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# COMPARISON OF FUZZY REASONING METHODS*

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L.A. Zadeh, E.H. Mamdani, and M. Mizumoto et al. have proposed methods for fuzzy reasoning in which the antecedent involves a fuzzy conditional proposition 'If $x$ is $A$ then $y$ is $B$,' with $A$ and $B$ being fuzzy concepts. Mizumoto *et al.* have investigated the properties of their methods in the case of 'generalized modus ponens'.

This paper deals with the properties of their methods in the case of 'generalized modus tollens', and investigates the other new fuzzy reasoning methods obtained by introducing the implication rules of many valued logic systems. Finally, the properties of syllogism and contrapositive are investigated under each fuzzy reasoning method.

*Keywords*: Fuzzy reasoning, Fuzzy conditional inference, Generalized modus ponens, Generalized modus tollens, Syllogism, Contrapositive.

## 1. Introduction

In our daily life we often make inferences whose antecedents and consequences contain fuzzy concepts. Such an inference can not be made adequately by the methods which are based either on classical two valued logic or on many valued logic. In order to make such an inference, Zadeh [1] suggested an inference rule called 'compositional rule of inference'. Using this inference rule, he, Mamdani [2] and Mizumoto et al. [3–6] suggested several methods for fuzzy reasoning in which the antecedent contains a conditional proposition with fuzzy concepts:

$$\begin{array}{ll} \text{Ant 1:} & \text{If } x \text{ is } A \text{ then } y \text{ is } B \\ \text{Ant 2:} & x \text{ is } A' \\ \hline \text{Cons:} & y \text{ is } B' \end{array} \tag{1}$$

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where $A$, $A'$, $B$, $B'$ are fuzzy concepts. An example of the fuzzy reasoning is the following.

$$\begin{array}{ll} \text{Ant 1:} & \text{If a tomato is } \textit{red} \text{ then the tomato is } \textit{ripe} \\ \text{Ant 2:} & \text{This tomato is } \textit{very red} \\ \hline \text{Cons:} & \text{This tomato is } \textit{very ripe} \end{array} \tag{2}$$

In [4–6] we have pointed out that for the type of fuzzy reasoning in (1) called 'generalized modus ponens', the consequences inferred by Zadeh's and Mamdani's methods are not always reasonable and suggested several new methods $R_s$, $R_g$, $R_{ss}$, $R_{sg}$, $R_{gs}$ and $R_{gg}$ which coincide with our intuition with respect to several criteria.

As continuation of our studies, this paper investigates the properties of their fuzzy reasoning methods in the case of 'generalized modus tollens'. Moreover, by introducing the implication rules of many valued logic systems [7–9], we discuss the newly obtained fuzzy reasoning methods in the cases of generalized modus ponens and generalized modus tollens. Finally, we discuss the properties of syllogism and contrapositive under each fuzzy reasoning method.

## 2. Fuzzy reasoning methods

We shall first consider the following form of inference in which a fuzzy conditional proposition is contained.

$$\begin{array}{ll} \text{Ant 1:} & \text{If } x \text{ is } A \text{ then } y \text{ is } B \\ \text{Ant 2:} & x \text{ is } A' \\ \hline \text{Cons:} & y \text{ is } B' \end{array} \tag{3}$$

where $x$ and $y$ are the names of objects, and $A$, $A'$, $B$ and $B'$ are fuzzy concepts represented by fuzzy sets in universes of discourse $U$, $U$, $V$ and $V$, respectively. This form of inference may be viewed as a *generalized modus ponens* which reduces to modus ponens when $A' = A$ and $B' = B$.

Moreover, the following form of inference is also possible which also contains a fuzzy conditional proposition.

$$\begin{array}{ll} \text{Ant 1:} & \text{If } x \text{ is } A \text{ then } y \text{ is } B \\ \text{Ant 2:} & y \text{ is } B' \\ \hline \text{Cons:} & x \text{ is } A' \end{array} \tag{4}$$

This inference can be considered as a *generalized modus tollens* which reduces to modus tollens when $B' = \textit{not } B$ and $A' = \textit{not } A$.

The Ant 1 of the form "If $x$ is $A$ then $y$ is $B$" in (3) and (4) may represent a certain relationship between $A$ and $B$. From this point of view, several methods were proposed for this form of fuzzy conditional proposition: 'If $x$ is $A$ then $y$ is $B$'.

Let $A$ and $B$ be fuzzy sets in $U$ and $V$, respectively, which are represented as

$$A=\int_U \mu_A(u)/u, \qquad B=\int_V \mu_B(v)/v \tag{5}$$

and let $\times$, $\cup$, $\cap$, $\neg$ and $\oplus$ be cartesian product, union, intersection, complement and bounded-sum for fuzzy sets, respectively. Then the following fuzzy relations in $U\times V$ can be derived from the fuzzy conditional proposition "If $x$ is $A$ then y is $B$" in Ant 1 of (3) and (4). The fuzzy relations $R_m$ and $R_a$ were proposed by Zadeh [1], $R_c$ by Mamdani [2], and $R_s$, $R_g$, $R_{sg}$, $R_{gg}$, $R_{gs}$ and $R_{ss}$ are by Mizumoto et al. [3–6].

$$R_m=(A\times B)\cup(\neg A\times V)$$
$$=\int_{U\times V}(\mu_A(u)\wedge\mu_B(v))\vee(1-\mu_A(u))/(u,v). \tag{6}$$

$$R_a=(\neg A\times V)\oplus(U\times B)$$
$$=\int_{U\times V}1\wedge(1-\mu_A(u)+\mu_B(v))/(u,v). \tag{7}$$

$$R_c=A\times B$$
$$=\int_{U\times V}\mu_A(u)\wedge\mu_B(v)/(u,v). \tag{8}$$

$$R_s=A\times V\underset{s}{\Rightarrow}U\times B$$
$$=\int_{U\times V}[\mu_A(u)\underset{s}{\rightarrow}\mu_B(v)]/(u,v), \tag{9}$$

where

$$\mu_A(u)\underset{s}{\rightarrow}\mu_B(v)=\begin{cases}1 & \mu_A(u)\le\mu_B(v),\\ 0 & \mu_A(u)>\mu_B(v).\end{cases}$$

$$R_g=A\times V\underset{g}{\Rightarrow}U\times B$$
$$=\int_{U\times V}[\mu_A(u)\underset{g}{\rightarrow}\mu_B(v)]/(u,v), \tag{10}$$

where

$$\mu_A(u)\underset{g}{\rightarrow}\mu_B(v)=\begin{cases}1 & \mu_A(u)\le\mu_B(v),\\ \mu_B(v) & \mu_A(u)>\mu_B(v).\end{cases}$$

$$R_{sg}=(A\times V\underset{s}{\Rightarrow}U\times B)\cap(\neg A\times V\underset{g}{\Rightarrow}U\times\neg B)$$
$$=\int_{U\times V}[\mu_A(u)\underset{s}{\Rightarrow}\mu_B(v)]\wedge[1-\mu_A(u)\underset{g}{\rightarrow}1-\mu_B(v)]/(u,v). \tag{11}$$

$$R_{gg}=(A\times V\underset{g}{\Rightarrow}U\times B)\cap(\neg A\times V\underset{g}{\Rightarrow}U\times\neg B)$$
$$=\int_{U\times V}[\mu_A(u)\underset{g}{\rightarrow}\mu_B(v)]\wedge[1-\mu_A(u)\underset{g}{\rightarrow}1-\mu_B(v)]/(u,v). \tag{12}$$

$$R_{gs} = (A \times V \underset{g}{\Rightarrow} U \times B) \cap (\neg A \times V \underset{s}{\Rightarrow} U \times \neg B)$$
$$= \int_{U \times V} [\mu_A(u) \underset{g}{\rightarrow} \mu_B(v)] \wedge [1 - \mu_A(u) \underset{s}{\rightarrow} 1 - \mu_B(v)]/(u, v). \quad (13)$$

$$R_{ss} = (A \times V \underset{s}{\Rightarrow} U \times B) \cap (\neg A \times V \underset{s}{\Rightarrow} U \times \neg B)$$
$$= \int_{U \times V} [\mu_A(u) \underset{s}{\rightarrow} \mu_B(v)] \wedge [1 - \mu_A(u) \underset{s}{\rightarrow} 1 - \mu_B(v)]/(u, v). \quad (14)$$

Note that the implications $a \underset{s}{\rightarrow} b$ and $a \underset{g}{\rightarrow} b$ are the implication rules in 'Standard sequence' $S_{\aleph}$ and 'Gödelian sequence' $G_{\aleph}$, respectively [7]. *Ra* is based on the implication rule in Lukasiewicz's logic $L_{\aleph}$.

In addition to the above fuzzy relations (6)–(14), it is also possible to define new fuzzy relations for the proposition "If $x$ is $A$ then y is $B$" by introducing the implication rules of many valued logic systems [7–9]. These implication rules and the implication rules used in (6), (7), (9) and (10) are discussed in detail in [7–9]. In the following we shall discuss some new fuzzy relations.

$$R_b = (\neg A \times V) \cup (U \times B)$$
$$= \int_{U \times V} (1 - \mu_A(u)) \vee \mu_B(v)/(u, v). \quad (15)$$

$$R_\Delta = A \times V \underset{\Delta}{\Rightarrow} U \times B$$
$$= \int_{U \times V} [\mu_A(u) \underset{\Delta}{\rightarrow} \mu_B(v)]/(u, v), \quad (16)$$

where

$$\mu_A(u) \underset{\Delta}{\rightarrow} \mu_B(v) = \begin{cases} 1 & \mu_A(u) \leq \mu_B(v), \\ \dfrac{\mu_B(v)}{\mu_A(u)} & \mu_A(u) > \mu_B(v). \end{cases}$$

$$R_\blacktriangle = A \times V \underset{\blacktriangle}{\Rightarrow} U \times B$$
$$= \int_{U \times V} [\mu_A(u) \underset{\blacktriangle}{\rightarrow} \mu_B(v)]/(u, v), \quad (17)$$

where

$$\mu_A(u) \underset{\blacktriangle}{\rightarrow} \mu_B(v) = [\mu_A(u) \underset{\Delta}{\rightarrow} \mu_B(v)] \wedge [1 - \mu_B(v) \underset{\Delta}{\rightarrow} 1 - \mu_A(u)]$$
$$= \begin{cases} 1 \wedge \dfrac{\mu_B(v)}{\mu_A(u)} \wedge \dfrac{1 - \mu_A(u)}{1 - \mu_B(v)} & \mu_A(u) > 0,\ 1 - \mu_B(v) > 0, \\ 1 & \mu_A(u) = 0 \text{ or } 1 - \mu_B(v) = 0. \end{cases}$$

$$R_* = A \times V \underset{*}{\Rightarrow} U \times B$$
$$= \int_{U \times X} [\mu_A(u) \underset{*}{\rightarrow} \mu_B(v)]/(u, v), \quad (18)$$

where

$$\mu_A(u) \underset{*}{\rightarrow} \mu_B(v) = 1 - \mu_A(u) + \mu_A(u)\mu_B(v).$$

$$R_{\#} = A \times V \underset{\#}{\Rightarrow} U \times B = \int_{U \times V} [\mu_A(u) \underset{\#}{\rightarrow} \mu_B(v)]/(u, v), \tag{19}$$

where

$$\begin{aligned}\mu_A(u) \underset{\#}{\rightarrow} \mu_B(v) &= (\mu_A(u) \wedge \mu_B(v)) \vee (1 - \mu_A(u) \wedge 1 - \mu_B(v)) \\ &\quad \vee (\mu_B(v) \wedge 1 - \mu_A(u)) \\ &= (1 - \mu_A(u) \vee \mu_B(v)) \wedge (\mu_A(u) \vee 1 - \mu_A(u)) \\ &\quad \wedge (\mu_B(v) \vee 1 - \mu_B(v)).\end{aligned}$$

$$R_{\square} = A \times V \underset{\square}{\Rightarrow} U \times B = \int_{U \times V} [\mu_A(u) \underset{\square}{\rightarrow} \mu_B(v)]/(u, v), \tag{20}$$

where

$$\mu_A(u) \underset{\square}{\rightarrow} \mu_B(v) = \begin{cases} 1 & \mu_A(u) < 1 \text{ or } \mu_B(v) = 1, \\ 0 & \mu_A(u) = 1,\ \mu_B(v) < 1 \end{cases}$$

Fig. 1 shows the graphs of the 15 fuzzy relations obtained in (6)–(20). In each graph the symbols $\mu_A$ and $\mu_B$ are used instead of $\mu_A(u)$ and $\mu_B(v)$ for simplicity. Each left figure with parameter $\mu_B$ will be found to be useful to discuss the generalized modus ponens in (3), and each right figure with parameter $\mu_A$ is useful to analyze the generalized modus tollens in (4).

In the generalized modus ponens of (3), the consequence $B'$ in Cons can be deduced from Ant 1 and Ant 2 using the max–min composition '∘' [1] of the fuzzy set $A'$ and the fuzzy relation obtained in (6)–(20). For example, we can have

$$B'_m = A' \circ Rm = A' \circ [(A \times B) \cup (\neg A \times V)]. \tag{21}$$

The membership function of the fuzzy set $B'_m$ in $V$ is given as

$$\mu_{B'_m}(v) = \bigvee_u \{\mu_{A'}(u) \wedge [(\mu_A(u) \wedge \mu_B(v)) \vee (1 - \mu_A(u))]\}. \tag{22}$$

In the same way, we have

$$B'_a = A' \circ R_a = A' \circ [(\neg A \times V) \oplus (U \times B)], \tag{23}$$

$$B'_c = A' \circ R_c = A' \circ (A \times B), \tag{24}$$

$$B'_s = A' \circ R_s = A' \circ [A \times V \underset{s}{\Rightarrow} U \times B], \tag{25}$$

⋮

Similarly, in the generalized modus tollens of (4), the consequence $A'$ in Cons can be deduced using the composition '∘' of the relation and the fuzzy set $B'$.

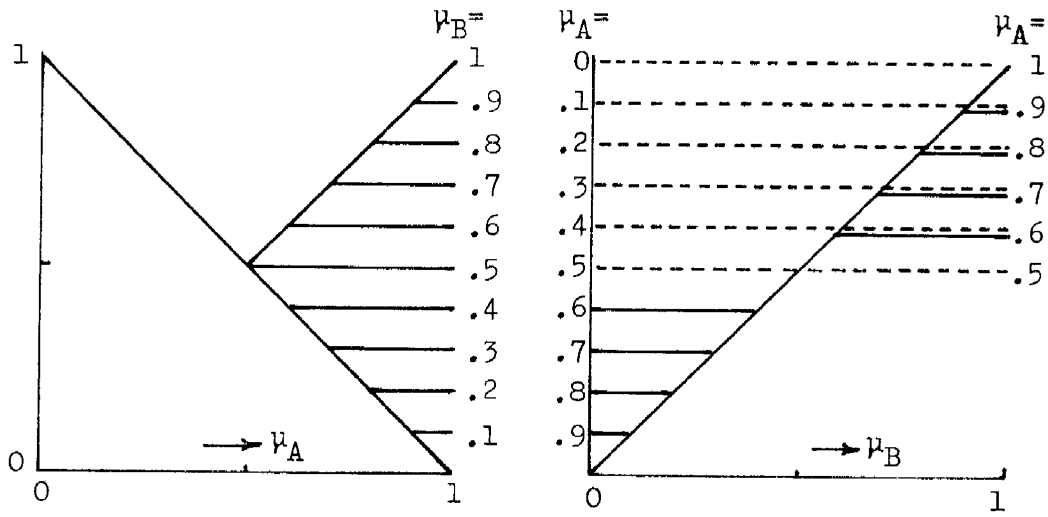


Fig. 1(i). $R_m$: $(\mu_A \wedge \mu_B) \vee (1-\mu_A)$.

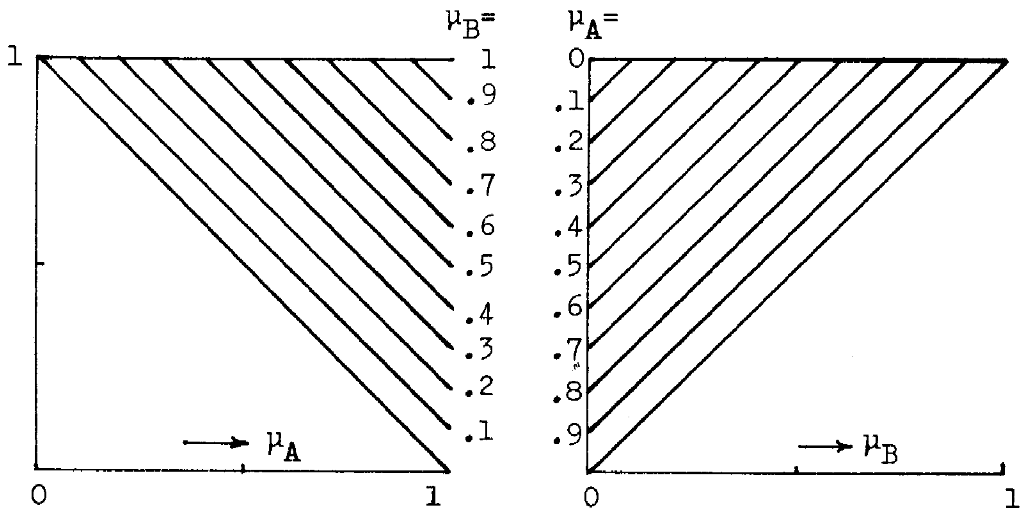


Fig. 1(ii). $R_a$: $1 \wedge (1-\mu_A+\mu_B)$.

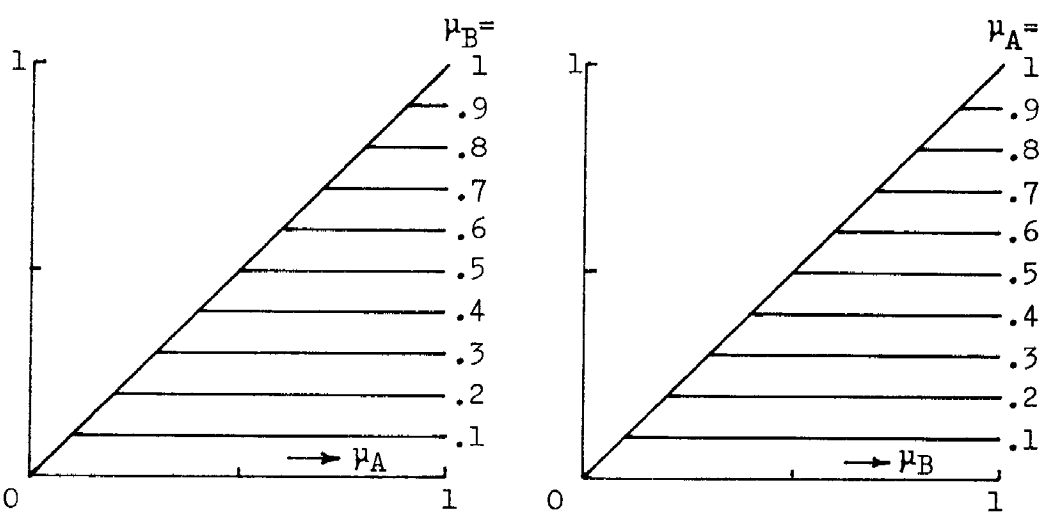


Fig. 1(iii). $R_c$ : $\mu_A \wedge \mu_B$.

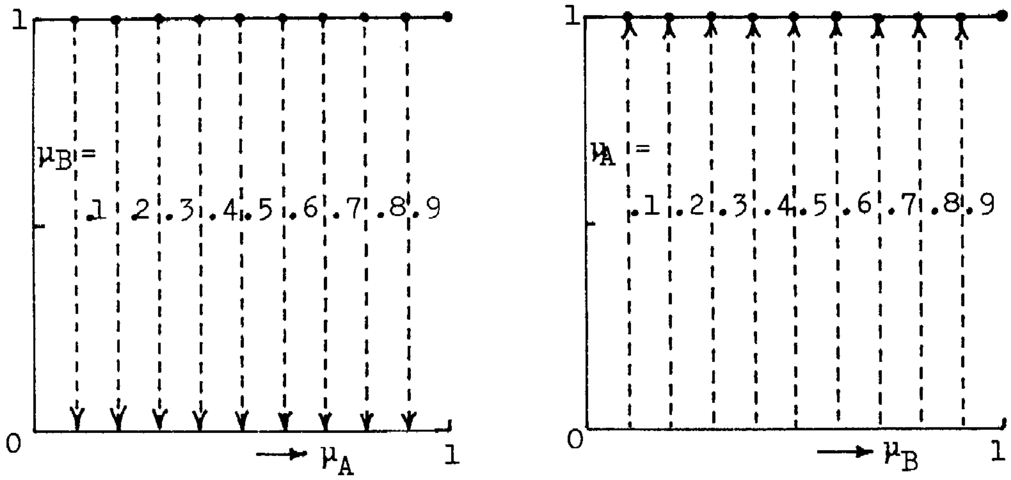


Fig. 1(iv). $R_s$: $\mu_A \underset{s}{\rightarrow} \mu_B$ (see (9)).

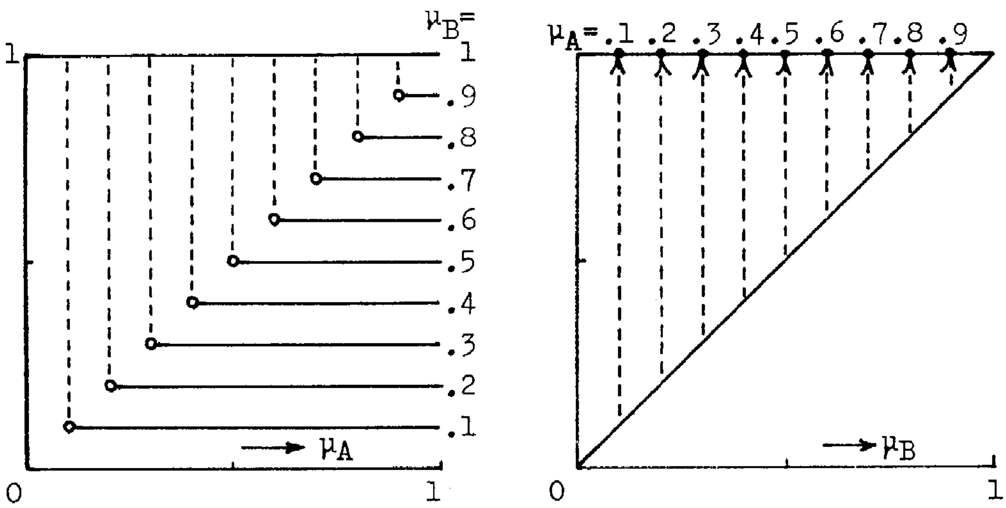


Fig. 1(v). $R_g$: $\mu_A \underset{g}{\rightarrow} \mu_B$ (see (10)).

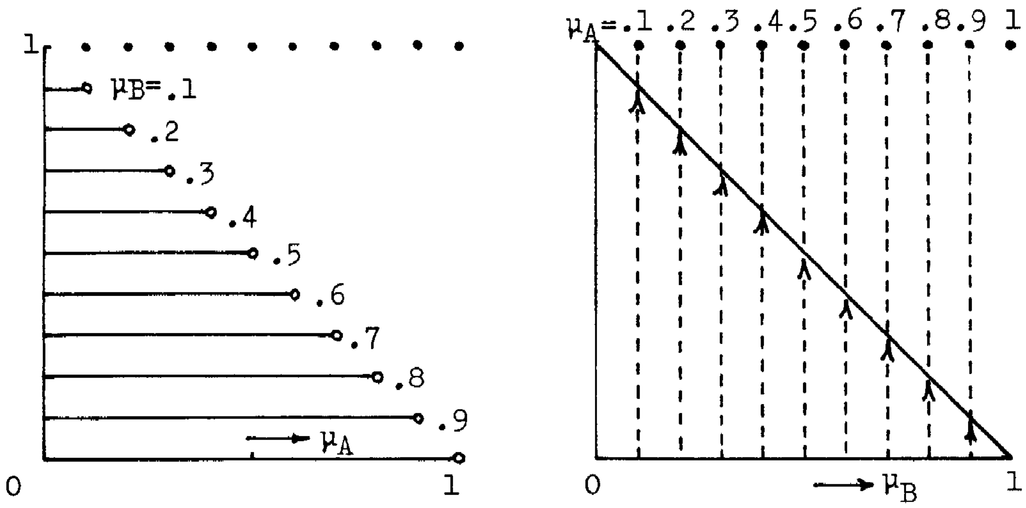


Fig. 1(vi). $R_{sg}$: $(\mu_A \underset{s}{\rightarrow} \mu_B) \wedge (1-\mu_A \underset{g}{\rightarrow} 1-\mu_B)$.

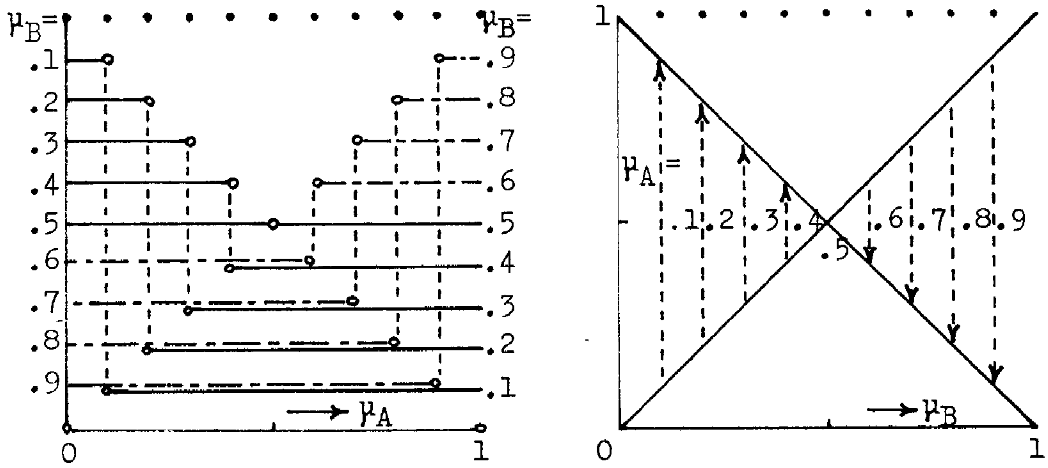


Fig. 1(vii). $R_{gg}$: $(\mu_A \underset{g}{\rightarrow} \mu_B) \wedge (1-\mu_A \underset{g}{\rightarrow} 1-\mu_B)$.

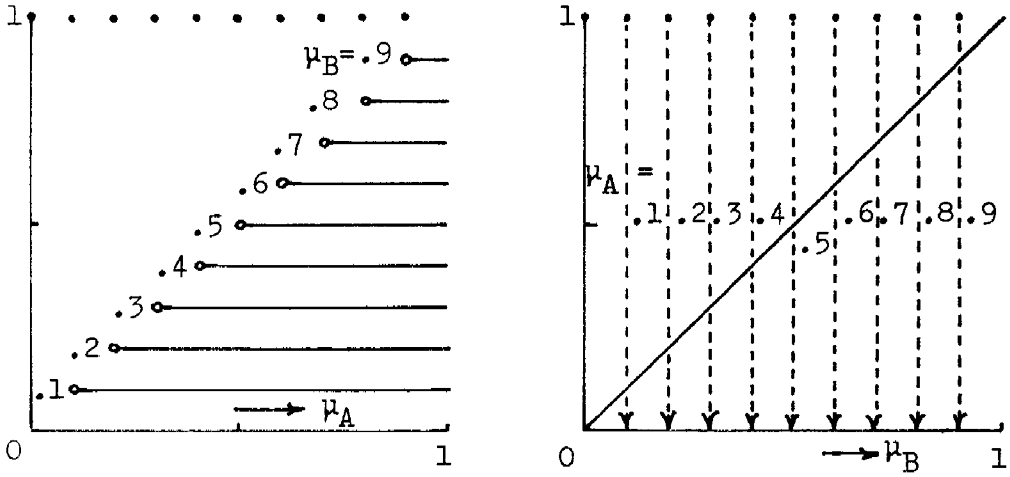


Fig. 1(viii). $R_{gs}$: $(\mu_A \underset{g}{\rightarrow} \mu_B) \wedge (1-\mu_A \underset{s}{\rightarrow} 1-\mu_B)$.

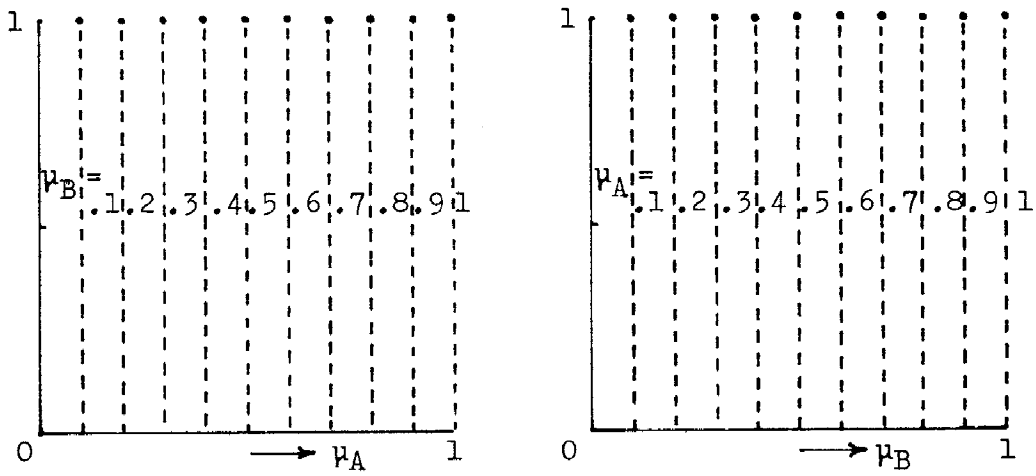


Fig. 1(ix). $R_{ss}$: $(\mu_A \underset{s}{\rightarrow} \mu_B) \wedge (1-\mu_A \underset{s}{\rightarrow} 1-\mu_B)$.

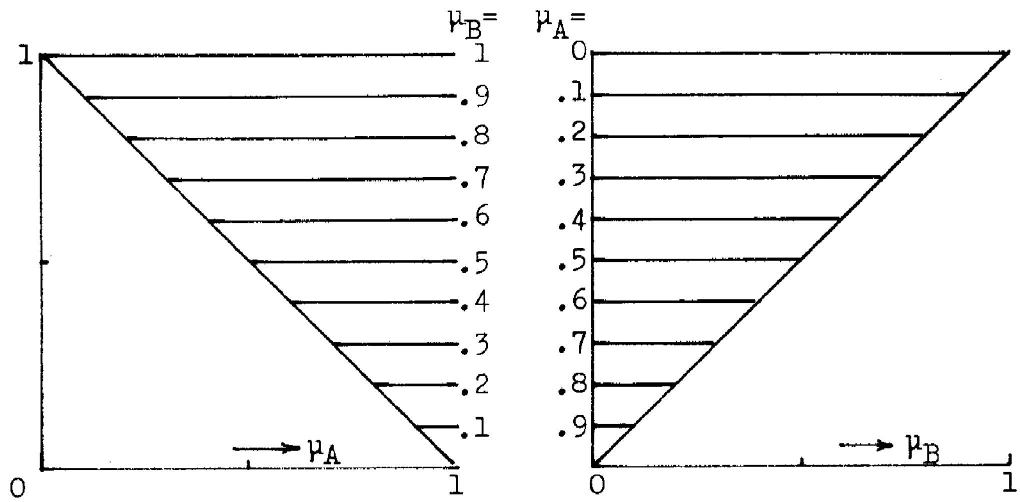


Fig. 1(x). $R_b$: $(1-\mu_A)\vee\mu_B$.

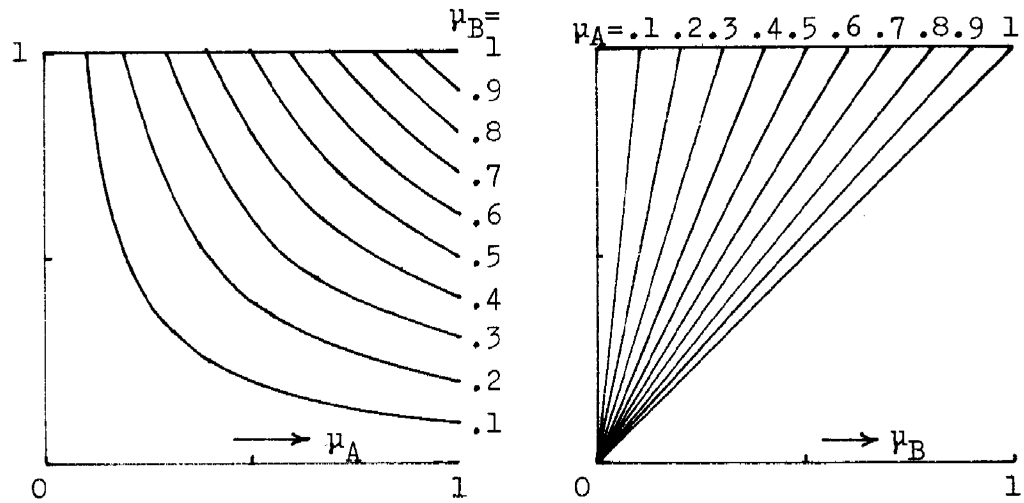


Fig. 1(xi). $R_\triangle$: $\mu_A \underset{\triangle}{\rightarrow} \mu_B$ (see (16)).

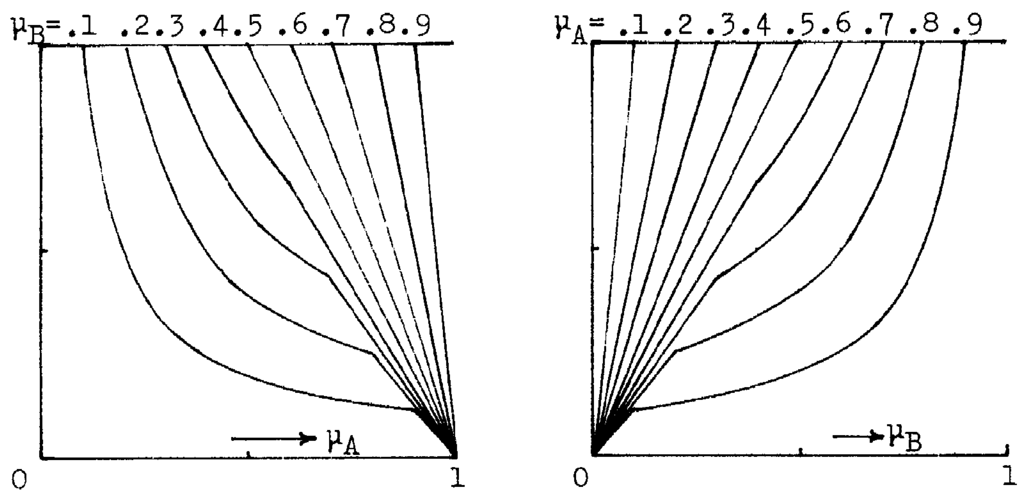


Fig. 1(xii). $R_\blacktriangle$: $\mu_A \underset{\blacktriangle}{\rightarrow} \mu_B$ (see (17)).

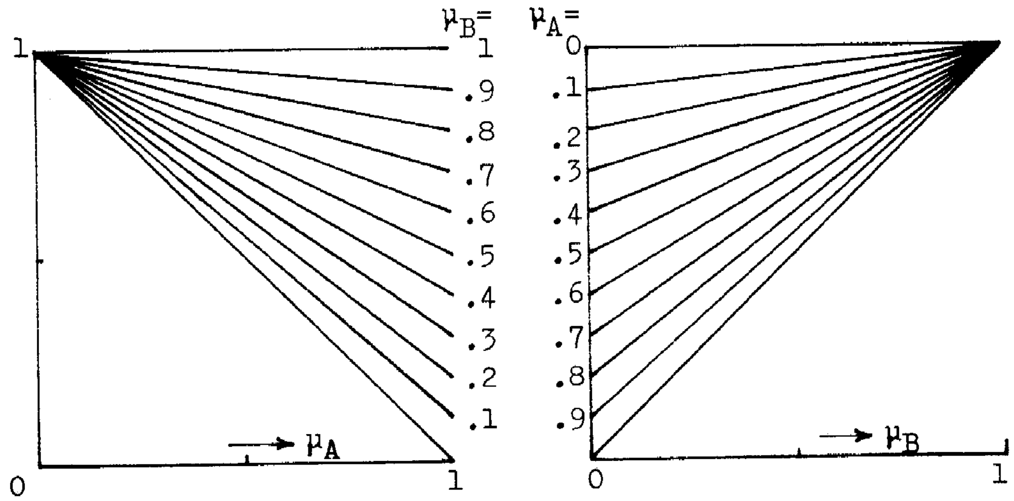


Fig. 1(xiii). $R_*$: $1-\mu_A+\mu_A\mu_B$.

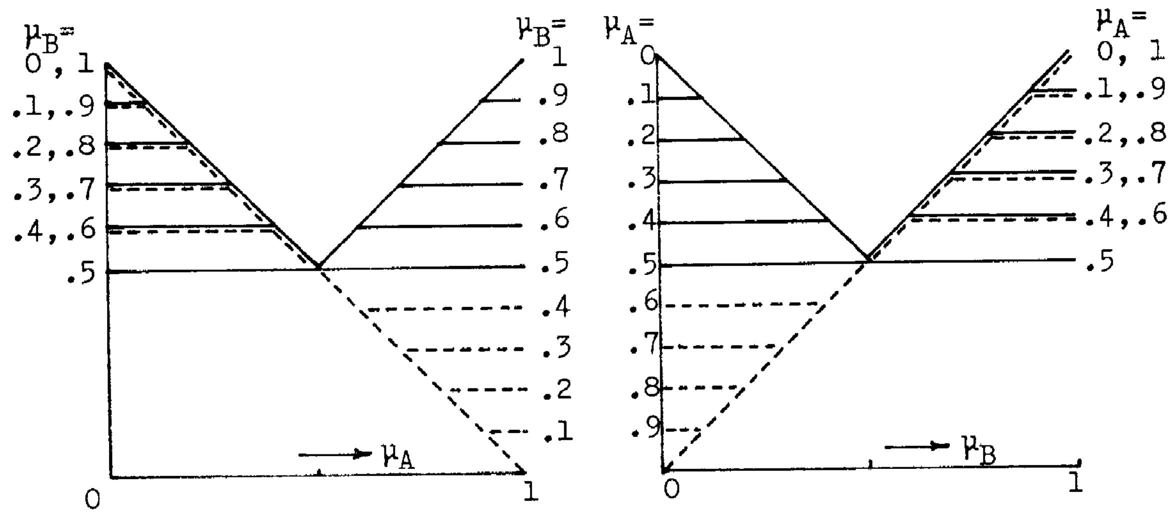


Fig. 1(xiv). $R_{\#}$: $\mu_A \underset{\#}{\rightarrow} \mu_B$ (see (19)).

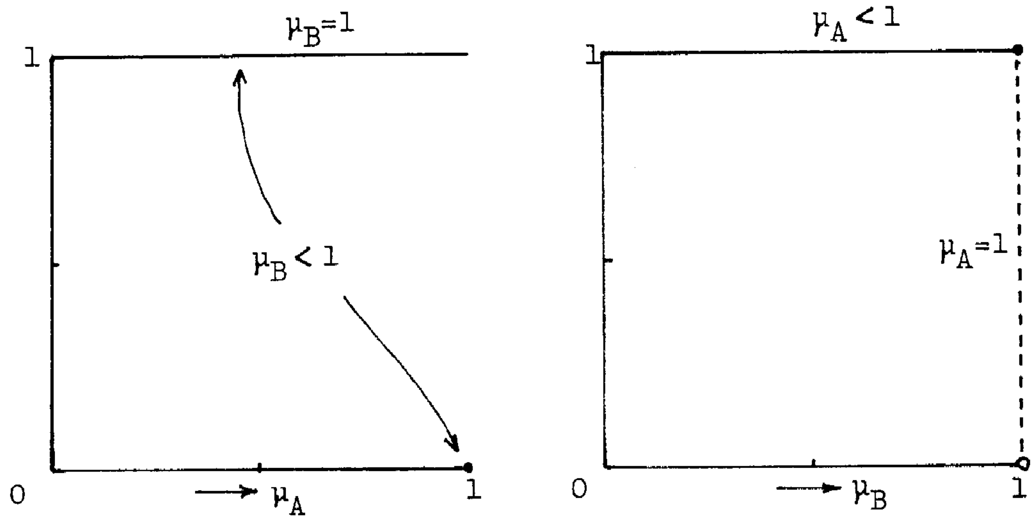


Fig. 1(xv). $R_{\square}$: $\mu_A \underset{\square}{\rightarrow} \mu_B$ (see (20)).

Namely,

$$\begin{aligned} A'_m &= R_m \circ B' \\ &= [(A \times B) \cup (\neg A \times V)] \circ B' \\ &= \int_U \bigvee_v \{[(\mu_A(u) \wedge \mu_B(v)) \vee (1-\mu_A(u))] \wedge \mu_{B'}(v)\}/u, \end{aligned} \tag{26}$$

$$A'_a = R_a \circ B' = [(\neg A \times V) \oplus (U \times B)] \circ B', \tag{27}$$

$$A'_c = R_c \circ B' = (A \times B) \circ B', \tag{28}$$

$$A'_s = R_s \circ B' = [A \times V \underset{s}{\Rightarrow} U \times B] \circ B', \tag{29}$$

$$\vdots$$

## 3. Comparison of fuzzy reasoning methods

In this section we shall make comparisons between the fuzzy reasoning methods obtained above by applying the 15 fuzzy relations (6)–(20) to the generalized modus ponens (3) and the generalized modus tollens (4).

In the generalized modus ponens, we shall show what the consequences $B'_m$, $B'_a$, $B'_c, \ldots$ will be when using the max–min composition (as in (21)–(25)) of the fuzzy set $A'$ and the fuzzy relation, where the fuzzy set $A'$ is

$$A' = A = \int_U \mu_A(u)/u, \tag{30}$$

$$A' = \textit{very } A = A^2 = \int_U \mu_A(u)^2/u, \tag{31}$$

$$A' = \textit{more or less } A = A^{0.5} = \int_U \mu_A(u)^{0.5}/u, \tag{32}$$

$$A' = \textit{not } A = \neg A = \int_U 1-\mu_A(u)/u, \tag{33}$$

which are typical examples of $A'$.

Similarly, in the generalized modus tollens we shall show what the consequences $A'_m, A'_a, A'_c, \ldots.$ will be when using the composition (as in (26)–(29)) of the relation and the fuzzy set $B'$, where $B'$ is

$$B' = \textit{not } B = \neg B = \int_V 1-\mu_B(v)/v, \tag{34}$$

$$B' = \textit{not very } B = \neg B^2 = \int_V 1-\mu_B(v)^2/v, \tag{35}$$

$$B' = \textit{not more or less } B = \neg B^{0.5} = \int_V 1-\mu_B(v)^{0.5}/v, \tag{36}$$

$$B' = B = \int_V \mu_B(v)/v. \tag{37}$$

The consequences inferred by all the fuzzy reasoning methods are summarized in Table 1 (The case of generalized modus ponens) and Table 2 (The case of generalized modus tollens), in which $\mu_B$ and $\mu_A$ stand for $\mu_B(v)$ and $\mu_A(u)$, respectively. These results are also depicted in Fig. 2 in order to make their comparison more transparent. Each left figure shows the results inferred by $R_m$, $R_c$, $R_s$, $R_g$, $R_{sg}$, $R_{gg}$, $R_{gs}$, $R_{ss}$, $R_b$ and $R_{\#}$, and each right figure shows the results by $R_a$, $R_\Delta$, $R_\blacktriangle$ and $R_*$. The figure of $R_\square$ is omitted because of its simplicity.

In Table 1 the inference results by the method $R_\blacktriangle$ at $A' = A$, *very A* and *more or less A* can be more precisely rewritten as follows.

$$\sqrt{\mu_B} \wedge \frac{1}{2-\mu_B} = \begin{cases} \mu_B & \mu_B \leqslant \dfrac{3-\sqrt{5}}{2} \ (=0.3819\ldots), \\[2mm] \dfrac{1}{2-\mu_B} & \mu_B \geqslant \dfrac{3-\sqrt{5}}{2}. \end{cases}$$

$$\mu_B^{2/3} \wedge \left[\frac{\sqrt{5-4\mu_B}-1}{2(1-\mu_B)}\right]^2 = \begin{cases} \mu_B^{2/3} & \mu_B \leqslant b_0, \\[2mm] \left[\dfrac{\sqrt{5-4\mu_B}-1}{2(1-\mu_B)}\right]^2 & \mu_B \geqslant b_0, \end{cases}$$

Table 1. Inference results by each method (case of generalized modus ponens)

| | $A$ | *very A* | *more or less A* | *not A* |
|---|---|---|---|---|
| $R_m$ | $0.5 \vee \mu_B$ | $\frac{3-\sqrt{5}}{2} \vee \mu_B$ | $\frac{\sqrt{5}-1}{2} \vee \mu_B$ | 1 |
| $R_a$ | $\frac{1+\mu_B}{2}$ | $\frac{3+2\mu_B-\sqrt{5+4\mu_B}}{2}$ | $\frac{\sqrt{5+4\mu_B}-1}{2}$ | 1 |
| $R_c$ | $\mu_B$ | $\mu_B$ | $\mu_B$ | $0.5 \wedge \mu_b$ |
| $R_s$ | $\mu_B$ | $\mu_B^2$ | $\sqrt{\mu_B}$ | 1 |
| $R_g$ | $\mu_B$ | $\mu_B$ | $\sqrt{\mu_B}$ | 1 |
| $R_{sg}$ | $\mu_B$ | $\mu_B^2$ | $\sqrt{\mu_B}$ | $1-\mu_B$ |
| $R_{gg}$ | $\mu_B$ | $\mu_B$ | $\sqrt{\mu_B}$ | $1-\mu_B$ |
| $R_{gs}$ | $\mu_B$ | $\mu_B$ | $\sqrt{\mu_B}$ | $1-\mu_B$ |
| $R_{ss}$ | $\mu_B$ | $\mu_B^2$ | $\sqrt{\mu_B}$ | $1-\mu_B$ |
| $R_b$ | $0.5 \vee \mu_B$ | $\frac{3-\sqrt{5}}{2} \vee \mu_B$ | $\frac{\sqrt{5}-1}{2} \vee \mu_B$ | 1 |
| $R_\Delta$ | $\sqrt{\mu_B}$ | $\mu_B^{2/3}$ | $\mu_B^{1/3}$ | 1 |
| $R_\blacktriangle$ | $\sqrt{\mu_B} \wedge \frac{1}{2-\mu_B}$ | $\mu_B^{2/3} \wedge \left[\frac{\sqrt{5-4\mu_B}-1}{2(1-\mu_B)}\right]^2$ | $\mu_B^{1/3} \wedge \frac{\sqrt{\mu_B^2-2\mu_B+5}+\mu_B-1}{2}$ | 1 |
| $R_*$ | $\frac{1}{2-\mu_B}$ | $\left[\frac{\mu_B-1+\sqrt{(1-\mu_B)^2+4}}{2}\right]^2$ | $\frac{\sqrt{5-4\mu_B}-1}{2(1-\mu_B)}$ | 1 |
| $R_{\#}$ | $0.5 \vee \mu_B$ | $\frac{3-\sqrt{5}}{2} \vee \mu_B$ | $\mu_B \vee \left[(1-\mu_B) \wedge \frac{\sqrt{5}-1}{2}\right]$ | $\mu_B \vee (1-\mu_B)$ |
| $R_\square$ | 1 | 1 | 1 | 1 |

where

$$b_0 = 1 - \sqrt[3]{\frac{1}{2} + \frac{\sqrt{93}}{18}} - \sqrt[3]{\frac{1}{2} - \frac{\sqrt{93}}{18}} = 0.3177\ldots \tag{38}$$

$$\mu_B^{1/3} \wedge \frac{\sqrt{\mu_B^2 - 2\mu_B + 5} + \mu_B - 1}{2} = \begin{cases} \mu_B^{1/3} & \mu_B \leq b_0', \\ \frac{\sqrt{\mu_B^2 - 2\mu_B + 5} + \mu_B - 1}{2} & \mu_B \geq b_0', \end{cases}$$

where

$$b_0' = \frac{2}{3} - \frac{1}{3}\left(\sqrt[3]{\frac{11 + 3\sqrt{69}}{2}} + \sqrt[3]{\frac{11 - 3\sqrt{69}}{2}}\right) = 0.4301\ldots \tag{39}$$

Table 2. Inference results by each method (case of generalized modus tollens)

| | *not B* | *not very B* | *not more or less B* | *B* |
|---|---|---|---|---|
| $R_m$ | $0.5 \vee (1-\mu_A)$ | $(1-\mu_A) \vee \left(\frac{\sqrt{5}-1}{2} \wedge \mu_A\right)$ | $\frac{3-\sqrt{5}}{2} \vee (1-\mu_A)$ | $\mu_A \vee (1-\mu_A)$ |
| $R_a$ | $1 - \frac{\mu_A}{2}$ | $\frac{1 - 2\mu_A + \sqrt{1+4\mu_A}}{2}$ | $\frac{3 - \sqrt{1+4\mu_A}}{2}$ | 1 |
| $R_c$ | $0.5 \wedge \mu_A$ | $\frac{\sqrt{5}-1}{2} \wedge \mu_A$ | $\frac{3-\sqrt{5}}{2} \wedge \mu_A$ | $\mu_A$ |
| $R_s$ | $1-\mu_A$ | $1-\mu_A^2$ | $1-\sqrt{\mu_A}$ | 1 |
| $R_g$ | $0.5 \vee (1-\mu_A)$ | $\frac{\sqrt{5}-1}{2} \vee (1-\mu_A^2)$ | $\frac{3-\sqrt{5}}{2} \vee (1-\sqrt{\mu_A})$ | 1 |
| $R_{sg}$ | $1-\mu_A$ | $1-\mu_A^2$ | $1-\sqrt{\mu_A}$ | $0.5 \vee \mu_A$ |
| $R_{gg}$ | $0.5 \vee (1-\mu_A)$ | $\frac{\sqrt{5}-1}{2} \vee (1-\mu_A^2)$ | $\frac{3-\sqrt{5}}{2} \vee (1-\sqrt{\mu_A})$ | $0.5 \vee \mu_A$ |
| $R_{gs}$ | $0.5 \vee (1-\mu_A)$ | $\frac{\sqrt{5}-1}{2} \vee (1-\mu_A^2)$ | $\frac{3-\sqrt{5}}{2} \vee (1-\sqrt{\mu_A})$ | $\mu_A$ |
| $R_{ss}$ | $1-\mu_A$ | $1-\mu_A^2$ | $1-\sqrt{\mu_A}$ | $\mu_A$ |
| $R_b$ | $0.5 \vee (1-\mu_A)$ | $\frac{\sqrt{5}-1}{2} \vee (1-\mu_A)$ | $\frac{3-\sqrt{5}}{2} \vee (1-\mu_A)$ | 1 |
| $R_\triangle$ | $\frac{1}{1+\mu_A}$ | $\frac{\sqrt{1+4\mu_A^2}-1}{2\mu_A^2}$ | $\frac{2+\mu_A - \sqrt{\mu_A^2 + 4\mu_A}}{2}$ | 1 |
| $R_\blacktriangle$ | $\frac{1}{1+\mu_A} \wedge \sqrt{1-\mu_A}$ | see (40) | see (42) | 1 |
| $R_*$ | $\frac{1}{1+\mu_A}$ | $1 - \frac{\mu_A(\mu_A + 2 - \sqrt{\mu_A^2 + 4\mu_A})}{2}$ | $\frac{2\mu_A + 1 - \sqrt{1+4\mu_A^2}}{2\mu_A}$ | 1 |
| $R_\#$ | $0.5 \vee (1-\mu_A)$ | $(1-\mu_A) \vee \left(\mu_A \wedge \frac{\sqrt{5}-1}{2}\right)$ | $\frac{3-\sqrt{5}}{2} \vee (1-\mu_A)$ | $\mu_A \vee (1-\mu_A)$ |
| $R_\square$ | $\begin{cases} 1 & \mu_A < 1 \\ 0 & \mu_A = 1 \end{cases}$ | $\begin{cases} 1 & \mu_A < 1 \\ 0 & \mu_A = 1 \end{cases}$ | $\begin{cases} 1 & \mu_A < 1 \\ 0 & \mu_A = 1 \end{cases}$ | 1 |

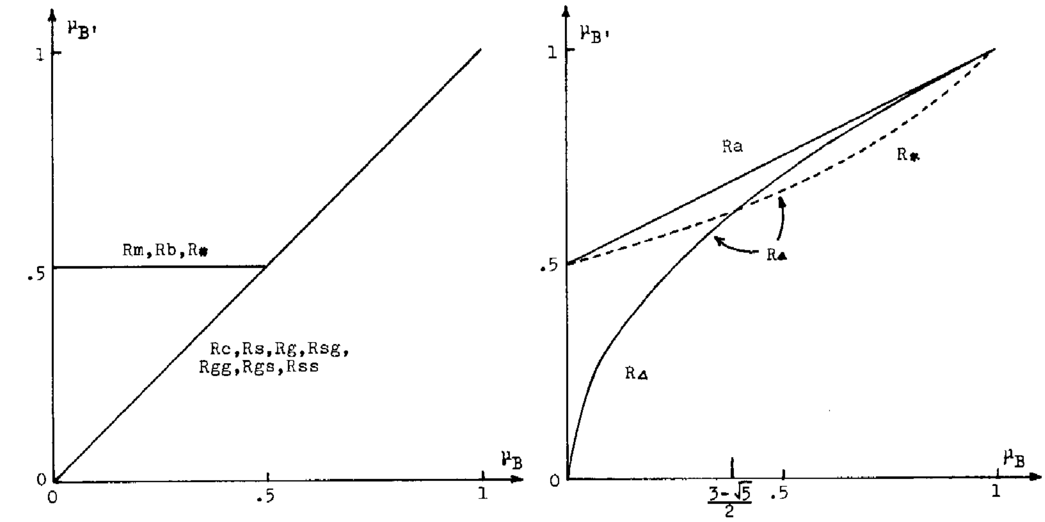


Fig. 2(a). At $A' = A$.

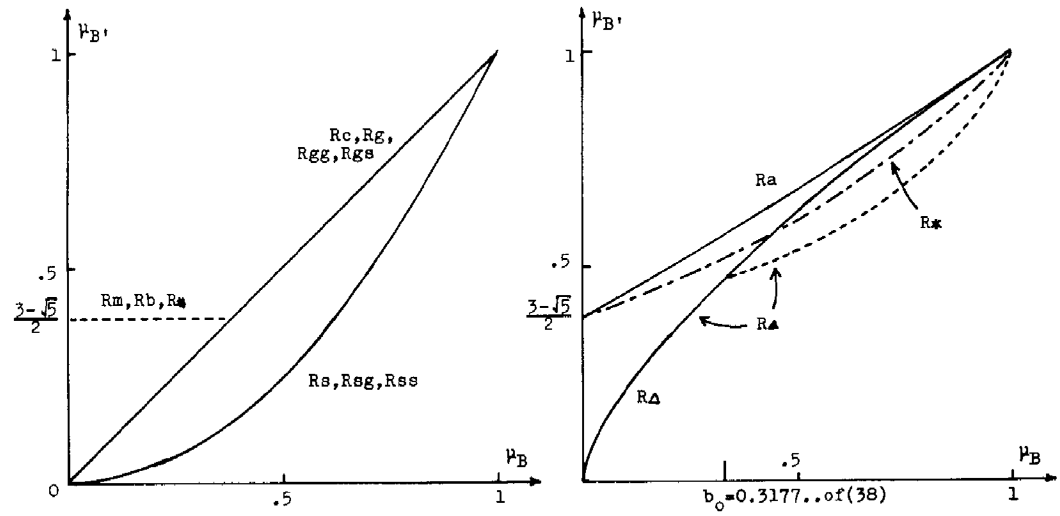


Fig. 2(b). At $A' = \textit{very } A \; (= A^2)$.

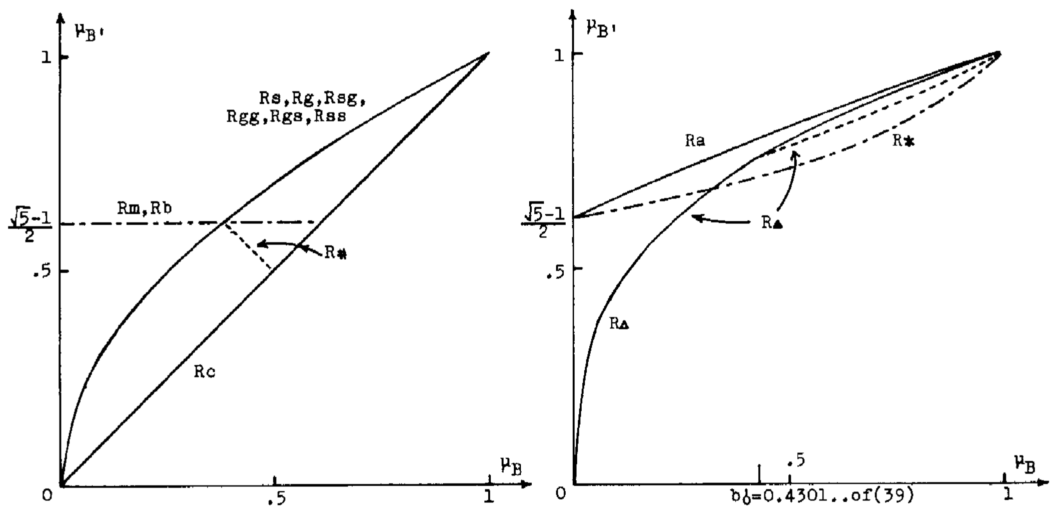


Fig. 2(c). At $A' = \textit{more or less } A \; (= A^{0.5})$.

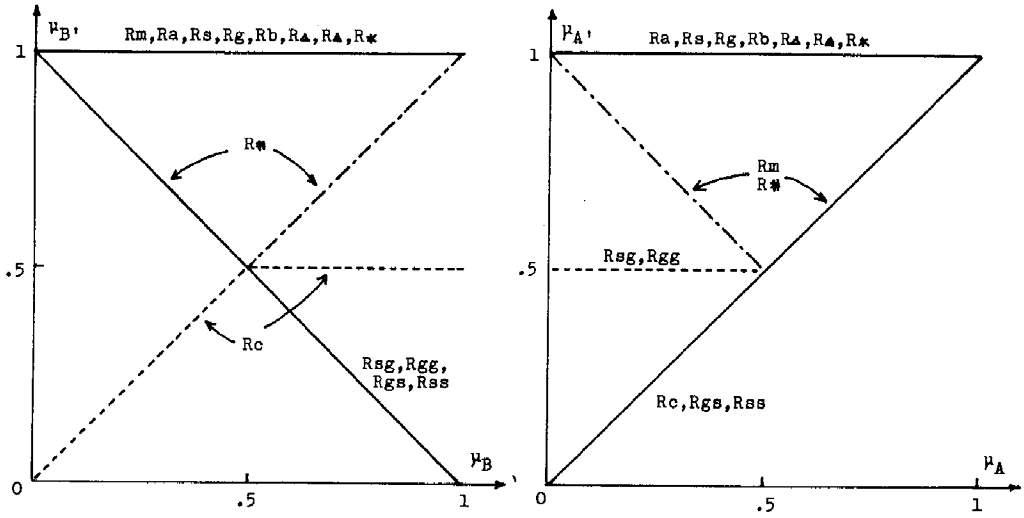


Fig. 2(d). At $A' = not\ A\ (= \neg A)$; (e) at $B' = B$.

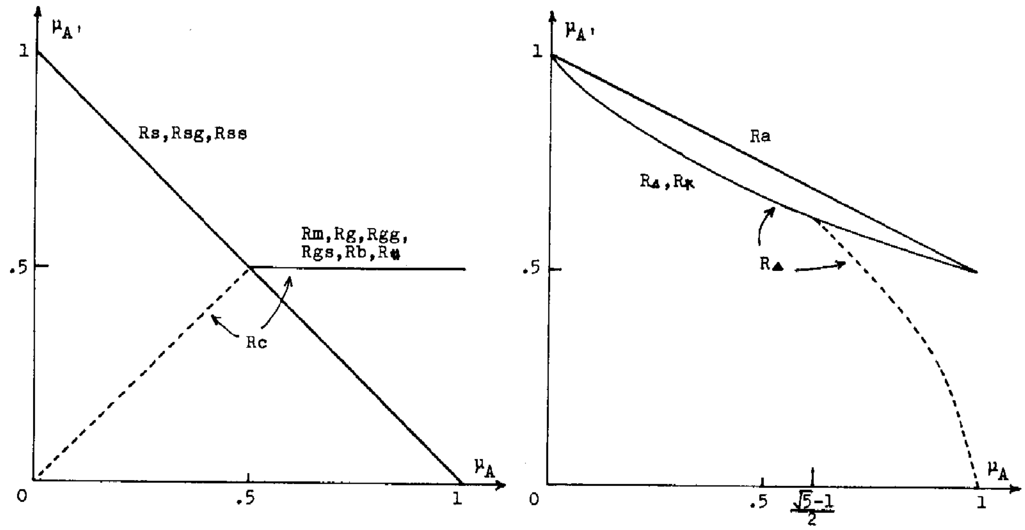


Fig. 2(f). At $B' = not\ B\ (= \neg B)$.

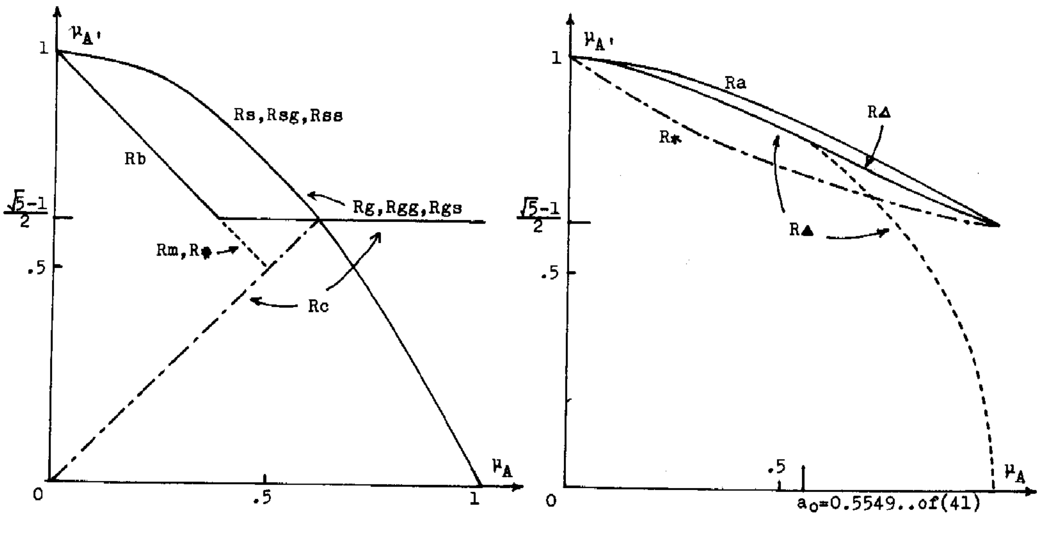


Fig. 2(g). At $B = not\ very\ B\ (= \neg B^2)$.

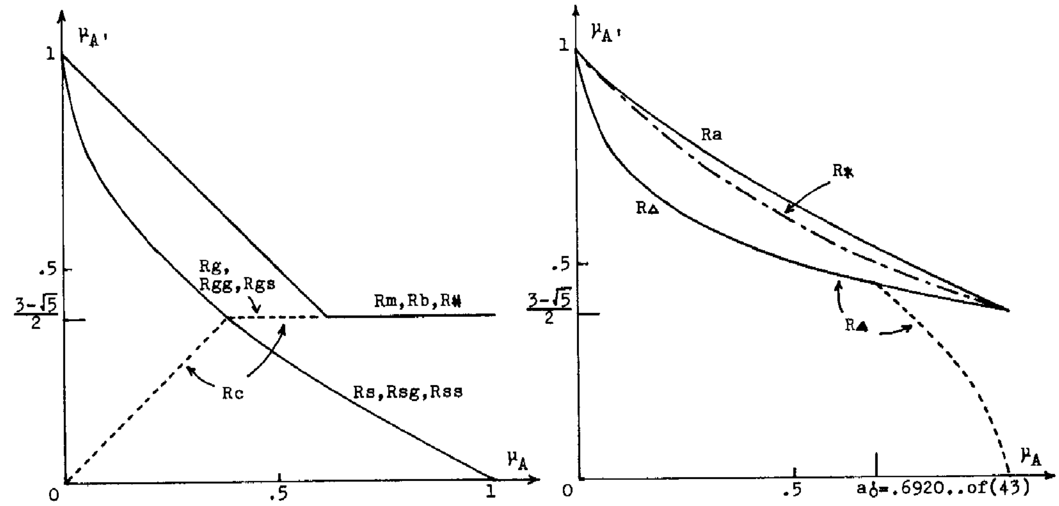


Fig. 2(h). At $B' =$ *not more or less* $B(= \neg B^{0.5})$.

Moreover, in Table 2 the results inferred by the method $R_\blacktriangle$ at $B' =$ *not very B* and *not more or less B* are given as follows.

*At $B' =$ not very B by $R_\blacktriangle$:*

$$\frac{\sqrt{1+4\mu_A^2}-1}{2\mu_A^2} \wedge \left[1-\left(\frac{1-4\cos\left(\frac{\theta+4\pi}{3}\right)}{3}\right)^2\right]$$

$$= \begin{cases} \dfrac{\sqrt{1+4\mu_A^2}-1}{2\mu_A^2} & \mu_A \leq a_0, \\ 1-\left(\dfrac{1-4\cos\left(\frac{\theta+4\pi}{3}\right)}{3}\right)^2 & \mu_A \geq a_0, \end{cases} \tag{40}$$

where

$$\theta = \cos^{-1}\left(\frac{27\mu_A - 11}{16}\right),$$

$$a_0 = \frac{2}{3}\left[1+\sqrt{7}\cos\left(\frac{\theta'+4\pi}{3}\right)\right] = 0.5549\ldots \quad \text{with } \theta' = \cos^{-1}\left(\frac{\sqrt{7}}{14}\right). \tag{41}$$

*At $B' =$ not more or less B by $R_\blacktriangle$:*

$$\frac{2+\mu_A-\sqrt{\mu_A^2+4\mu_A}}{2} \wedge \left[1-\sqrt{\frac{3-2\sqrt{6(1-\mu_A)}\cos(\frac{1}{3}\varphi)}{3}}\right]$$

$$= \begin{cases} \dfrac{2+\mu_A-\sqrt{\mu_A^2+4\mu_A}}{2} & \mu_A \leq a_0', \\ 1-\sqrt{\dfrac{3-2\sqrt{6(1-\mu_A)}\cos(\frac{1}{3}\varphi)}{3}} & \mu_A \geq a_0', \end{cases} \tag{42}$$

where

$$\varphi = \cos^{-1}\left(-\frac{3\sqrt{6(1-\mu_A)}}{8}\right),$$

$$a_0' = \frac{2\sqrt{21}\cos(\frac{1}{3}\varphi') - 3}{3} = 0.6920\ldots \quad \text{with } \varphi' = \cos^{-1}\left(-\frac{3\sqrt{21}}{14}\right). \tag{43}$$

Using Fig. 1 we shall show how to obtain the results of Tables 1 and 2. However, we shall discuss only the case of $R_b$ (15) at $A' = very\ A$ and $B' = not\ very\ B$ because of limitations of space. The methods of obtaining the consequences for $R_m$, $R_a, \ldots, R_{sg}$ and $R_{gg}$ of (6)–(12) in the case of generalized modus ponens are found in [4–6]. The other consequences can be obtained in the same way as in the case of $R_b$, though we must solve a cubic equation, particularly in the case of $R_\blacktriangle$ in (17).

(i) *The case of $R_b$ at $A' = very\ A$*

The consequence $B_b'$, which is inferred by taking the composition of $A'$ and $R_b$ as in (21)–(25), is given by

$$B_b' = A' \circ R_b = A' \circ [(\neg A \times V) \cup (U \times B)].$$

Then the membership function of $B_b'$ at $A' = very\ A$ (31) is

$$\mu_{B_b'}(v) = \bigvee_u \{\mu_A(u)^2 \wedge [(1-\mu_A(u)) \vee \mu_B(v)]\}.$$

This expression becomes the following by omitting '$(u)$' and '$(v)$' for simplicity:

$$\mu_{B_b'} = \bigvee_u \{\mu_A^2 \wedge [(1-\mu_A) \vee \mu_B]\}. \tag{44}$$

For example, if $\mu_B = 0.2$, the expression in (44)

$$\mu_A^2 \wedge [(1-\mu_A) \vee \mu_B] \tag{45}$$

is indicated by the broken line '– – – –' in Fig. 3(a) whose figure comes from the left figure of Fig. 1(x). The value of $\mu_{B_b'}$ at $\mu_B = 0.2$ becomes $(3-\sqrt{5})/2$ by taking the maximum of this line (i.e. (45)) by virtue of (44). Thus, in general, we can have

$$\mu_{B_b'} = \frac{3-\sqrt{5}}{2} \quad \text{at } \mu_B \leqslant \frac{3-\sqrt{5}}{2}\ (=0.3819\ldots).$$

On the other hand, when $\mu_B = 0.7$ ($\geqslant (3-\sqrt{5})/2$), (45) is shown by the dotted line '–·–·' and then the value $\mu_{B_b'}$ (the maximum value of this line) is 0.7 when $\mu_B = 0.7$. Thus, in general

$$\mu_{B_b'} = \mu_B \quad \text{at } \mu_B \geqslant \frac{3-\sqrt{5}}{2}.$$

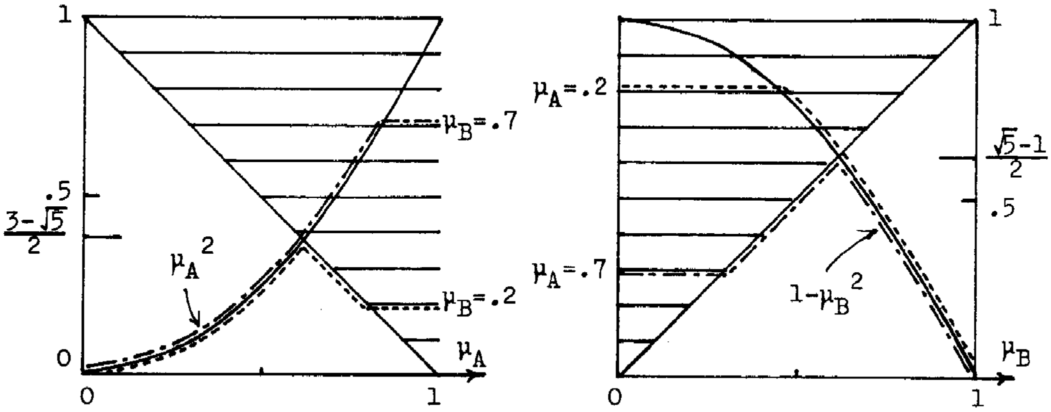


Fig. 3. The way of obtaining $B_b'$ and $A_b'$: (a) $\mu_{B_b'}$ at $\mu_{A'}=\mu_A^2$; (b) $\mu_{A_b'}$ at $\mu_{B'}=1-\mu_B^2$.

Therefore, we obtain

$$\mu_{B_b'}=\begin{cases}\dfrac{3-\sqrt{5}}{2} & \mu_B\leqslant\dfrac{3-\sqrt{5}}{2},\\ \mu_B & \mu_B\geqslant\dfrac{3-\sqrt{5}}{2},\end{cases}$$

which leads to

$$\mu_{B_b'}=\frac{3-\sqrt{5}}{2}\vee\mu_B.$$

The same method is applicable to $A'=A$, *more or less* $A$, and *not* $A$.

(ii) *The case of $R_b$ at $B'$ = not very B*

The consequence $A_b'$, which is obtained by taking the composition of $R_b$ and $B'$ as in (26)–(29), is given by

$$A_b'=R_b\circ B'=[(\neg A\times V)\cup(U\times B)]\circ B'.$$

The membership function of $A_b'$ at $B'$ = *not very B* (35) is

$$\mu_{A_b'}=\bigvee_v\{[(1-\mu_A)\vee\mu_B]\wedge(1-\mu_B^2)\} \tag{46}$$

by omitting '($u$)' and '($v$)'. The expression in (46)

$$[(1-\mu_A)\vee\mu_B]\wedge(1-\mu_B^2) \tag{47}$$

at $\mu_A=0.2$ ($\leqslant(\sqrt{5}-1)/2=0.6180\ldots$) is shown by the line '– – – –' in Fig. 3(b) which comes from the right figure of Fig. 1(x). The maximum value of this line becomes 0.8 (= 1–0.2). When $\mu_A=0.7$ ($\geqslant(\sqrt{5}-1)/2$), (47) is indicated by the line '–·–·' and its maximum value is $(\sqrt{5}-1)/2$. Thus we have in general

$$\mu_{A_b'}=\begin{cases}1-\mu_A & \mu_A\leqslant\dfrac{\sqrt{5}-1}{2},\\ \dfrac{\sqrt{5}-1}{2} & \mu_A\geqslant\dfrac{\sqrt{5}-1}{2}.\end{cases}$$

Namely,

$$\mu_{A_b'} = \frac{\sqrt{5}-1}{2} \vee (1-\mu_A).$$

The same way is applicable to $B' =$ *not B*, *not more or less B*, and *B*.

**Example.** Using Tables 1 and 2 and Fig. 2, we shall present a simple example of fuzzy reasoning in Fig. 4. Fig. 4(a) shows fuzzy sets $A$ and $B$, and Fig. 4(b) includes fuzzy sets '*not A*', '*not very A*', . . . , '*unknown*' in order to compare with the inference results of Fig. 4(c)–(j).

In the forms of fuzzy conditional inferences (3) and (4), it seems according to our intuitions that the relations between $A'$ in Ant 2 and $B'$ in Cons of the generalized modus ponens (3) ought to be satisfied as shown in Table 3. Similarly, the relations between $B'$ in Ant 2 and $A'$ in Cons of the generalized modus tollens (4) ought to be satisfied as in Table 4.

Relation I in Table 3 corresponds to the modus ponens. Relation II-2 has a consequence different from that of Relation II-1, but if there is not a strong causal relation between "$x$ is $A$" and "$y$ is $B$" in the proposition "If $x$ is $A$ then $y$ is $B$", the satisfaction of Relation II-2 will be permitted. Relation IV-1 asserts that when $x$ is *not A*, any information about $y$ is not conveyed from Ant 1. The satisfaction of Relation IV-2 is demanded when the fuzzy proposition "If $x$ is $A$ then $y$ is $B$" means tacitly the proposition "If $x$ is $A$ then $y$ is $B$ else $y$ is *not B*". Although this relation may not be accepted in ordinary logic, in our daily life we often encounter the situation in which this relation can hold. Relation V corresponds to modus tollens. Relation VIII is discussed as in the case of Relation IV.

In Table 5, the satisfaction (0) or failure (x) of each criterion in Tables 3 and 4 under each fuzzy reasoning method is indicated by using the consequence results of Tables 1 and 2.

Under these criteria it is found that $R_m$ and $R_a$ are neither very suitable for the fuzzy conditional inference in the case of generalized modus tollens nor in the case of generalized modus ponens. $R_c$ is not bad. $R_s$, $R_g$, $R_{sg}$, . . . , $R_{ss}$ are satisfactory. $R_b$, . . . , $R_{\Box}$ are not very good.

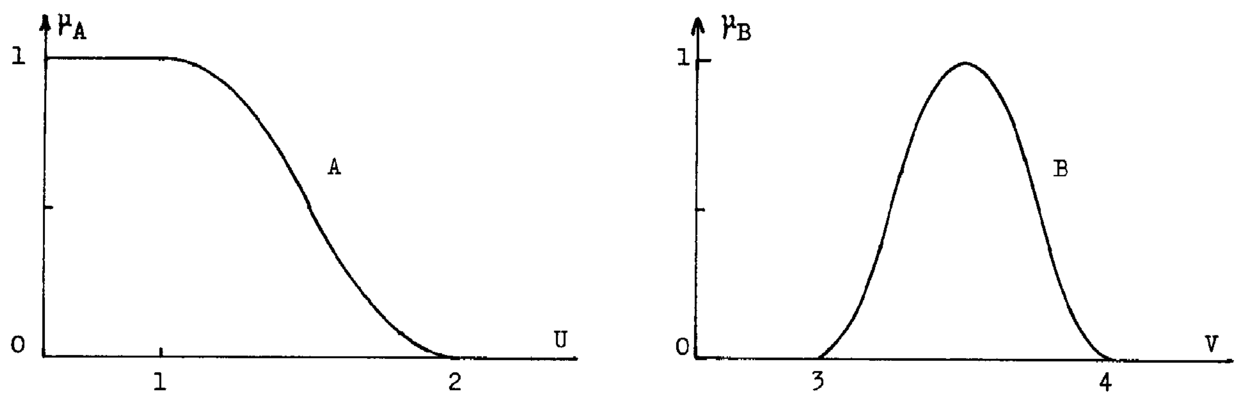


Fig. 4(a). Fuzzy sets $A$ and $B$.

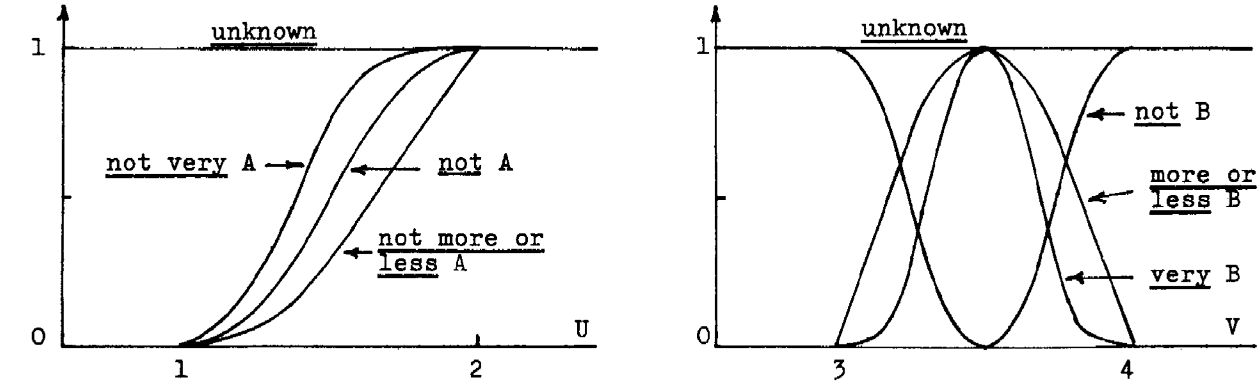


Fig. 4(b). *not A, not very A, not more or less A, not B, very B, more or less B,* and *unknown.*

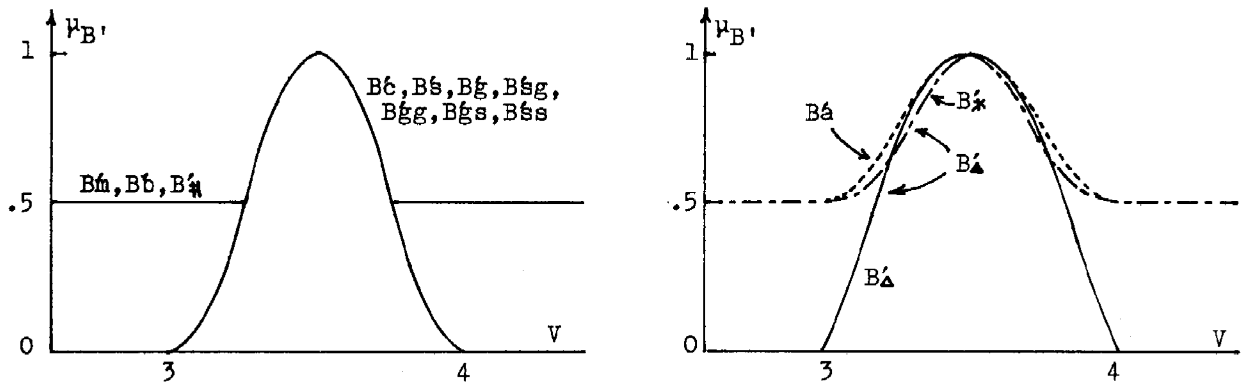


Fig. 4(c). Inference results at $A' = A$.

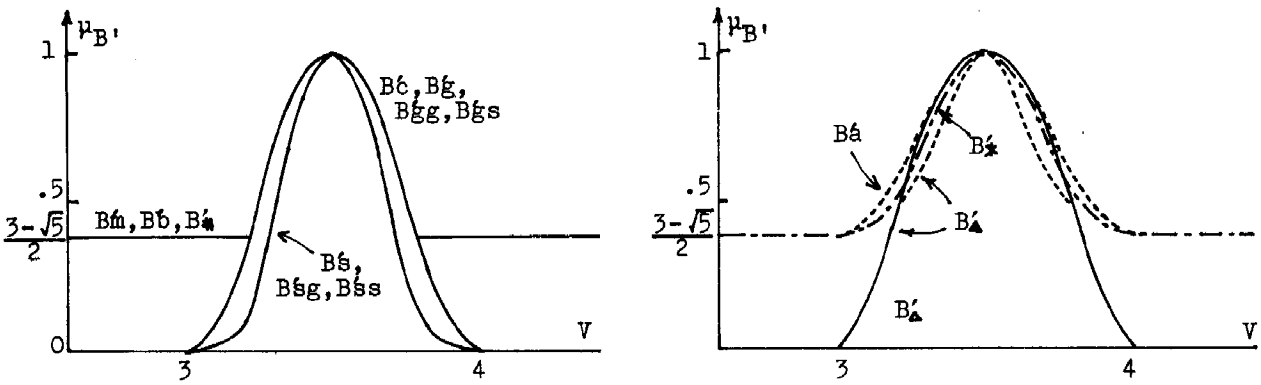


Fig. 4(d). Inference results at $A' = very\ A$.

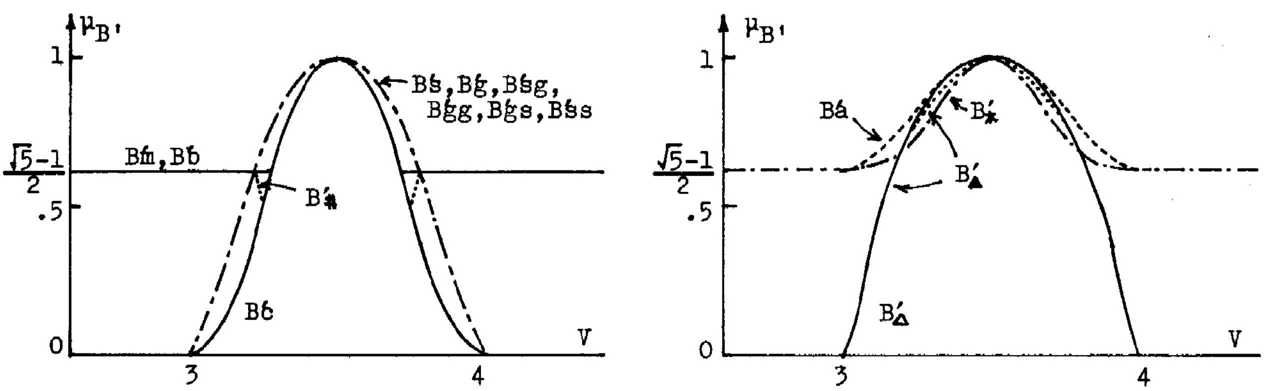


Fig. 4(e). Inference results at $A' = more\ or\ less\ A$.

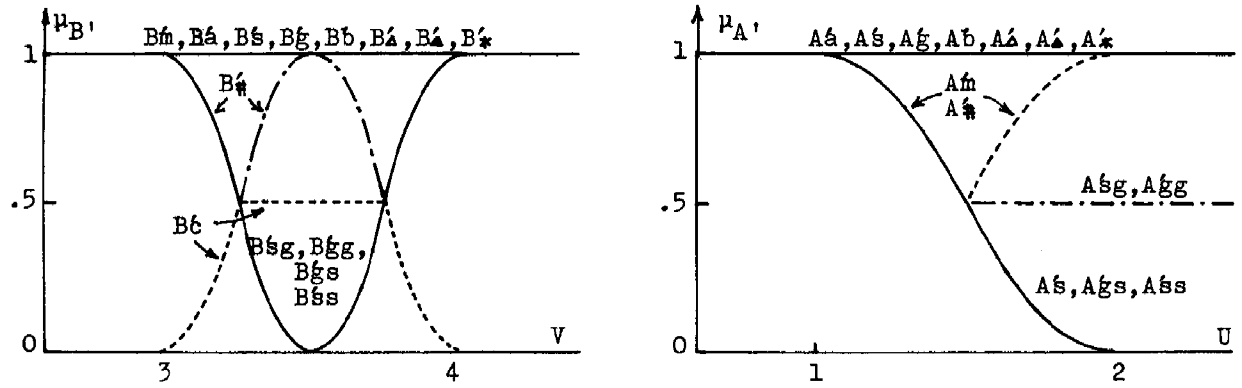

Fig. 4(f). Inference results at $A' = not\ A$; (g) at $B' = B$.

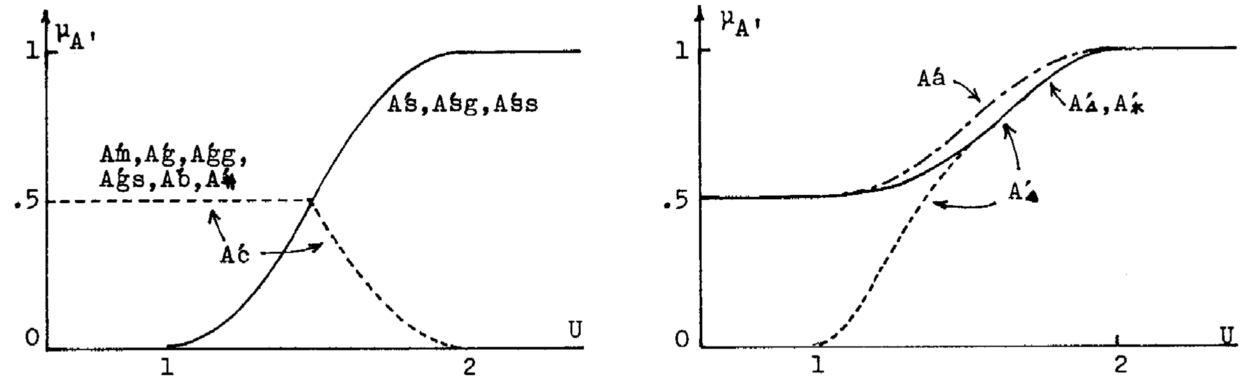

Fig. 4(h). Inference results at $B' = not\ B$.

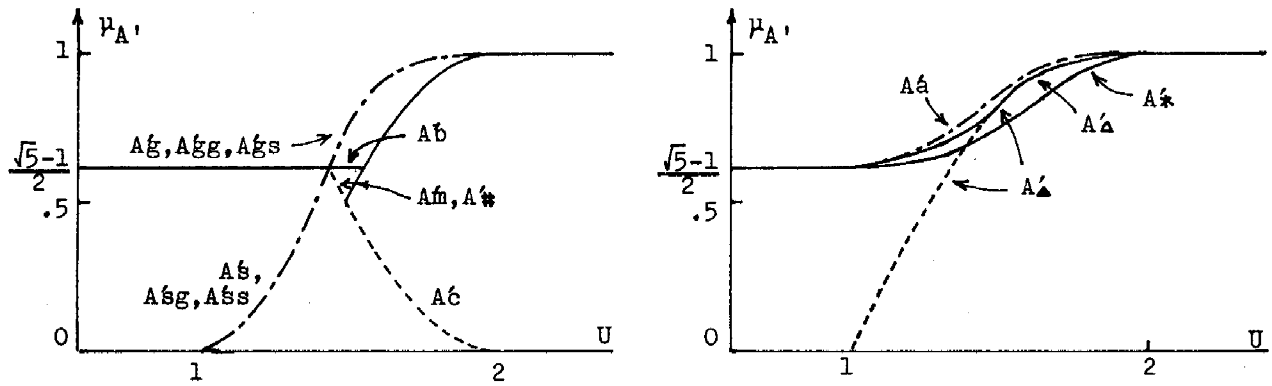

Fig. 4(i). Inference results at $B' = not\ very\ B$.

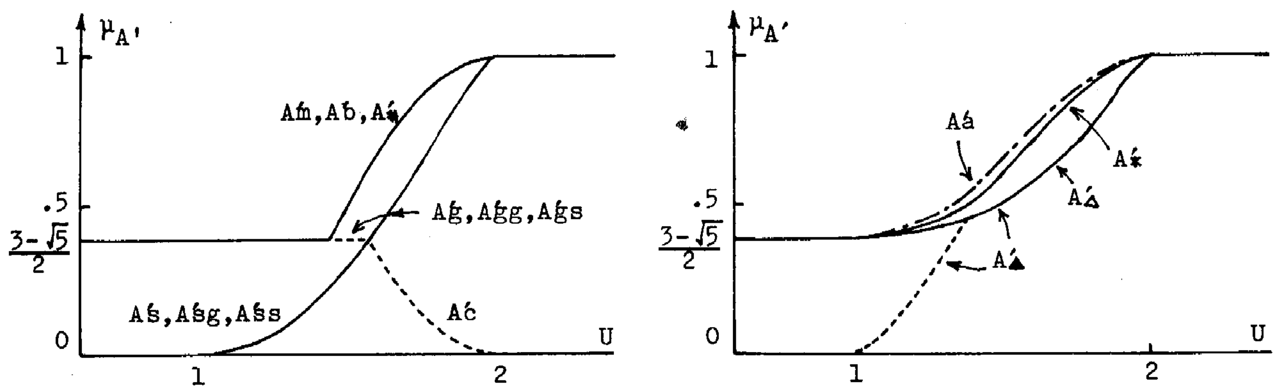

Fig. 4(j). Inference results at $B' = not\ more\ or\ less\ B$.

Table 3. Relations between Ant 2 and Cons under Ant 1 for the generalized modus ponens in (3)

| | $x$ is $A'$ (Ant 2) | $y$ is $B'$ (Cons) |
|---|---|---|
| Relation I (modus ponens) | $x$ is $A$ | $y$ is $B$ |
| Relation II-1 | $x$ is *very* $A$ | $y$ is *very* $B$ |
| Relation II-2 | $x$ is *very* $A$ | $y$ is $B$ |
| Relation III-1 | $x$ is *more or less* $A$ | $y$ is *more or less* $B$ |
| Relation III-2 | $x$ is *more or less* $A$ | $y$ is $B$ |
| Relation IV-1 | $x$ is *not* $A$ | $y$ is *unknown* |
| Relation IV-2 | $x$ is *not* $A$ | $y$ is *not* $B$ |

Table 4. Relations between Ant 2 and Cons under Ant 1 for the generalized modus tollens in (4)

| | $y$ is $B'$ (Ant 2) | $x$ is $A'$ (Cons) |
|---|---|---|
| Relation V (modus tollens) | $y$ is *not* $B$ | $x$ is *not* $A$ |
| Relation VI | $y$ is *not very* $B$ | $x$ is *not very* $A$ |
| Relation VII | $y$ is *not more or less* $B$ | $x$ is *not more or less* $A$ |
| Relation VIII-1 | $y$ is $B$ | $x$ is *unknown* |
| Relation VIII-2 | $y$ is $B$ | $x$ is $A$ |

Table 5. Satisfaction of each Relation in Tables 3 and 4 under each method

| | Ant 2 | Cons | $R_m$ | $R_a$ | $R_c$ | $R_s$ | $R_g$ | $R_{sg}$ | $R_{gg}$ | $R_{gs}$ | $R_{ss}$ | $R_b$ | $R_\triangle$ | $R_\blacktriangle$ | $R_*$ | $R_\#$ | $5_\square$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| Relation I (modus ponens) | $A$ | $B$ | × | × | 0 | 0 | 0 | 0 | 0 | 0 | 0 | × | × | × | × | × | × |
| Relation II-1 | *very* $A$ | *very* $B$ | × | × | × | 0 | × | 0 | × | × | 0 | × | × | × | × | × | × |
| Relation II-2 | *very* $A$ | $B$ | × | × | 0 | × | 0 | × | 0 | 0 | × | × | × | × | × | × | × |
| Relation III-1 | *more or less* $A$ | *more or less* $B$ | × | × | × | 0 | 0 | 0 | 0 | 0 | 0 | × | × | × | × | × | × |
| Relation III-2 | *more or less* $A$ | $B$ | × | × | 0 | × | × | × | × | × | × | × | × | × | × | × | × |
| Relation IV-1 | *not* $A$ | *unknown* | 0 | 0 | × | 0 | 0 | × | × | × | × | 0 | 0 | 0 | 0 | × | 0 |
| Relation IV-2 | *not* $A$ | *not* $B$ | × | × | × | × | × | 0 | 0 | 0 | 0 | × | × | × | × | × | × |
| Relation V (modus tollens) | *not* $B$ | *not* $A$ | × | × | × | 0 | × | 0 | × | × | 0 | × | × | × | × | × | × |
| Relation VI | *not very* $B$ | *not very* $A$ | × | × | × | 0 | × | 0 | × | × | 0 | × | × | × | × | × | × |
| Relation VII | *not more or less* $B$ | *not more or less* $A$ | × | × | × | 0 | × | 0 | × | × | 0 | × | × | × | × | × | × |
| Relation VIII-1 | $B$ | *unknown* | × | 0 | × | 0 | 0 | × | × | × | × | 0 | 0 | 0 | 0 | × | 0 |
| Relation VIII-2 | $B$ | $A$ | × | × | 0 | × | × | × | × | 0 | 0 | × | × | × | × | × | × |

## 4. Syllogism and contrapositive under each fuzzy reasoning method

In this section we shall investigate two interesting concepts of 'syllogism' and 'contrapositive' under each fuzzy reasoning method obtained in Section 2.

Let $P_1$, $P_2$ and $P_3$ be fuzzy conditional propositions such as

$P_1$: If $x$ is $A$ then $y$ is $B$,
$P_2$: If $y$ is $B$ then $z$ is $C$,
$P_3$: If $x$ is $A$ then $z$ is $C$,

where $A$, $B$ and $C$ are fuzzy sets in $U$, $V$ and $W$, respectively. If the proposition $P_3$ is deduced from the propositions $P_1$ and $P_2$, i.e. the following holds:

$$\begin{array}{ll} P_1\text{:} & \text{If } x \text{ is } A \text{ then } y \text{ is } B \\ P_2\text{:} & \text{If } y \text{ is } B \text{ then } z \text{ is } C \\ \hline P_3\text{:} & \text{If } x \text{ is } A \text{ then } z \text{ is } C \end{array} \tag{48}$$

then it is said that the syllogism holds.

Let $R(A, B)$, $R(B, C)$ and $R(A, C)$ be fuzzy relations in $U \times V$, $V \times W$ and $U \times W$, respectively, which are obtained from the propositions $P_1$, $P_2$ and $P_3$, respectively. If the following equality holds, the syllogism holds:

$$R(A, B) \circ R(B, C) = R(A, C). \tag{49}$$

That is to say,

$$\begin{array}{llcl} P_1\text{:} & \text{If } x \text{ is } A \text{ then } y \text{ is } B & \rightarrow & R(A, B) \\ P_2\text{:} & \text{If } y \text{ is } B \text{ then } z \text{ is } C & \rightarrow & R(B, C) \\ \hline P_3\text{:} & \text{If } x \text{ is } A \text{ then } z \text{ is } C & \leftarrow & R(A, B) \circ R(B, C) \end{array} \tag{50}$$

where '$\circ$' is the max–min composition of $R(A, B)$ and $R(B, C)$, and the membership function of $R(A, B) \circ R(B, C)$ is given by

$$\mu_{R(A,B)\circ R(B,C)}(u, w) = \bigvee_{v} [\mu_{R(A,B)}(u, v) \wedge \mu_{R(B,C)}(v, w)]. \tag{51}$$

Now we shall obtain $R(A, B) \circ R(B, C)$ under each fuzzy reasoning method and then show whether the syllogism holds or not.

We shall begin with the method $R_a$. The fuzzy relations $R_a(A, B)$ and $R_a(B, C)$ are obtained from propositions $P_1$ and $P_2$ by using (7):

$$R_a(A, B) = (\neg A \times V) \oplus (U \times B),$$

$$R_a(B, C) = (\neg B \times W) \oplus (V \times C).$$

Thus, the composition of $R_a(A, B)$ and $R_a(B, C)$ will be

$$R_a(A, B) \circ R_a(B, C) = [(\neg A \times V) \oplus (U \times B)] \circ [(\neg B \times W) \oplus (V \times C)]$$

and its membership function becomes as follows.

$$\mu_{R_a(A,B)\circ R_a(B,C)}(u, w)$$
$$= \bigvee_v \{[1 \wedge (1-\mu_A(u)+\mu_B(v))] \wedge [1 \wedge (1-\mu_B(v)+\mu_C(w)]\}$$
$$= \bigvee_v \{(\mathrm{i}) \wedge (\mathrm{ii})\}. \tag{52}$$

The function (i), i.e. $1 \wedge (1-\mu_A(u)+\mu_B(v))$, can be depicted by using the parameter $\mu_A(u)$ as in Fig. 5(a) and the function (ii), $1 \wedge (1-\mu_B(v)+\mu_C(w))$, is shown by using the parameter $\mu_C(w)$ as in Fig. 5(b). These figures base on Fig. 1(ii). From these figures, the function $(\mathrm{i}) \wedge (\mathrm{ii})$ in (52) with both parameters $\mu_A(u)=a$ and $\mu_C(w)=c$ will be shown by the broken line '- - - -' in Fig. 5(c) and its maximum value (by virtue of (52)) is $0.5+(1-a+c)/2$. On the other hand, if the parameter $\mu_A(u)$ is taken to be $a'$ as in Fig. 5(c), the maximum value of its line '-·-·' becomes 1. Therefore, in general, for any parameters $a$ and $c$, the maximum value of $(\mathrm{i}) \wedge (\mathrm{ii})$ is shown to be $1 \wedge (0.5+(1-a+c)/2)$. Therefore, the membership function $\mu_{R_a(A,B)\circ R_a(B,C)}(u, w)$ of (52) becomes:

$$\mu_{R_a(A,B)\circ R_a(B,C)}(u, w) = 1 \wedge (0.5+(1-\mu_A(u)+\mu_C(w))/2).$$

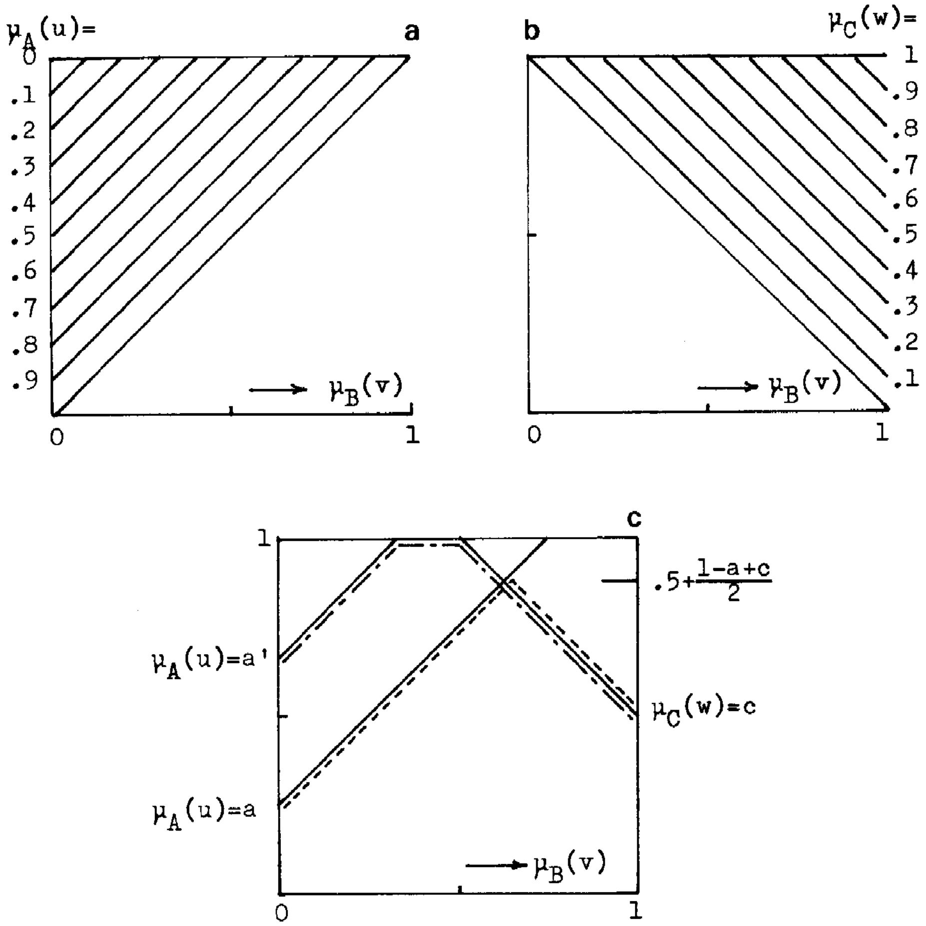


Fig. 5. The way of obtaining (52); (a) $1 \wedge (1-\mu_A(u)+\mu_B(v))$; (b) $1 \wedge (1-\mu_B(v)+\mu_C(w))$; (c) $1 \wedge (1-\mu_A(u)+\mu_B(v)) \wedge (1-\mu_B(v)+\mu_C(w))$.

From this result, we can have

$$R_a(A, B) \circ R_a(B, C)$$
$$= \int_{U\times W} 1 \wedge (0.5 + (1 - \mu_A(u) + \mu_C(w))/2)/(u, w)$$
$$\neq R_a(A, C) \ \left( = \int_{U\times W} 1 \wedge (1 - \mu_A(u) + \mu_C(w))/(u, w) \right). \tag{53}$$

Hence, we can conclude that the fuzzy reasoning method $R_a$ does not satisfy the syllogism.

Similarly, we can obtain $R(A, B) \circ R(B, C)$ under other fuzzy reasoning methods and we shall list them in the following.

$$R_m(A, B) \circ R_m(B, C)$$
$$= \int_{U\times W} 0.5 \vee (\mu_A(u) \wedge \mu_C(w)) \vee (1 - \mu_A(u))/(u, w)$$
$$\neq R_m(A, C) \ \left( = \int_{U\times W} (\mu_A(u) \wedge \mu_C(w)) \vee (1 - \mu_A(u))/(u, w) \right). \tag{54}$$

$$R_c(A, B) \circ R_c(B, C) = \int_{U\times W} \mu_A(u) \wedge \mu_C(w)/(u, w)$$
$$= R_c(A, C). \tag{55}$$

$$R_s(A, B) \circ R_s(B, C) = \int_{U\times W} \mu_A(u) \underset{s}{\rightarrow} \mu_C(w)/(u, w)$$
$$= R_s(A, C). \tag{56}$$

$$R_g(A, B) \circ R_g(B, C) = \int_{U\times W} \mu_A(u) \underset{g}{\rightarrow} \mu_C(w)/(u, w)$$
$$= R_g(A, C). \tag{57}$$

$$R_{sg}(A, B) \circ R_{sg}(B, C)$$
$$= \int_{U\times W} [\mu_A(u) \underset{s}{\rightarrow} \mu_C(w)] \wedge [1 - \mu_A(u) \underset{g}{\rightarrow} 1 - \mu_C(w)]/(u, w)$$
$$= R_{sg}(A, C). \tag{58}$$

$$R_{gg}(A, B) \circ R_{gg}(B, C)$$
$$= \int_{U\times W} [\mu_A(u) \underset{g}{\rightarrow} \mu_C(w)] \wedge [1 - \mu_A(u) \underset{g}{\rightarrow} 1 - \mu_C(w)]/(u, w)$$
$$= R_{gg}(A, C). \tag{59}$$

$$R_{gs}(A, B) \circ R_{gs}(B, C)$$
$$= \int_{U\times W} [\mu_A(u) \underset{g}{\rightarrow} \mu_C(w)] \wedge [1 - \mu_A(u) \underset{s}{\rightarrow} 1 - \mu_C(w)]/(u, w)$$
$$= R_{gs}(A, C). \tag{60}$$

$$R_{ss}(A,B)\circ R_{ss}(B,C)$$
$$=\int_{U\times W}[\mu_A(u)\underset{s}{\rightarrow}\mu_C(w)]\wedge[1-\mu_A(u)\underset{s}{\rightarrow}1-\mu_C(w)]/(u,w)$$
$$=R_{ss}(A,C). \tag{61}$$

$$R_b(A,B)\circ R_b(B,C)$$
$$=\int_{U\times W}0.5\vee(1-\mu_A(u))\vee\mu_C(w)/(u,w)$$
$$\neq R_b(A,C)\ \left(=\int_{U\times W}(1-\mu_A(u))\vee\mu_C(w)/(u,w)\right). \tag{62}$$

$$R_\Delta(A,B)\circ R_\Delta(B,C)$$
$$=\int_{U\times W}[\mu_A(u)\underset{\Delta'}{\rightarrow}\mu_C(w)]/(u,w)$$
$$\neq R_\Delta(A,C)\ \left(=\int_{U\times W}[\mu_A(u)\underset{\Delta}{\rightarrow}\mu_C(w)]/(u,w)\right), \tag{63}$$

where

$$\mu_A(u)\underset{\Delta'}{\rightarrow}\mu_C(w)=\begin{cases}1 & \mu_A(u)\leq\mu_C(w),\\ \sqrt{\dfrac{\mu_C(w)}{\mu_A(u)}} & \mu_A(u)>\mu_C(w),\end{cases}$$

$$\mu_A(u)\underset{\Delta}{\rightarrow}\mu_C(w)=\begin{cases}1 & \mu_A(u)\leq\mu_C(w),\\ \dfrac{\mu_C(w)}{\mu_A(u)} & \mu_A(u)>\mu_C(w).\end{cases}$$

$$R_\blacktriangle(A,B)\circ R_\blacktriangle(B,C)$$
$$=\int_{U\times W}[\mu_A(u)\underset{\blacktriangle'}{\rightarrow}\mu_C(w)]/(u,w)$$
$$\neq R_\blacktriangle(A,C)\ \left(=\int_{U\times W}[\mu_A(u)\underset{\blacktriangle}{\rightarrow}\mu_C(w)]/(u,w)\right), \tag{64}$$

where

$$\mu_A(u)\underset{\blacktriangle'}{\rightarrow}\mu_C(w)=\begin{cases}1\wedge\sqrt{\dfrac{\mu_C(w)}{\mu_A(u)}}\wedge\sqrt{\dfrac{1-\mu_A(u)}{1-\mu_C(w)}}\wedge(1-\mu_A(u)+\mu_C(w)), & \\ \qquad\qquad \mu_A(u)>0,\ 1-\mu_C(w)>0 & \\ 1 \qquad\qquad \mu_A(u)=0 \text{ or } 1-\mu_C(w)=0, & \end{cases}$$

$$\mu_A(u)\underset{\blacktriangle}{\rightarrow}\mu_C(w)=\begin{cases}1\wedge\dfrac{\mu_C(w)}{\mu_A(u)}\wedge\dfrac{1-\mu_A(u)}{1-\mu_C(w)} & \mu_A(u)>0,\ 1-\mu_C(w)>0,\\ 1 & \mu_A(u)=0 \text{ or } 1-\mu_C(w)=0.\end{cases}$$

$$R_*(A, B) \circ R_*(B, C)$$
$$= \int_{U \times W} \frac{1 - \mu_C(w) + \mu_A(u)\mu_C(w)}{1 - \mu_C(w) + \mu_A(u)} \Big/ (u, w)$$
$$\neq R_*(A, C) \left( = \int_{U \times W} 1 - \mu_A(u) + \mu_A(u)\mu_C(w)/(u, w) \right). \tag{65}$$

$$R_\#(A, B) \circ R_\#(B, C)$$
$$= \int_{U \times W} (0.5 \vee 1 - \mu_A(u) \vee \mu_C(w)) \wedge (\mu_A(u) \vee 1 - \mu_A(u))$$
$$\wedge (1 - \mu_C(w) \vee \mu_C(w))/(u, w)$$
$$\neq R_\#(A, C) \left( = \int_{U \times W} (1 - \mu_A(u) \vee \mu_C(w)) \wedge (\mu_A(u) \vee 1 - \mu_A(u)) \right.$$
$$\left. \wedge (1 - \mu_C(w) \vee \mu_C(w))/(u, w) \right). \tag{66}$$

$$R_\square(A, B) \circ R_\square(B, C) = \int_{U \times W} [\mu_A(u) \underset{\square}{\to} \mu_C(w)]/(u, w)$$
$$= R_\square(A, C), \tag{67}$$

where

$$\mu_A(u) \underset{\square}{\to} \mu_C(w) = \begin{cases} 1 & \mu_A(u) < \text{ or } \mu_C(w) = 1, \\ 0 & \mu_A(u) = 1, \ \mu_C(v) < 1. \end{cases}$$

Using these results the satisfaction or failure of the syllogism under each method is listed in Table 6. The membership functions of $R(A, B) \circ R(B, C)$ whose method does not satisfy the syllogism are depicted in Fig. 6 using a parameter $\mu_C$ in order to make comparisons between $R(A, B) \circ R(B, C)$ and $R(A, C)$, where the membership function of $R(A, C)$ is obtained by replacing $\mu_B$ with $\mu_C$ in Fig. 1 (left figure).

Finally, we shall investigate the contrapositive of a fuzzy conditional proposition under each method.

For a fuzzy conditional proposition $P_1$:

$P_1$: If $x$ is $A$ then $y$ is $B$

and its contrapositive proposition $P_2$:

$P_2$: If $y$ is *not* $B$ then $x$ is *not* $A$,

we can obtain fuzzy relations $R(A, B)$ in $U \times V$ from $P_1$ and $R(\neg B, \neg A)$ in $V \times U$ from $P_2$ using each method, where $A$, $B$ are fuzzy sets in $U$, $V$, respectively. If the contrapositive holds, the following identity is satisfied.

Table 6. Satisfaction of syllogism and contrapositive

| | $R_m$ | $R_a$ | $R_c$ | $R_s$ | $R_g$ | $R_{sg}$ | $R_{gg}$ | $R_{gs}$ | $R_{ss}$ | $R_b$ | $R_\triangle$ | $R_\blacktriangle$ | $R_*$ | $R_\#$ | $R_\square$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| Syllogism | × | × | 0 | 0 | 0 | 0 | 0 | 0 | 0 | × | × | × | × | × | 0 |
| Contrapositive | × | 0 | × | 0 | × | × | × | × | 0 | 0 | × | × | 0 | 0 | × |

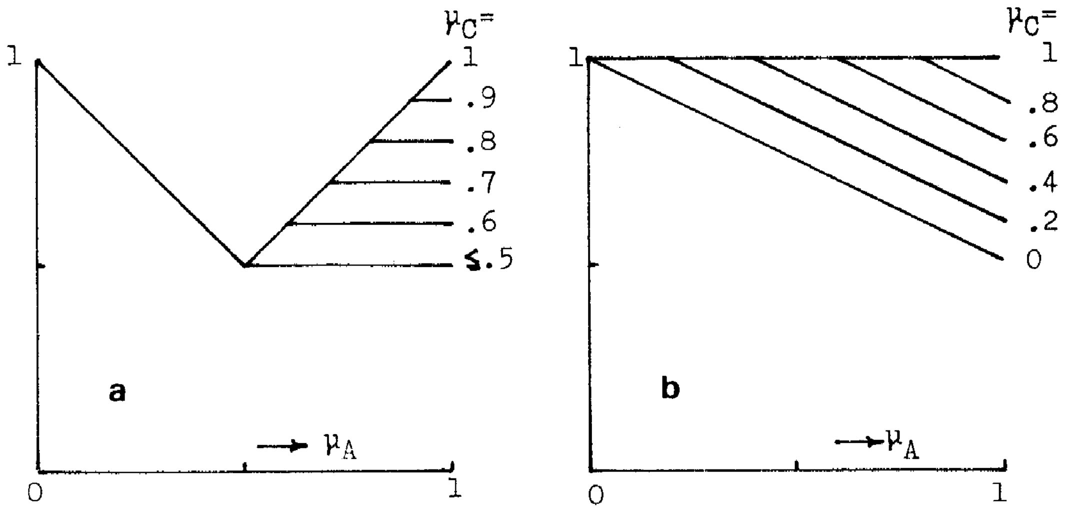


Fig. 6(a). $R_m(A, B) \circ R_m(B, C)$; (b) $R_a(A, B) \circ R_a(B, C)$.

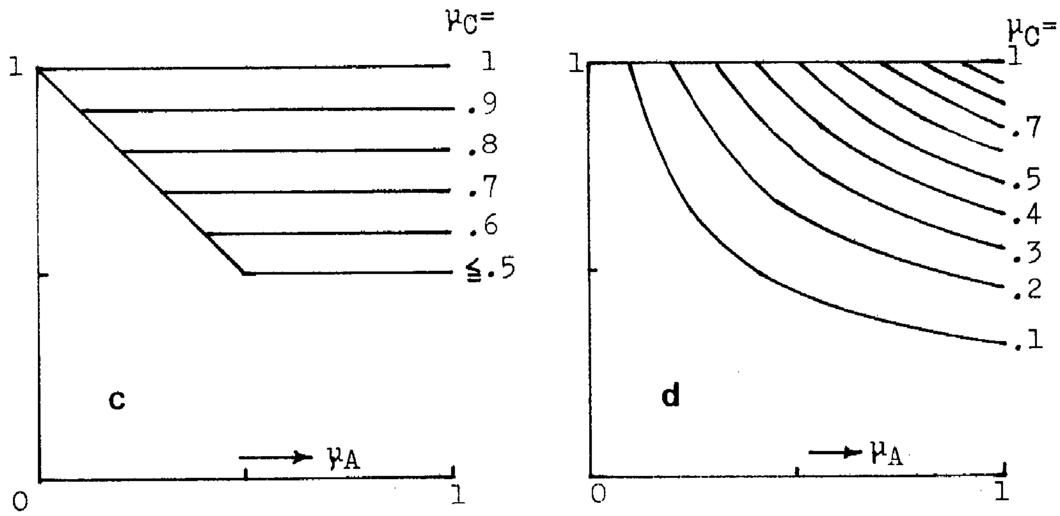


Fig. 6(c). $R_b(A, B) \circ R_b(B, C)$; (d) $R_{\triangle}(A, B) \circ R_{\triangle}(B, C)$.

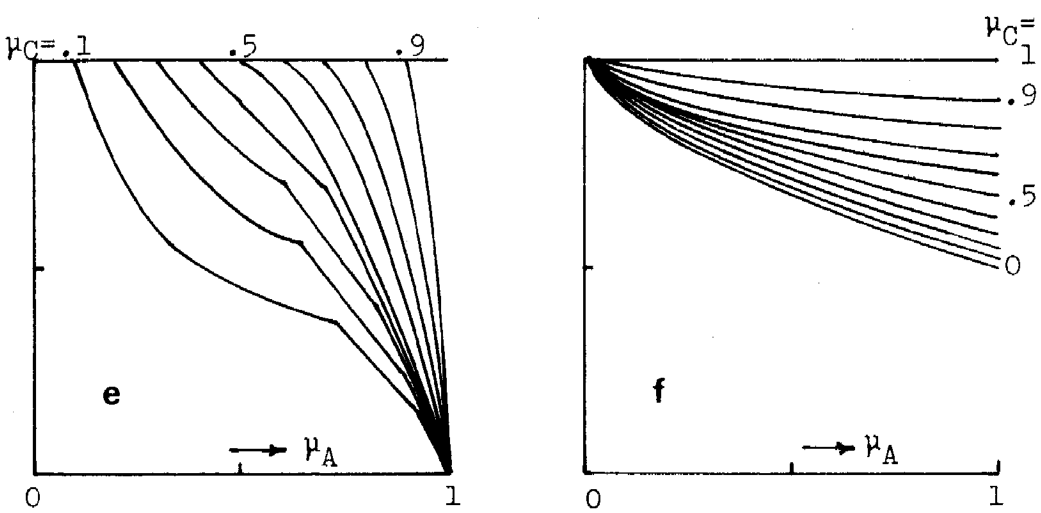


Fig. 6(e). $R_{\blacktriangle}(A, B) \circ R_{\blacktriangle}(B, C)$; (f) $R_{*}(A, B) \circ R_{*}(B, C)$.

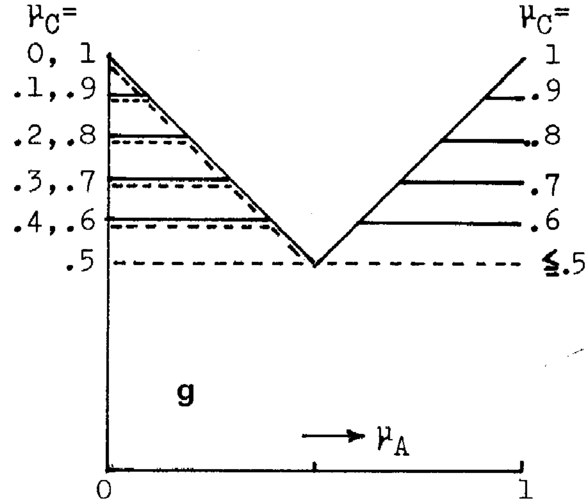


Fig. 6(g). $R_{\#}(A, B) \circ R_{\#}(B, C)$.

$$R(\neg B, \neg A) = \tilde{R}(A, B) \tag{68}$$

where $\tilde{R}(A, B)$ denotes the converse of $R(A, B)$.

For example, for the method $R_a$, we have

$$\begin{aligned} R_a(\neg B, \neg A) &= \int_{V\times U} 1 \wedge [1-(1-\mu_B(v)) + (1-\mu_A(u))]/(v, u) \\ &= \int_{V\times U} 1 \wedge (1-\mu_A(u) + \mu_B(v))/(v, u) \\ &= \tilde{R}_a(A, B). \end{aligned}$$

Therefore, the contrapositive holds under the method $R_a$.

For the method $R_m$,

$$\begin{aligned} &R_m(\neg B, \neg A) \\ &\quad = \int_{V\times U} [(1-\mu_B(v)) \wedge (1-\mu_A(u))] \vee [1-(1-\mu_B(v))]/(v, u) \\ &\quad \neq \tilde{R}_m(A, B) \left( = \int_{V\times U} (\mu_A(u) \wedge \mu_B(v)) \vee (1-\mu_A(u))/(v, u) \right). \end{aligned}$$

Hence, $R_m$ does not satisfy the contrapositive.

In the same way, we can see if the contrapositive holds or not under other methods (See Table 6).

## 5. Conclusion

We have investigated the inference results in the cases of generalized modus ponens and generalized modus tollens under the fuzzy reasoning methods which were proposed before and which are newly obtained by introducing the implication rules of many valued logics. Moreover, the syllogism and contrapositive are discussed under each method. From these results we can conclude that the methods $R_m$ and $R_a$ are not suitable for the fuzzy reasoning, since they do not satisfy the criteria which are quite reasonable demands. $R_c$ is not a bad method and $R_s$, $R_g, \ldots$, $R_{ss}$ are suitable methods. The new methods $R_b$, $R_\Delta, \ldots, R_\square$ which are based on the implication rules of many valued logic systems are not good.

We have discussed the compositional rule of inference which uses the max–min composition. The possibility of using new compositions other than max–min composition is given in [10, 11].

The formalization of inference methods for the more complicated form of inference such as

$$\begin{array}{l} \text{If } x \text{ is } A \text{ then } y \text{ is } B \text{ else } y \text{ is } C \\ x \text{ is } A' \\ \hline y \text{ is } D \end{array}$$

$$\begin{array}{l} \text{If } x \text{ is } A_1 \text{ then } y \text{ is } B_1 \text{ else} \\ \text{If } x \text{ is } A_2 \text{ then } y \text{ is } B_2 \text{ else} \\ \qquad\qquad \vdots \\ \text{If } x \text{ is } A_n \text{ then } y \text{ is } B_n \\ x \text{ is } A' \\ \hline y \text{ is } B' \end{array}$$

would be the future subjects of investigation.

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