Approximate Reasoning in Expert Systems
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EXTENDED FUZZY REASONING

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As an extension of ordinary fuzzy reasoning, the following form of fuzzy reasoning is discussed in which each of the premises consists of several fuzzy propositions combined with "and".

Prem 1: If x_1 is A_1 and x_2 is A_2 and ... and x_n is A_n then y is B
Prem 2: x_1 is A_1' and x_2 is A_2' and ... and x_n is A_n'
Cons: y is B'

We obtain inference results for the extended fuzzy reasoning under a number of translating rules translated from the fuzzy conditional proposition. All of the translating rules except Rc infer consequences which are given as the union of consequences of ordinary fuzzy reasoning. A translating rule Rc infers consequences equal to the intersection of consequences of ordinary fuzzy reasoning. It is shown that translating rules Rc, Rs, Rg, Rsg, Rgg, Rgs and Rss can infer reasonable inference results for the extended fuzzy reasoning.

Keywords: Fuzzy sets, fuzzy reasoning, extended fuzzy reasoning, compositional rule of inference, approximate reasoning.

INTRODUCTION

In our daily life we often make inferences of the form:

Prem 1: If x is A then y is B
Prem 2: x is A'
Cons: y is B'

where A, A', B and B' are fuzzy concepts. In order to make such an inference with fuzzy concepts, Zadeh suggested an inference rule called the "compositional rule of inference", which infers B' of Cons from Prem 1 and Prem 2 by taking the max-min composition of A' and the fuzzy relation which is translated from the fuzzy conditional proposition "If x is A then y is B". In this connection, he [1], Mamdani [2] , and Mizumoto et al.[3] suggested several translating rules for translating the fuzzy proposition "If x is A then y is B" into a fuzzy relation. In [3,4] we have investigated the properties of their methods by using the compositional rule of inference.

In this paper, as an extension of the ordinary fuzzy reasoning given above, the following form of fuzzy reasoning is discussed in which each premise consists of several fuzzy propositions combined using the connective "and".

Prem 1: If x_1 is A_1 and x_2 is A_2 and ... and x_n is A_n then y is B
Prem 2: x_1 is A_1' and x_2 is A_2' and ... and x_n is A_n'
Cons: y is B'

We obtain and compare the inference results for the extended fuzzy reasoning under several translating rules which are methods of translating the above fuzzy conditional proposition into a fuzzy relation.

EXTENDED FUZZY REASONING

We shall first review the following form of ordinary fuzzy reasoning.

$$\begin{array}{ll} \text{Prem 1:} & \text{If } x \text{ is } A \text{ then } y \text{ is } B \\ \text{Prem 2:} & x \text{ is } A' \\ \hline \text{Cons:} & y \text{ is } B' \end{array} \qquad (1)$$

where x and y are the names of objects, and A, A', B and B' are fuzzy concepts represented by fuzzy sets in universes of discourse U, U, V and V, respectively.

The fuzzy proposition "If x is A then y is B" represents a certain relationship between A and B. From this point of view, a number of translating rules have been proposed for translating the fuzzy conditional proposition "If x is A then y is B" into a fuzzy relation in U x V. For example, Zadeh [1] proposed a translating rule Ra called "arithmetic rule".

Let A and B be fuzzy sets in U and V, respectively, then the arithmetic rule is given as

$$Ra(A,B) = (\bar{A} \times V) \oplus (U \times B)$$

$$= \int_{U \times V} 1 \wedge (1 - \mu_A(u) + \mu_B(v)) / (u,v) \qquad (2)$$

It is noted that the arithmetic rule is based on the implication rule of Lukasiewicz's logic, i.e.,

$$a \to b = 1 \wedge (1 - a + b) \quad a, b \in [0, 1] \qquad (3)$$

It is possible to introduce other implication rules of many-valued logic systems to a translating rule for the fuzzy conditional proposition (see [3]).

$$Rm: \quad a \to b = (a \wedge b) \vee (1 - a) \qquad (4)$$

$$Rc: \quad a \to b = a \wedge b \qquad (5)$$

$$Rs: \quad a \underset{s}{\to} b = \begin{cases} 1 & \ldots a \leqq b \\ 0 & \ldots a > b \end{cases} \qquad (6)$$

$$Rg: \quad a \underset{g}{\to} b = \begin{cases} 1 & \ldots a \leqq b \\ b & \ldots a > b \end{cases} \qquad (7)$$

$$Rsg: \quad a \underset{sg}{\to} b = (a \underset{s}{\to} b) \wedge (1\text{-}a \underset{g}{\to} 1\text{-}b) \qquad (8)$$

$$Rgg: \quad a \underset{gg}{\to} b = (a \underset{g}{\to} b) \wedge (1\text{-}a \underset{g}{\to} 1\text{-}b) \qquad (9)$$

$$Rgs: \quad a \underset{gs}{\to} b = (a \underset{g}{\to} b) \wedge (1\text{-}a \underset{s}{\to} 1\text{-}b) \qquad (10)$$

$$Rss: \quad a \underset{ss}{\to} b = (a \underset{s}{\to} b) \wedge (1\text{-}a \underset{s}{\to} 1\text{-}b) \qquad (11)$$

$$Rb: \quad a \to b = (1 - a) \vee b \qquad (12)$$

$$R_{\triangle}: \quad a \to b = \begin{cases} 1 & \ldots a \leqq b \\ b/a & \ldots a > b \end{cases} \qquad (13)$$

$$R_{\blacktriangle}: \quad a \to b = \begin{cases} 1 \wedge b/a \wedge (1\text{-}a)/(1\text{-}b) & \ldots a > 0,\ 1\text{-}b > 0 \\ 1 & \ldots a = 0 \text{ or } 1\text{-}b = 0 \end{cases} \qquad (14)$$

$$R_{*}: \quad a \to b = 1 - a + ab \qquad (15)$$

$$R_{\#}: \quad a \to b = (1-a \wedge b) \vee (a \wedge 1-a) \vee (b \wedge 1-b) \tag{16}$$

$$R_{\square}: \quad a \to b = \begin{cases} 1 & \ldots \; a < 1 \;\text{ or }\; b = 1 \\ 0 & \ldots \; a = 1, \;\; b < 1 \end{cases} \tag{17}$$

The consequence B' of fuzzy reasoning (1) can be deduced from Prem 1 and Prem 2 by taking the max-min composition "o" of the fuzzy set A' and the fuzzy relation obtained above (the <u>compositional rule of inference</u>). For example, we have for Ra

$$Ba' = A' \circ Ra(A,B) = A' \circ [(\bar{A} \times V) \oplus (U \times B)] \tag{18}$$

$$\mu_{Ba'}(v) = \bigvee_u \{\mu_{A'}(u) \wedge [1 \wedge (1 - \mu_A(u) + \mu_B(v))]\} \tag{19}$$

For example, when A' = A, the arithmetic rule Ra infers such a consequence as

$$Ba' = A \circ Ra(A,B) = \int_V \frac{1 + \mu_B(v)}{2} / v \quad \neq \quad B \tag{20}$$

This inference result indicates that the arithmetic rule does not satisfy the modus ponens which is quite reasonable demand in the fuzzy reasoning.

$$\begin{array}{l} \text{If } x \text{ is } A \text{ then } y \text{ is } B \\ \underline{x \text{ is } A \qquad\qquad\qquad} \\ y \text{ is } B \end{array} \qquad \text{(modus ponens)} \tag{21}$$

In the same way, we can obtain a consequence by each of the translating rules based on the implications in (4)-(17). The inference results by these translating rules are given in [3] when A' = A, <u>very</u> A, <u>more or less</u> A and <u>not</u> A.

We shall next consider the following form of inference in which the hypothesis of a fuzzy conditional proposition "If ... then ..." contains two fuzzy propositions "x_1 is A_1" and "x_2 is A_2" combined using the connective "and".

$$\begin{array}{ll} \text{Prem 1:} & \text{If } x_1 \text{ is } A_1 \text{ and } x_2 \text{ is } A_2 \text{ then } y \text{ is } B \\ \text{Prem 2:} & x_1 \text{ is } A_1' \text{ and } x_2 \text{ is } A_2' \\ \hline \text{Cons:} & y \text{ is } B' \end{array} \tag{22}$$

where x_1, x_2 and y are the names of objects, and A_i, A_i' , B and B' are fuzzy concepts represented by fuzzy sets in universes of discourse U_i, U_i, V and V, respectively.

The following is an example of the fuzzy reasoning whose form is often used in the treatment of fuzzy control problems.

Prem 1: If the pressure is <u>big positive</u> and the change in this error is <u>zero or small</u>.
then the heat change is <u>small negative</u>.
Prem 2: This pressure error is <u>rather positive</u> and the change in the error is <u>very small</u>

Cons: This heat change is <u>zero negative</u>.

The fuzzy proposition "x_1 is A_1' and x_2 is A_2'" of Prem 2 in (22) consists of two fuzzy propositions "x_1 is A_1'" and "x_2 is A_2'" which are connected with "and", where A_1' and A_2' are fuzzy sets in U_1 and U_2, respectively. This compound fuzzy proposition is denoted as $A_1' \cap A_2'$, i.e.,

$$x_1 \text{ is } A_1' \text{ and } x_2 \text{ is } A_2' \equiv A_1' \cap A_2' \tag{23}$$

The notation $A_1' \cap A_2'$ stands for the intersection of $A_1' \times U_2$ and $U_1 \times A_2'$, and, in other words, a fuzzy product $A_1' \times A_2'$.

$$\begin{aligned} A_1' \cap A_2' &= (A_1' \times U_2) \cap (U_1 \times A_2') \\ &= A_1' \times A_2' \\ &= \int_{U_1 \times U_2} \mu_{A_1'}(u_1) \wedge \mu_{A_2'}(u_2) \,/\, (u_1,u_2) \end{aligned} \tag{24}$$

where $\wedge$ means "min".

On the other hands, the fuzzy conditional proposition "If x_1 is A_1 and x_2 is A_2 then y is B" of Prem 1 in (22) is translated into a fuzzy relation $R(A_1,A_2;B)$ in $U_1 \times U_2 \times V$, where A_1, A_2 , and B are fuzzy sets in U_1, U_2, and V, respectively.

For example, $Ra(A_1,A_2;B)$ is defined as the extension of "arithmetic rule" of (2) by using the implication of Lukasiewicz's logic of (3).

$$Ra(A_1,A_2;B) = (\overline{A_1 \cap A_2} \times V) \oplus (U_1 \times U_2 \times B) \tag{25}$$

$$= \int 1 \wedge [1 - (\mu_{A_1}(u_1) \wedge \mu_{A_2}(u_2)) + \mu_B(v)] \,/\, (u_1,u_2,v) \tag{26}$$

In the same way, we can define a number of translating rules for translating "If x_1 is A_1 and x_2 is A_2 then y is B" into a fuzzy relation in $U_1 \times U_2 \times V$ by rewriting $a \rightarrow b$ in (4)-(17) as $(\mu_{A_1}(u_1) \wedge \mu_{A_2}(u_2)) \rightarrow \mu_B(v)$. For example, we have

$$Rm(A_1,A_2;B) \tag{27}$$

$$= \int [(\mu_{A_1}(u_1) \wedge \mu_{A_2}(u_2)) \wedge \mu_B(v)] \vee [1 - (\mu_{A_1}(u_1) \wedge \mu_{A_2}(u_2))] \,/\, (u_1,u_2,v)$$

$$Rc(A_1,A_2;B) = \int (\mu_{A_1}(u_1) \wedge \mu_{A_2}(u_2)) \wedge \mu_B(v) \,/\, (u_1,u_2,v) \tag{28}$$

$$Rs(A_1,A_2;B) = \int (\mu_{A_1}(u_1) \wedge \mu_{A_2}(u_2)) \underset{s}{\rightarrow} \mu_B(v) \,/\, (u_1,u_2,v) \tag{29}$$

where

$$(\mu_{A_1}(u_1) \wedge \mu_{A_2}(u_2)) \underset{s}{\rightarrow} \mu_B(v) = \begin{cases} 1 & \dots (\mu_{A_1}(u_1) \wedge \mu_{A_2}(u_2)) \leqq \mu_B(v) \\ 0 & \dots (\mu_{A_1}(u_1) \wedge \mu_{A_2}(u_2)) > \mu_B(v) \end{cases}$$

Similarly, we can obtain other fuzzy relations $Rg(A_1,A_2;B)$, $Rsg(A_1,A_2;B)$, $Rgg(A_1,A_2;B)$, $Rgs(A_1,A_2;B)$, $Rss(A_1,A_2;B)$, $Rb(A_1,A_2;B)$, $R_\triangle(A_1,A_2;B)$, $R_\blacktriangle(A_1,A_2;B)$, $R_*(A_1,A_2;B)$, $R_\#(A_1,A_2;B)$, and $R_\square(A_1,A_2;B)$ by using the implications in (7)-(17).

We shall next obtain inference result B' of (22) under each of translating rules given above. The consequence B' can be deduced from Prem 1 and Prem 2 by taking the max-min composition "$\circ$" of the fuzzy set $A_1' \cap A_2'$ in (24) and the fuzzy relation given above. For example, the consequence Ba' by the translating rule Ra is obtained as

$$Ba' = (A_1' \cap A_2') \circ Ra(A_1, A_2; B) \tag{30}$$

$$\mu_{Ba'}(v) \tag{31}$$
$$= \bigvee_{u_1,u_2} \{(\mu_{A_1'}(u_1) \wedge \mu_{A_2'}(u_2)) \wedge [1 \wedge (1 - (\mu_{A_1}(u_1) \wedge \mu_{A_2}(u_2)) + \mu_B(v))] \}$$

In the same way, we can have the consequences B' by other translating rules Rm, Rc, Rs, ..., $R_{\square}$. For example,

$$Bm' = (A_1' \cap A_2') \circ Rm(A_1, A_2; B) \tag{32}$$

$$Bc' = (A_1' \cap A_2') \circ Rc(A_1, A_2; B) \tag{33}$$

$$Bs' = (A_1' \cap A_2') \circ Rs(A_1, A_2; B) \tag{34}$$

$$\vdots$$

As a simple case, we shall first discuss the method Ra when $A_1' = A_1$ and $A_2' = A_2$. We assume in the discussion of inference results that $\mu_{A_1}(u_1)$ and $\mu_{A_2}(u_2)$ take all values in [0, 1] as u_1 and u_2 vary over U_1 and U_2, respectively, that is, μ_{A_1} and μ_{A_2} are functions onto [0, 1].

We have the consequence Ba' at $A_1' = A_1$ and $A_2' = A_2$ from (31) as follows.

$$\mu_{Ba'}(v)$$
$$= \bigvee_{u_1,u_2} \{(\mu_{A_1}(u_1) \wedge \mu_{A_2}(u_2)) \wedge [1 \wedge (1 - (\mu_{A_1}(u_1) \wedge \mu_{A_2}(u_2)) + \mu_B(v))] \}$$
$$= \bigvee_{u_1,u_2} \{(\mu_{A_1}(u_1) \wedge \mu_{A_2}(u_2)) \wedge [1 \wedge [(1 - \mu_{A_1}(u_1) + \mu_B(v)) \vee (1 - \mu_{A_2}(u_2) + \mu_B(v))]] \}$$
$$= \bigvee_{u_1,u_2} \{(\mu_{A_1}(u_1) \wedge \mu_{A_2}(u_2)) \wedge [1 \wedge (1 - \mu_{A_1}(u_1) + \mu_B(v))]\}$$
$$\vee \bigvee_{u_1,u_2} \{(\mu_{A_1}(u_1) \wedge \mu_{A_2}(u_2)) \wedge [1 \wedge (1 - \mu_{A_2}(u_2) + \mu_B(v))]\}$$
$$= \bigvee_{u_1} \{\mu_{A_1}(u_1) \wedge [1 \wedge (1 - \mu_{A_1}(u_1) + \mu_B(v))]\} \quad \ldots\ldots \quad \bigvee_{u_1} \mu_{A_1}(u_1) = \bigvee_{u_2} \mu_{A_2}(u_2) = 1$$
$$\vee \bigvee_{u_2} \{\mu_{A_2}(u_2) \wedge [1 \wedge (1 - \mu_{A_2}(u_2) + \mu_B(v))]\}$$

The last two terms correspond to the inference results $A_1 \circ R(A_1;B)$ and $A_2 \circ R(A_2;B)$ of ordinary fuzzy reasoning (1) under the method Ra given in (18). Thus, the consequence Ba' becomes

$$\mu_{Ba'}(v) = \frac{1 + \mu_B(v)}{2} \vee \frac{1 + \mu_B(v)}{2} = \frac{1 + \mu_B(v)}{2}$$

Hence, we have the consequence Ba' at $A_1' = A_1$ and $A_2' = A_2$ as

$$Ba' = (A_1 \cap A_2) \circ Ra(A_1, A_2; B) = \int_V \frac{1 + \mu_B(v)}{2} / v \qquad (35)$$

According to our intuition, it seems that the consequence B' in (22) should be B when $A_1' = A_1$ and $A_2' = A_2$. Namely, the following inference may be a quite natural demand.

Prem 1:	If x_1 is A_1 and x_2 is A_2 then y is B	
Prem 2:	x_1 is A_1 and x_2 is A_2	(36)
Cons:	y is B	

It is noted that the inference result Ba' of (35) by the method Ra does not satisfy this natural criterion.

We shall next investigate the inference result by the translating rule Rc (28) proposed by Mamdani [2] which is often used in the discussion of fuzzy control problems.
The consequence Bc' at $A_1' = A_1$ and $A_2' = A_2$ is given from (33) as

$$\mu_{Bc'}(v)$$

$$= \bigvee_{u_1,u_2} \{(\mu_{A_1}(u_1) \wedge \mu_{A_2}(u_2)) \wedge [(\mu_{A_1}(u_1) \wedge \mu_{A_2}(u_2)) \wedge \mu_B(v)]\}$$

$$= \bigvee_{u_1,u_2} \{\mu_{A_1}(u_1) \wedge \mu_{A_2}(u_2) \wedge \mu_B(v)\}$$

$$= \mu_B(v) \qquad \cdots\cdot \qquad \bigvee_{u_1} \mu_{A_1}(u_1) = \bigvee_{u_2} \mu_{A_2}(u_2) = 1$$

Therefore, we have

$$Bc' = (A_1 \cap A_2) \circ Rc(A_1, A_2; B) = B \qquad (37)$$

which leads to the satisfaction of the criterion of (36).

We shall next obtain inference results under other translating rules $Rm(A_1,A_2;B)$, $Rs(A_1,A_2;B)$, ..., $R_\square(A_1,A_2;B)$. At first, we consider the implications in (3)-(17) which satisfy the property:

$$(a_1 \wedge a_2) \rightarrow b = (a_1 \rightarrow b) \vee (a_2 \rightarrow b), \qquad a_1, a_2, b \in [0, 1] \qquad (38)$$

The implications satisfying this property are Ra, Rs, Rg, Rb, $R_\triangle$, $R_\blacktriangle$, $R_\#$ and $R_\square$. For the translating rules $Ra(A_1, A_2;B)$, $Rs(A_1, A_2;B)$, ..., $R_\square(A_1,A_2;B)$ which are based on these implications, we have in general from (38) as

$$\mu_{R(A_1,A_2;B)}(u_1,u_2,v) = [\mu_{A_1}(u_1) \wedge \mu_{A_2}(u_2)] \rightarrow \mu_B(v)$$

$$= [\mu_{A_1}(u_1) \rightarrow \mu_B(v)] \vee [\mu_{A_2}(u_2) \rightarrow \mu_B(v)]$$

$$= \mu_{R(A_1;B)}(u_1, v) \vee \mu_{R(A_2;B)}(u_2,v)$$

Therefore, the inference results B' in (22) under these translating rules are obtained as follows.

$$B' = (A_1' \cap A_2') \circ R(A_1,A_2;B)$$

$$\mu_{B'}(v) = \bigvee_{u_1,u_2} \{(\mu_{A_1'}(u_1) \wedge \mu_{A_2'}(u_2)) \wedge \mu_{R(A_1,A_2;B)}(u_1,u_2,v)\}$$

$$= \bigvee_{u_1,u_2} \{(\mu_{A_1'}(u_1) \wedge \mu_{A_2'}(u_2)) \wedge [\mu_{R(A_1,B)}(u_1,v) \vee \mu_{R(A_2,B)}(u_2,v)]\}$$

$$= \bigvee_{u_1,u_2} \{\mu_{A_1'}(u_1) \wedge \mu_{A_2'}(u_2) \wedge \mu_{R(A_1;\ B)}(u_1, v)\}$$

$$\vee \bigvee_{u_1,u_2} \{\mu_{A_1'}(u_1) \wedge \mu_{A_2'}(u_2) \wedge \mu_{R(A_2;\ B)}(u_2, v)\}$$

$$= \bigvee_{u_1} \{ \mu_{A_1'}(u_1) \wedge \mu_{R(A_1;B)}(u_1,v)\} \vee \bigvee_{u_2} \{ \mu_{A_2'}(u_2) \wedge \mu_{R(A_2;B)}(u_2,v)\}$$

$$= \mu_{A_1' \circ R(A_1;B)}(v) \vee \mu_{A_2' \circ R(A_2;B)}(v) \tag{39}$$

Thus, we obtain

$$B' = (A_1' \cap A_2') \circ R(A_1,A_2;B)$$

$$= (A_1' \circ R(A_1;B)) \cup (A_2' \circ R(A_2;B)) \tag{40}$$

for the translating rules $Ra(A_1,A_2;B)$, $Rs(A_1,A_2;B)$, $Rg(A_1,A_2;B)$, $Rb(A_1,A_2;B)$, $R_\triangle(A_1,A_2;B)$, $R_\blacktriangle(A_1,A_2;B)$, $R_*(A_1,A_2;B)$, and $R_\square(A_1,A_2;B)$. It is noted that the consequence B' is given as the union of the consequences B_1' and B_2' of ordinary fuzzy reasoning such that

$$\begin{array}{ll} \text{If } x_1 \text{ is } A_1 \text{ then } y \text{ is } B & \text{If } x_2 \text{ is } A_2 \text{ then } y \text{ is } B \\ \underline{x_1 \text{ is } A_1'} & \underline{x_2 \text{ is } A_2'} \\ y \text{ is } B_1' (= A_1' \circ R(A_1;B)) & y \text{ is } B_2' (= A_2' \circ R(A_2;B)) \end{array} \tag{41}$$

We shall next investigate the translating rule Rc of (28) which is based on the implication $a \rightarrow b = a \wedge b$ which does not satisfy the property (38).

The consequence Bc' is inferred as

$$Bc' = (A_1' \cap A_2') \circ Rc(A_1,A_2;B)$$

$$\mu_{Bc'}(v) = \bigvee_{u_1,u_2} \{\mu_{A_1'}(u_1) \wedge \mu_{A_2'}(u_2) \wedge [\mu_{A_1}(u_1) \wedge \mu_{A_2}(u_2) \wedge \mu_B(v)] \}$$

$$= \bigvee_{u_1} \{\mu_{A_1'}(u_1) \wedge \mu_{A_1}(u_1) \wedge \mu_B(v) \wedge \bigvee_{u_2} [\mu_{A_2'}(u_2) \wedge \mu_{A_2}(u_2) \wedge \mu_B(v)] \}$$

$$= \bigvee_{u_1} \{\mu_{A_1'}(u_1) \wedge \mu_{A_1}(u_1) \wedge \mu_B(v) \wedge \mu_{A_2' \circ Rc(A_2;B)}(v)\}$$

$$= \mu_{A_1' \circ Rc(A_1;B)}(v) \wedge \mu_{A_2' \circ Rc(A_2;B)}(v) \tag{42}$$

Therefore, the consequence Bc' is given as the intersection of B_1' and B_2' of ordinary fuzzy reasoning in (41). Namely, we have for the translating rule Rc

$$Bc' = (A_1' \cap A_2') \circ Rc(A_1,A_2;B)$$

$$= (A_1' \circ Rc(A_1;B)) \cap (A_2' \circ Rc(A_2;B)) \tag{43}$$

As for other translating rules Rm, Rsg, Rgg, Rgs, Rss and R# which have not been discussed above, we can not treat them uniformly. So we shall investigate them in the case where fuzzy sets A_i' (i = 1, 2) in Prem 2 in (22) are restricted to such fuzzy sets as

$$A_i' = A_i \tag{44}$$

$$A_i' = \underline{\text{very}}\ A_i = A_i^2 = \int \mu_{A_i}(u_i)^2 / u_i \tag{45}$$

$$A_i' = \underline{\text{more or less}}\ A_i = A_i^{0.5} = \int \sqrt{\mu_{A_i}(u_i)} / u_i \tag{46}$$

$$A_i' = \underline{\text{not}}\ A_i = \int 1 - \mu_{A_i}(u_i) / u_i \tag{47}$$

which are typical fuzzy sets of A_i'.

We shall discuss the translating rule $Rm(A_1,A_2;B)$ of (27) only in the case of A_1' = very A_1 and A_2' = more or less A_2 because of limitations of space.

The consequence Bm' by $Rm(A_1,A_2;B)$ is defined as

$$Bm' = (\underline{\text{very}}\ A_1 \cap \underline{\text{more or less}}\ A_2) \circ Rm(A_1,A_2;B) \tag{48}$$

$$\mu_{Bm'}(v) = \bigvee_{u_1,u_2} \{(\mu_{A_1}(u_1)^2 \wedge \sqrt{\mu_{A_2}(u_2)})$$
$$\wedge [(\mu_{A_1}(u_1) \wedge \mu_{A_2}(u_2) \wedge \mu_B(v)) \vee (1 - (\mu_{A_1}(u_1) \wedge \mu_{A_2}(u_2)))]\} \tag{49}$$

From the assumption denoted above that μ_{A_1} and μ_{A_2} are functions onto [0, 1], the values $\mu_{A_1}(u_1)^2$ and $\sqrt{\mu_{A_2}(u_2)}$ take all values in [0, 1]. Thus, for simplicity, we rewrite (49) as

$$bm' = \bigvee_{a_1,a_2} \{(a_1^2 \wedge \sqrt{a_2}) \wedge [(a_1 \wedge a_2 \wedge b) \vee (1 - (a_1 \wedge a_2))]\} \tag{50}$$

by letting

$$bm' = \mu_{Bm'}(v), \quad a_1 = \mu_{A_1}(u_1), \quad a_2 = \mu_{A_2}(u_2), \quad b = \mu_B(v) \tag{51}$$

Thus, bm' is as follows.

bm'

$$= \bigvee_{a_1, a_2} \{(a_1^2 \wedge \sqrt{a_2} \wedge a_1 \wedge a_2 \wedge b) \vee (a_1^2 \wedge \sqrt{a_2} \wedge (1 - (a_1 \wedge a_2)))\}$$

$$= \bigvee_{a_1, a_2} \{a_1^2 \wedge a_2 \wedge b\} \vee \bigvee_{a_1, a_2} \{a_1^2 \wedge \sqrt{a_2} \wedge [(1 - a_1) \vee (1 - a_2)]\}$$

$$= b \vee \bigvee_{a_1, a_2} \{[a_1^2 \wedge \sqrt{a_2} \wedge (1 - a_1)] \vee [a_1^2 \wedge \sqrt{a_2} \wedge (1 - a_2)]\}$$

$$= b \vee \bigvee_{a_2} \{\sqrt{a_2} \wedge \bigvee_{a_1} [a_1^2 \wedge (1 - a_1)]\} \vee \bigvee_{a_1} \{a_1^2 \wedge \bigvee_{a_2} [\sqrt{a_2} \wedge (1 - a_2)]\}$$

$$= b \vee \bigvee_{a_2} \{\sqrt{a_2} \wedge \frac{3 - \sqrt{5}}{2}\} \vee \bigvee_{a_1} \{a_1^2 \wedge \frac{\sqrt{5} - 1}{2}\}$$

$$= b \vee \frac{3 - \sqrt{5}}{2} \vee \frac{\sqrt{5} - 1}{2}$$

$$= (b \vee \frac{3 - \sqrt{5}}{2}) \vee (b \vee \frac{\sqrt{5} - 1}{2}) \tag{52}$$

It is noted that the consequences B_1' and B_2' of ordinary fuzzy reasoning of (41) under the method Rm are as follows when A_1' = very A_1 and A_2' = more or less A_2 (see [3]).

$$\mu_{B_1'}(v) = \mu_{\text{very } A_1} \circ Rm(A_1;B)(v) = \mu_B(v) \vee \frac{3 - \sqrt{5}}{2}$$

$$\mu_{B_2'}(v) = \mu_{\text{more or less } A_2} \circ Rm(A_2;B)(v) = \mu_B(v) \vee \frac{\sqrt{5} - 1}{2}$$

Therefore, using the notations of (51), the consequence Bm' is obtained from (52) as

$$\mu_{Bm'}(v) = (\mu_B(v) \vee \frac{3 - \sqrt{5}}{2}) \vee (\mu_B(v) \vee \frac{\sqrt{5} - 1}{2})$$

$$= \mu_{\text{very } A_1} \circ Rm(A_1;B)(v) \vee \mu_{\text{more or less } A_2} \circ Rm(A_2;B)(v)$$

$$= \mu_B(v) \vee \frac{\sqrt{5} - 1}{2}$$

Similarly, we can obtain the consequeces Bm' for other A_i' in (44)-(47). It is found from the inference results that the consequence Bm'is given as the union of the consequences of (41), that is,

$$Bm' = (A_1' \cap A_2') \circ Rm(A_1, A_2; B)$$

$$= (A_1' \circ Rm(A_1;B)) \cup (A_2' \circ Rm(A_2;B)) \tag{53}$$

for any fuzzy sets A_i' in (44)-(47).

In the same way, we can obtain inference results under Rsg, Rgg, Rgs, Rss and $R_{\#}$. These results show that

$$B' = (A_1' \cap A_2') \circ R(A_1, A_2 ;B)$$
$$= (A_1' \circ R(A_1,B)) \cup (A_2' \circ R(A_2,B)) \qquad (54)$$

holds for any fuzzy sets A_i' in (44)-(47) and $Rsg(A_1,A_2;B)$, $Rgg(A_1,A_2;B)$, $Rgs(A_1,A_2;B)$, $Rss(A_1,A_2;B)$ and $R_{\#}(A_1,A_2;B)$.

Therefore, if we restrict ourselves on the fuzzy sets A_i' in (44)-(47), the inference result B' in (22) can be represented as the union of the consequences of ordinary fuzzy reasoning of (41) as in the case of Ra, Rs, Rg, Rb, $R_{\triangle}$, $R_{\blacktriangle}$, $R_{\ddagger}$ and $R_{\square}$. It should be noted that only Rc gets the consequence as the intersection of the consequences of ordinary fuzzy reasoning.

Table 1 shows the inference results by all the translating rules when the fuzzy sets A_i' are equal to A_i, <u>very</u> A_i, <u>more or less</u> A_i, and <u>not</u> A_i in (44)-(47). It is found from the table that translating rules Rc, Rs, Rg, Rsg, Rgg, Rgs and Rss infer the consequence B' = B when $A_1' = A_1$ and $A_2' = A_2$. Namely, they satisfy the reasonable criterion of (36) for the fuzzy reasoning.

Finally, we shall investigate fuzzy reasoning of the form:

Prem 1:	If x_1 is A_1 and x_2 is A_2 and ... and x_n is A_n then y is B	
Prem 2:	x_1 is A_1' and x_2 is A_2' and ...and x_n is A_n'	(55)
Cons:	y is B'	

which is an extension of the fuzzy reasoning of (22), where A_i, A_i' (i=1,...,n), B and B' are fuzzy sets in U_i, U_i, V and V, respectively.

The fuzzy proposition "x_1 is A_1' and x_2 is A_2' and ...and x_n is A_n'" is denoted as $A_1' \cap A_2' \cap \ldots \cap A_n'$ which stands for the fuzzy product of A_1', A_2', ..., and A_n', that is,

$$A_1' \cap A_2' \cap \ldots \cap A_n'$$
$$= A_1' \times A_2' \times \ldots \times A_n'$$
$$= \int_{U_1 \times U_2 \times \ldots \times U_n} \mu_{A_1'}(u_1) \wedge \mu_{A_2'}(u_2) \wedge \ldots \wedge \mu_{A_n'}(u_n) \ / \ (u_1,u_2,\ldots,u_n) \qquad (56)$$

The fuzzy conditional proposition "If x_1 is A_1 and x_2 is A_2 and ... and x_n is A_n then y is B" is translated into a fuzzy relation $R(A_1,A_2,\ldots,A_n;B)$ in $U_1 \times U_2 \times \ldots \times U_n \times V$. For example, as the extension of arithmetic rule of (25), $Ra(A_1,A_2,\ldots,A_n;B)$ is defined as

$$Ra(A_1,A_2,\ldots,A_n;B)$$
$$= (\overline{A_1 \cap A_2 \cap \ldots \cap A_n} \times V) \oplus (U_1 \times U_2 \times \ldots \times U_n \times B)$$
$$= 1 \wedge [1 - (\mu_{A_1}(u_1) \wedge \mu_{A_2}(u_2) \wedge \ldots \wedge \mu_{A_n}(u_n)) + \mu_B(v)] \ / \ (u_1,u_2,\ldots,u_n)$$

In the same way, we can define $R(A_1,A_2,\ldots,A_n;B)$ for Rm, Rs, Rg, ..., $R_{\square}$ by using the implications in (4)-(17).

Table 1 Inference Results $B' = (A_1' \cap A_2') \circ R(A_1, A_2; B)$

A_1'	A_2'	Ra	Rm	Rc	Rs	Rg	Rsg	Rgg	Rgs	Rss	Rb
A_1	A_2	$\frac{1+\mu_B}{2}$	$0.5 \vee \mu_B$	μ_B	μ_B	μ_B	μ_B	μ_B	μ_B	μ_B	$0.5 \vee \mu_B$
A_1	very A_2	$\frac{1+\mu_B}{2}$	$0.5 \vee \mu_B$	μ_B	μ_B	μ_B	μ_B	μ_B	μ_B	μ_B	$0.5 \vee \mu_B$
A_1	more or less A_2	$\frac{\sqrt{5+4\mu_B}-1}{2}$	$\frac{\sqrt{5}-1}{2} \vee \mu_B$	μ_B	$\sqrt{\mu_B}$	$\sqrt{\mu_B}$	$\sqrt{\mu_B}$	$\sqrt{\mu_B}$	$\sqrt{\mu_B}$	$\sqrt{\mu_B}$	$\frac{\sqrt{5}-1}{2} \vee \mu_B$
A_1	not A_2	1	1	$0.5 \wedge \mu_B$	1	1	$1-\mu_B$	$1-\mu_B$	$1-\mu_B$	$1-\mu_B$	1
very A_1	very A_2	$\frac{3+2\mu_B-\sqrt{5+4\mu_B}}{2}$	$\frac{3-\sqrt{5}}{2} \vee \mu_B$	μ_B	μ_B^2	μ_B	μ_B^2	μ_B	μ_B	μ_B^2	$\frac{3-\sqrt{5}}{2} \vee \mu_B$
very A_1	more or less A_2	$\frac{\sqrt{5+4\mu_B}-1}{2}$	$\frac{\sqrt{5}-1}{2} \vee \mu_B$	μ_B	$\sqrt{\mu_B}$	$\sqrt{\mu_B}$	$\sqrt{\mu_B}$	$\sqrt{\mu_B}$	$\sqrt{\mu_B}$	$\sqrt{\mu_B}$	$\frac{\sqrt{5}-1}{2} \vee \mu_B$
very A_1	not A_2	1	1	$0.5 \wedge \mu_B$	1	1	$1-\mu_B$	$1-\mu_B$	$1-\mu_B$	$1-\mu_B$	1
more or less A_1	more or less A_2	$\frac{\sqrt{5+4\mu_B}-1}{2}$	$\frac{\sqrt{5}-1}{2} \vee \mu_B$	μ_B	$\sqrt{\mu_B}$	$\sqrt{\mu_B}$	$\sqrt{\mu_B}$	$\sqrt{\mu_B}$	$\sqrt{\mu_B}$	$\sqrt{\mu_B}$	$\frac{\sqrt{5}-1}{2} \vee \mu_B$
more or less A_1	not A_2	1	1	$0.5 \wedge \mu_B$	1	1	$1-\mu_B$	$1-\mu_B$	$1-\mu_B$	$1-\mu_B$	1
not A_1	not A_2	1	1	$0.5 \wedge \mu_B$	1	1	$1-\mu_B$	$1-\mu_B$	$1-\mu_B$	$1-\mu_B$	1

Table 1 (continued)

A_1'	A_2'	$R_\triangle$	$R_\blacktriangle$	R_*	$R_\#$	$R_\square$
A_1	A_2	$\sqrt{\mu_B}$	$\sqrt{\mu_B} \wedge \frac{1}{2-\mu_B}$	$\frac{1}{2-\mu_B}$	$0.5 \vee \mu_B$	1
A_1	very A_2	$\sqrt{\mu_B}$	$\sqrt{\mu_B} \wedge \frac{1}{2-\mu_B}$	$\frac{1}{2-\mu_B}$	$0.5 \vee \mu_B$	1
A_1	more or less A_2	$\mu_B^{\frac{1}{3}}$	$\mu_B^{\frac{1}{3}} \wedge \frac{\sqrt{\mu_B^2 - 2\mu_B + 5} + \mu_B - 1}{2}$	$\frac{\sqrt{5-4\mu_B} - 1}{2(1-\mu_B)}$	$\mu_B \vee [(1-\mu_B) \wedge \frac{\sqrt{5}-1}{2}]$	1
A_1	not A_2	1	1	1	$\mu_B \vee (1-\mu_B)$	1
very A_1	very A_2	$\mu_B^{\frac{2}{3}}$	$\mu_B^{\frac{2}{3}} \wedge \left[\frac{\sqrt{5-4\mu_B} - 1}{2(1-\mu_B)}\right]^2$	$\left[\frac{\mu_B - 1 + \sqrt{(1-\mu_B)^2 + 4}}{2}\right]^2$	$\frac{3-\sqrt{5}}{2} \vee \mu_B$	1
very A_1	more or less A_2	$\mu_B^{\frac{1}{3}}$	$\mu_B^{\frac{1}{3}} \wedge \frac{\sqrt{\mu_B^2 - 2\mu_B + 5} + \mu_B - 1}{2}$	$\frac{\sqrt{5-4\mu_B} - 1}{2(1-\mu_B)}$	$\mu_B \vee [(1-\mu_B) \wedge \frac{\sqrt{5}-1}{2}]$	1
very A_1	not A_2	1	1	1	$\mu_B \vee (1-\mu_B)$	1
more or less A_1	more or less A_2	$\mu_B^{\frac{1}{3}}$	$\mu_B^{\frac{1}{3}} \wedge \frac{\sqrt{\mu_B^2 - 2\mu_B + 5} + \mu_B - 1}{2}$	$\frac{\sqrt{5-4\mu_B} - 1}{2(1-\mu_B)}$	$\mu_B \vee [(1-\mu_B) \wedge \frac{\sqrt{5}-1}{2}]$	1
more or less A_1	not A_2	1	1	1	$\mu_B \vee (1-\mu_B)$	1
not A_1	not A_2	1	1	1	$\mu_B \vee (1-\mu_B)$	1

The consequence B' of (55) is obtained by taking the max-min composition of $A_1' \cap A_2' \cap .. \cap A_n'$ and $R(A_1, A_2, ..., A_n; B)$, that is,

$$B' = (A_1' \cap A_2' \cap .... \cap A_n') \circ R(A_1, A_2, ..., A_n; B) \qquad (57)$$

$$\mu_{B'}(v) \qquad (58)$$

$$= \bigvee_{u_1, u_2, .., u_n} \{(\mu_{A_1'}(u_1) \wedge \mu_{A_2'}(u_2) \wedge \cdots \wedge \mu_{A_n'}(u_n)) \wedge \mu_{R(A_1, A_2, ..., A_n; B)}(u_1, u_2, ..., u_n, v)\}$$

We shall next obtain inference results B' under each of translating rules. At first, we shall consider the translating rules Ra, Rs, Rg, Rb, $R_\triangle$, $R_\blacktriangle$, $R_\#$, and $R_\square$ whose implication rules $a \rightarrow b$ satisfy the following property:

$$(a_1 \wedge a_2 \wedge \cdots \wedge a_n) \rightarrow b = (a_1 \rightarrow b) \vee (a_2 \rightarrow b) \vee \cdots \vee (a_n \rightarrow b) \qquad (59)$$

For these translating rules, we have in general

$$\mu_{R(A_1, A_2, ..., A_n; B)}(u_1, u_2, ..., u_n, v)$$

$$= [\mu_{A_1}(u_1) \wedge \mu_{A_2}(u_2) \wedge \cdots \wedge \mu_{A_n}(u_n)] \rightarrow \mu_B(v)$$

$$= [\mu_{A_1}(u_1) \rightarrow \mu_B(v)] \vee [\mu_{A_2}(u_2) \rightarrow \mu_B(v)] \vee \cdots \vee [\mu_{A_n}(u_n) \rightarrow \mu_B(v)]$$

$$= \mu_{R(A_1; B)}(u_1, v) \vee \mu_{R(A_2; B)}(u_2, v) \vee \cdots \vee \mu_{R(A_n; B)}(u_n, v)$$

where the notation $\mu_{R(A_i; B)}(u_i, v)$ stands for the membership function of $R(A_i; B)$ which is a translating rule for a fuzzy conditional proposition "If x_i is A_i then y is B".

Therefore, the consequence B' under these translating rules are obtained as*

$$\mu_{B'}(v)$$

$$= \bigvee_{u_1, ..., u_n} \{(\mu_{A_1'}(u_1) \wedge \cdots \wedge \mu_{A_n'}(u_n)) \wedge [\mu_{R(A_1; B)}(u_1, v) \vee \cdots \vee \mu_{R(A_n; B)}(u_n; v)]\}$$

$$= \bigvee_{u_1, ..., u_n} \{\mu_{A_1'}(u_1) \wedge \cdots \wedge \mu_{A_n'}(u_n) \wedge \mu_{R(A_1; B)}(u_1, v)\}$$

....

$$\vee \bigvee_{u_1, ..., u_n} \{\mu_{A_1'}(u_1) \wedge \cdots \wedge \mu_{A_n'}(u_n) \wedge \mu_{R(A_n; B)}(u_n, v)\}$$

---

* In this discussion we assume that $\mu_{A_i'}$ is a function onto [0, 1]. Thus, A_i' is a normal fuzzy set, i.e., $\bigvee_{u_i} \mu_{A_i'}(u_i) = 1$. Therefore, for example, we have

$$\bigvee_{u_2, ..., u_n} \mu_{A_2'}(u_2) \wedge \cdots \wedge \mu_{A_n'}(u_n) = 1 .$$

$$= \bigvee_{u_1} \{ \mu_{A_1'}(u_1) \wedge \mu_{R(A_1;B)}(u_1,v)\} \ \vee \ \cdots \ \vee \ \bigvee_{u_n} \{ \mu_{A_n'}(u_n) \wedge \mu_{R(A_n;B)}(u_n,v)\}$$

$$= \mu_{A_1'} \circ R(A_1;B)(v) \vee \cdots \vee \mu_{A_n'} \circ R(A_n;B)(v) \tag{60}$$

Therefore, the consequence B' is given as the union of $A_i' \circ R(A_i;B)$, that is,

$$B' = (A_1' \cap A_2' \cap \cdots \cap A_n') \circ R(A_1, A_2, \ldots, A_n; B)$$

$$= (A_1' \circ R(A_1;B)) \cup (A_2' \circ R(A_2;B)) \cup \cdots \cup (A_n' \circ R(A_n;B)) \tag{61}$$

for the translating rules Ra, Rs, Rg, Rb, $R_\triangle$, $R_\blacktriangle$, $R_\#$, and $R_\square$, where $A_i' \circ R(A_i;B)$ represents the consequence B_i' of ordinary fuzzy reasoning:

$$\begin{array}{l} \text{If } x_i \text{ is } A_i \text{ then } y \text{ is } B \\ x_i \text{ is } A_i' \\ \hline y \text{ is } B_i' \ (= A_i' \circ R(A_i;B)) \end{array} \tag{62}$$

We shall next discuss the translating rule $Rc(A_1, \ldots, A_n;B)$ whose implication does not satisfy the properties (59). We have the consequence Bc' as

$$Bc' = (A_1' \cap \cdots \cap A_n') \circ Rc(A_1, \ldots, A_n;B)$$

$$\mu_{Bc'}(v)$$

$$= \bigvee_{u_1,\ldots,u_n} \{\mu_{A_1'}(u_1) \wedge \cdots \wedge \mu_{A_n'}(u_n) \wedge [\mu_{A_1}(u_1) \wedge \cdots \wedge \mu_{A_n}(u_n) \wedge \mu_B(v)]\}$$

$$= \bigvee_{u_1} \{\mu_{A_1'}(u_1) \wedge \mu_{A_1}(u_1) \wedge \mu_B(v) \wedge \bigvee_{u_2} \{\mu_{A_2'}(u_2) \wedge \mu_{A_2}(u_2) \wedge \mu_B(v) \wedge \cdots$$

$$\wedge \bigvee_{u_n} \{\mu_{A_n'}(u_n) \wedge \mu_{A_n}(u_n) \wedge \mu_B(v)\} \cdots \}\}$$

$$= \mu_{A_1'} \circ Rc(A_1;B)(v) \wedge \mu_{A_2'} \circ Rc(A_2;B)(v) \wedge \cdots \wedge \mu_{A_n'} \circ Rc(A_n;B)(v)$$

Thus, the consequence Bc' is given as the intersection of $A_i' \circ Rc(A_i;B)$ for ordinary fuzzy reasoning in (62). Namely,

$$Bc' = (A_1' \cap A_2' \cap \cdots \cap A_n') \circ Rc(A_1, A_2, \ldots, A_n;B) \tag{63}$$

$$= (A_1' \circ Rc(A_1;B)) \cap (A_2' \circ Rc(A_2;B)) \cap \cdots \cap (A_n' \circ Rc(A_n;B))$$

For the translating rules Rm, Rsg, Rgg, Rgs, Rss and $R_\#$, we can have the consequence B' as the union of $A_i' \circ R(A_i;B)$ when A_i' is a fuzzy set in (44)-(47). Namely,

$$B' = (A_1' \cap A_2' \cap \cdots \cap A_n') \circ R(A_1, A_2, \ldots, A_n;B)$$

$$= (A_1' \circ R(A_1;B)) \cup (A_2' \circ R(A_2;B)) \cup \ldots \cup (A_n' \circ R(A_n;B)) \qquad (64)$$

holds for the translating rules Rm, Rsg, Rgg, Rgs, Rss and $R_\#$ when A_i' is restricted to A_i, <u>very</u> A_i, <u>more or less</u> A_i, and <u>not</u> A_i.

The following reasoning form is a quite reasonable one in the extended fuzzy reasoning. This criterion demands that the consequence B' should be B when A_i' is equal to A_i.

$$\begin{array}{l} \text{If } x_1 \text{ is } A_1 \text{ and } x_2 \text{ is } A_2 \text{ and } \ldots \text{ and } x_n \text{ is } A_n \text{ then } y \text{ is } B \\ x_1 \text{ is } A_1 \text{ and } x_2 \text{ is } A_2 \text{ and } \ldots \text{and } x_n \text{ is } A_n \\ \hline y \text{ is } B \end{array} \qquad (65)$$

The translating rules satisfying this criterion are Rc, Rs, Rg, Rsg, Rgg, Rgs and Rss. In the ordinary fuzzy reasoning in (62), they get $A_i \circ R(A_i;B) = B$ at $A_i' = A_i$ (see [3]), that is, they satisfy the modus ponens of (21). As in (61), (63) and (64), the consequence B' for the extended fuzzy reasoning is given as the union or intersection of $A_i \circ R(A_i;B)$ (=B). Hence, the consequence B' becomes B when $A_i' = A_i$ for any $i = 1,\ldots,n$, which leads to the satisfaction of the criterion of (65). Note that translating rules other than Rc, Rs, Rg, Rsg, Rgg, Rgs and Rss do not satisfy the criterion since they get the consequence $A_i \circ R(A_i;B)$ which is not equal to B.

CONCLUSION

As the extension of ordinary fuzzy reasoning, we have investigated the extended fuzzy reasoning under a number of translating rules. The inference results under them are obtained as the union or intersection of those of ordinary fuzzy reasoning. The translating rules Rc, Rs, Rg, Rsg, Rgg, Rgs and Rss are found to satisfy a reasonable criterion.

In this paper we have discussed the case where the connective "and" in the compound proposition "x_1 is A_1 and x_2 is A_2 and ... and x_n is A_n" stands for "min". It would be of interest if we would interpret "and" as other operations such as algebraic product and bounded-product.

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