using two same rules

$$
\begin{array}{ll}
\text{Rule 1:} & A \text{ and } B \Rightarrow z \\
\text{Rule 2:} & A \text{ and } B \Rightarrow z
\end{array}
\qquad (9)
$$

are used simultaneously, then the inference result obtained is double, i.e., 2ab in height as shown in Fig.3. Hence, by using the identical fuzzy rule twice, we can have an inference result at double height and thus double emphatic effect is given on the inference result. Therefore, the role of a fuzzy rule is enhanced every time it is used twice or more at the same time.

In general, to obtain emphatic effect of w times, it is suggested to use a fuzzy rule whose consequent part is not a real number z but a fuzzy singleton, that is, a real number z with grade w. Namely, we can define the following fuzzy rule of fuzzy singleton type:

$$A \text{ and } B \Rightarrow w/z \qquad (10)$$

where the notation "w/z" stands for a ***fuzzy singleton***, that is, a real number z with weight w. It is assumed that the weight w is a non-negative real number. Clearly, when $w = 1$, a fuzzy rule of (10) reduces to an ordinary fuzzy rule as in (7).

Hence we can define a fuzzy reasoning method called "***fuzzy singleton-type reasoning method***" (or "***simplified fuzzy reasoning method with weight***")[4] which treats the following fuzzy reasoning form:

$$
\begin{array}{ll}
\text{Rule 1:} & A_1 \text{ and } B_1 \Rightarrow w_1/z_1 \\
\text{Rule 2:} & A_2 \text{ and } B_2 \Rightarrow w_2/z_2 \\
 & \cdots\cdots\cdots \\
\text{Rule n:} & A_n \text{ and } B_n \Rightarrow w_n/z_n \\
\text{Fact:} & x_\circ \text{ and } y_\circ \\
\hline
\text{Cons:} & \qquad z_\circ
\end{array}
\qquad (11)
$$

where the weight w_i is a real number such as $w_i \geq 0$. $w_i > 1$ means that the corresponding rule A_i and $B_i \Rightarrow w_i/z_i$ is "emphatic," and $0 \leq w_i < 1$ means that the rule is "suppressive."

The consequence $z_\circ$ by the fuzzy singleton-type reasoning method is obtained as follows (see Fig.4). The degree of fitness h_i is given as (2), and the product $h_i \cdot w_i$ of the fitness h_i and the weight w_i of z_i is regarded as the degree to which z_i is obtained.

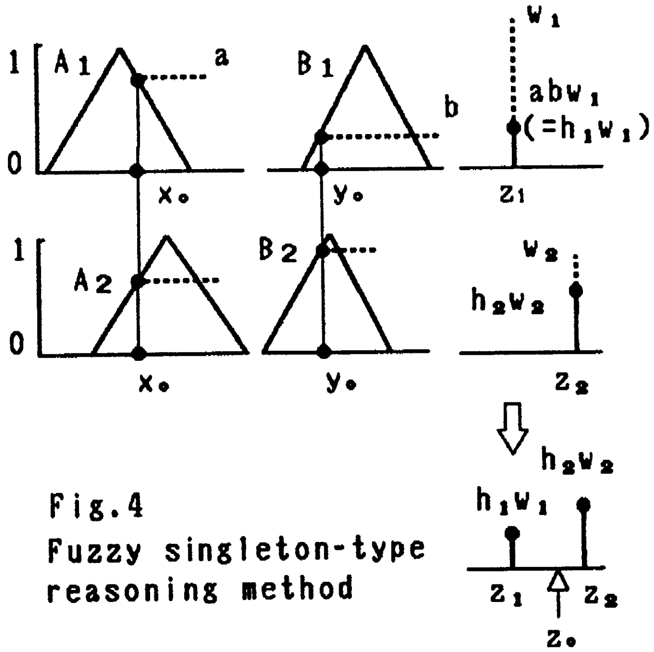


Fig.4 Fuzzy singleton-type reasoning method

Therefore, the final consequence z_0 of (11) is inferred as the weighted average of zi by the degree $h_i \cdot w_i$:

$$z_0 = \frac{h_1 \cdot w_1 \cdot z_1 + h_2 \cdot w_2 \cdot z_2 + \cdots + h_n \cdot w_n \cdot z_n}{h_1 \cdot w_1 + h_2 \cdot w_2 + \cdots + h_n \cdot w_n} \quad (12)$$

3. Equivalence of Product-Sum-Gravity Method, Fuzzy Singleton-type Reasoning Method and Simplified Fuzzy Reasoning Method

We shall first show the equivalence of product-sum-gravity method and fuzzy singleton-type reasoning method.

3.1 Equivalence of product-sum-gravity method and fuzzy singleton-type reasoning method

Let us consider again the method (6) of obtaining the representative point z_0 for a product-sum-gravity method.

We shall begin with the inference result Ci' given in (4). Let zi be the center of gravity of Ci' and let Si' be the area of Ci' as in Fig.5, then we have

$$z_i = \frac{\int z \cdot \mu_{C_i'}(z)dz}{\int \mu_{C_i'}(z)dz} = \frac{\int z \cdot \mu_{C_i'}(z)dz}{S_i'} \quad (13)$$

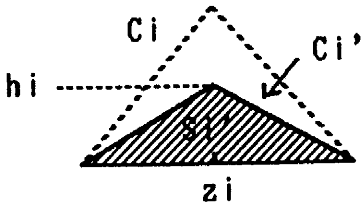


Fig.5 Area Si' and Gravity zi of Ci'

and thus

$$\int z \cdot \mu_{C_i'}(z)dz = S_i' \cdot z_i \quad (14)$$

The center of gravity z_0 of C' in (1) is given as

$$z_0 = \frac{\int z \cdot \mu_{C'}(z)dz}{\int \mu_{C'}(z)dz}$$

$$= \frac{\int z \cdot [\mu_{C_1'}(z) + \mu_{C_2'}(z) + \cdots + \mu_{C_n'}(z)]dz}{\int [\mu_{C_1'}(z) + \mu_{C_2'}(z) + \cdots + \mu_{C_n'}(z)]dz}$$

$$= \frac{\int z \cdot \mu_{C_1'}(z)dz + \cdots + \int z \cdot \mu_{C_n'}(z)dz}{\int \mu_{C_1'}(z)dz + \cdots + \int \mu_{C_n'}(z)dz}$$

$$= \frac{S_1' \cdot z_1 + S_2' \cdot z_2 + \cdots + S_n' \cdot z_n}{S_1' + S_2' + \cdots + S_n'}$$

$$= \frac{h_1 \cdot S_1 \cdot z_1 + h_2 \cdot S_2 \cdot z_2 + \cdots + h_n \cdot S_n \cdot z_n}{h_1 \cdot S_1 + h_2 \cdot S_2 + \cdots + h_n \cdot S_n} \quad (15)$$

where the area Si' of Ci' is given as

$$S_i' = h_i \cdot S_i \quad (16)$$

It is found that the way of obtaining the representative point z_0 has the same form as (12) with replacing wi with Si.

Thus, a fuzzy rule of product-sum-gravity method,

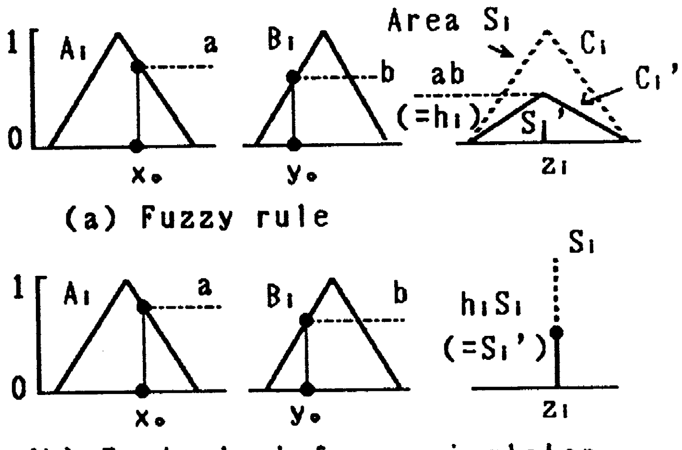
Ai and Bi => Ci (17)

is transformed to a fuzzy rule of fuzzy singleton-type reasoning method as

Ai and Bi => Si/zi (18)

(see Fig.6), where Si is the area of Ci and zi is the zenter of gravity of Ci.

Therefore, product-sum-gravity method is reduced to fuzzy singleton-type reasoning method by interpreting the area Si of the conclusion part Ci as the weight wi of fuzzy singleton-type reasoning method. Therefore, two fuzzy reasoning methods are shown to be equivalent.



(a) Fuzzy rule

(b) Equivalent fuzzy singleton -type rule

Fig.6 Equivalent Fuzzy rules

Rule 1: A_1 and B_1 => S_1/z_1
Rule 2: A_2 and B_2 => S_2/z_2
....
Rule n: A_n and B_n => S_n/z_n (19)
Fact: $x_\circ$ and $y_\circ$
Cons: $z_\circ$

Remark 1: Note that the above discussion holds for any types of fuzzy set Ci, though Ci in Fig.5 is of a triangular type.
Remark 2: Product-sum-gravity method is reduced to a simplified fuzzy reasoning method when the areas of Ci's are all equal, that is, $S_1 = S_2 = \cdots = S_n$.
Remark 3: The inference result Ci' in (3) is given by using algebraic product (·) twice. More generally, Ci' can be obtained by using two kinds of product operations $*_1$ and $*_2$ as follows:

$$\mu_{Ci'}(z) = [\mu_{Ai}(x_\circ) *_1 \mu_{Bi}(y_\circ)] *_2 \mu_{Ci}(z) \quad (20)$$

Such a fuzzy reasoning method is written as "($*_1/*_2$)-sum-gravity method." For example, product-sum-gravity method is represented as (product/product)-sum-gravity method. The fuzzy reasoning method "($*_1/*_2$)-sum-gravity method" can be represented by a fuzzy singleton-type reasoning method as long as $*_2$ is algebraic product, even if $*_1$ is arbitrary product operation such as min, bounded-product or, more generally, t-norms. In other words, ($*_1$/product)-sum-gravity method is equivalent to fuzzy singleton-type reasoning method in which $*_1$ operation is a t-norm. Note that min-max-gravity method is not equivalent to a fuzzy singleton-type reasoning method. More general discussion is found in [10].

In product-sum-gravity method, the extrapolative reasoning [8] can be executed by extending the range of membership functions of antecedent parts of fuzzy rules from [0,1] to $(-\infty,\infty)$. Furthermore, emphatic or suppressive effects [9] on fuzzy inference results can be realized by employing fuzzy rules whose consequent part is characterized by a membership function whose grades are in $(-\infty,\infty)$. These facts indicate that the membership functions of fuzzy rules should not be restricted but be more arbitrary.

In the following, we shall show that fuzzy rules whose fuzzy sets have the grades greater than 1 are reduced to fuzzy singleton-type rules.

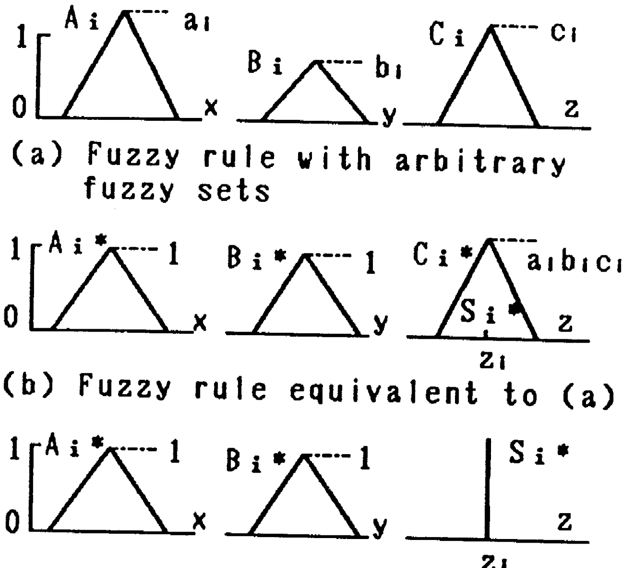
Now let us consider the fuzzy rule in Fig.7(a) whose antecedent part and consequent part consist of fuzzy sets whose heights are arbitrary, that is, may be greater than 1.

$$\text{Ai and Bi} \Rightarrow \text{Ci} \quad (21)$$

where ai, bi and ci are the heights of fuzzy sets Ai, Bi and Ci. Then the following equivalent rule can be obtained as

$$\text{Ai* and Bi*} \Rightarrow \text{Ci*} \quad (22)$$

where the heights of Ai* and Bi* are 1 and that of Ci*



(a) Fuzzy rule with arbitrary fuzzy sets

(b) Fuzzy rule equivalent to (a)

(c) Fuzzy singleton-type rule equivalent to (b)

Fig.7 Equivalent fuzzy rules

is aibici (Fig.7(b)). Moreover, this fuzzy rule is equivalent to the fuzzy singleton-type rule:

$$\text{Ai* and Bi*} \Rightarrow \text{Si*}/zi \quad (23)$$

where Si* is the area of Ci* and zi is the center of gravity of Ci*.

3.2 Equivalence of fuzzy singleton-type reasoning method and simplified fuzzy reasoning method

The weight wi in (11), or area Si in (19), of fuzzy singleton-type reasoning method is assumed to be a non-negative real number. But the weight can be normalized by dividing by, say, the sum or max of w_1, w_2, ⋯, wn in the following.

$$wi' = wi/(w_1 + w_2 + \cdots + wn) \quad (24)$$

$$wi' = wi/\max\{w_1, w_2, \cdots, wn\} \quad (25)$$

The inference result obtained under the new weights wi' which are in [0,1] is easily shown to be the same as the one in (12). Therefore, the weight is assumed to be normalized in the following discussion.

A fuzzy singleton-type rule of

$$\text{Ai and Bi} \Rightarrow wi/zi \quad (26)$$

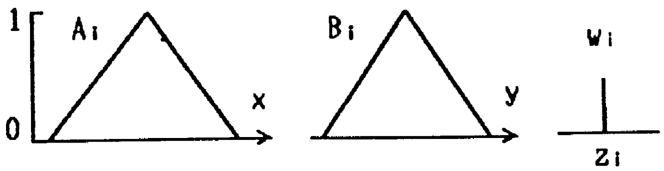
is equivalent to the following fuzzy rule of simplified fuzzy reasoning method.

$$\text{Ai' and Bi'} \Rightarrow zi \quad (27)$$

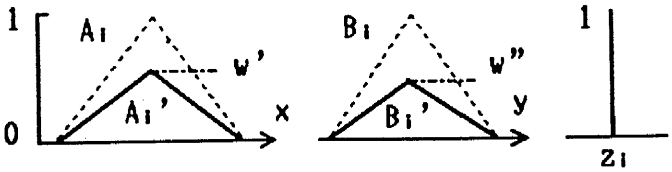
where the fuzzy set A' and Bi' are given as follows (Fig.8).

$$\mu_{Ai'}(x) = w' \cdot \mu_{Ai}(x) \quad (28)$$

$$\mu_{Bi'}(x) = w'' \cdot \mu_{Bi}(x) \quad (29)$$



(a) Fuzzy singleton-type rule



(b) Simplified fuzzy rule equivalent to (a)

Fig.8 Equivalent fuzzy rules

where w' and w'' are such that

$$w_i = w' \cdot w'' \qquad (30)$$

Hence, a fuzzy singleton-type rule is reduced to a fuzzy rule by multiplying by the weights w' and w'' the fuzzy sets of antecedent part of the simplified fuzzy reasoning method. Therefore, fuzzy singleton-type reasoning method is equivalent to simplified fuzzy reasoning method.

4. Conclusions

Product-sum-gravity method is known as a useful fuzzy control method as well as min-max-gravity method by Mamdani, both of which adopt fuzzy sets in the consequent parts of their fuzzy rules, while a fuzzy singleton-type reasoning method and a simplified reasoning method have singletons in the consequence parts. The fact that these three fuzzy reasoning methods are equivalent to each other indicates that the consequent parts are enough to be "singletons" as long as sum operation (+) is used in the aggregation of fuzzy inference results as in (5).

It should be noted that, although the reduced form of product-sum-gravity method is a fuzzy singleton-type reasoning method, there exist no reduced forms for min-max-gravity method.

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