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SOME CONSIDERATIONS ON FUZZY CONDITIONAL INFERENCE

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In much of human reasoning, the form of reasoning is approximate rather than exact as in 'A red apple is ripe and this apple is more or less red. Then this apple is more or less ripe.'

L.A. Zadeh and E.H. Mamdani suggested methods for such a fuzzy reasoning as an application of fuzzy set theory. The method involves an inference rule and a conditional proposition which contains fuzzy concepts.

In this paper we point out that the consequence inferred by their methods does not always fit our intuitions and we suggest the improved methods which fit our intuitions under several criteria.

Keywords: Compositional rule of inference, Fuzzy conditional inference, Implication

1. Introduction

In the semantics of natural language there exist vast amounts of fuzzy concepts, and we humans very often make such an inference whose antecedents and consequences contain fuzzy concepts. Therefore, from the standpoint of artificial intelligence, it seems that the formalization of inference methods for such inferences are very important. However, such inferences cannot be made sufficiently by the method which is based on classical two valued logic.

In order to make those inferences, L.A. Zadeh [4, 5] suggested an inference rule called compositional rule of inference instead of classical inference rule, i.e., modus-ponens. Using this inference rule, he suggested some methods for the inference in which the antecedent is a conditional proposition containing fuzzy concepts, and Mamdani [1] also has given a method for such an inference.

In this paper we point out that their methods do not give a consequence which fits our intuitions, and suggest improved methods for such an inference.

2. Fuzzy sets—notation, terminology and basic operations

We shall make a brief summary of the concept of fuzzy sets and fuzzy relations which will be needed in later sections.

Fuzzy sets. A *fuzzy set* A in a universe of discourse U is characterized by a membership function μ_A which takes the value in the interval [0, 1], i.e.,

$$\mu_A : U \to [0, 1].$$

When U is continuous, a fuzzy set A is represented as

$$A = \int_U \mu_A(u)/u.$$

When U is discrete, A is represented as

$$A = \mu_A(u_1)/u_1 + \mu_A(u_2)/u_2 + \cdots + \mu_A(u_n)/u_n.$$

Operations on fuzzy sets. If F and G are fuzzy sets in U, i.e.,

$$F = \int_U \mu_F(u)/u, \qquad G = \int_U \mu_G(u)/u,$$

the union $F \cup G$, intersection $F \cap G$, bounded-sum $F \oplus G$ and complement $\neg F$ are fuzzy sets in U defined by

$$F \cup G = \int_U \mu_F(u) \vee \mu_G(u)/u, \qquad F \cap G = \int_U \mu_F(u) \wedge \mu_G(u)/u,$$

$$F \oplus G = \int_U 1 \wedge (\mu_F(u) + \mu_G(u))/u, \qquad \neg F = \int_U 1 - \mu_F(u)/u,$$

where $\vee$ and $\wedge$ denote max and min, respectively.

If α is a real number, then F^α is defined by[1]

$$F^\alpha = \int_U (\mu_F(u))^\alpha/u = \int_U \mu_F^\alpha(u)/u.$$

[1] F^2 and $F^{0.5}$ may be used to approximate the effect of the linguistic modifiers **very** and **more or less**, thus, **very** $F = F^2$, **more or less** $F = F^{0.5}$.

Cartesian product. If $F_1, \ldots, F_n$ are fuzzy sets in $U_1, \ldots, U_n$, respectively, then the *Cartesian product* of $F_1, \ldots, F_n$ is a fuzzy set in $U_1 \times \cdots \times U_n$ defined by

$$F_1 \times \cdots \times F_n = \int_{U_1 \times \cdots \times U_n} (\mu_{F_1}(u_1) \wedge \cdots \wedge \mu_{F_n}(u_n))/(u_1, \ldots, u_n)$$

Fuzzy relations. An *n-ary fuzzy relation* R in $U_1 \times \cdots \times U_n$ $(n \geq 1)$ is a fuzzy set in $U_1 \times \cdots \times U_n$ and is defined by

$$R = \int_{U_1 \times \cdots \times U_n} \mu_R(u_1, \ldots, u_n)/(u_1, \ldots, u_n).$$

Composition of fuzzy relations. Let R and S be binary fuzzy relations in $U \times W$ and in $W \times V$, respectively, defined as

$$R = \int_{U \times W} \mu_R(u, w)/(u, w),$$

$$S = \int_{W \times V} \mu_S(w, v)/(w, v).$$

Then the *composition* of R and S is given by

$$R \circ S = \int_{U \times V} \bigvee_{w \in W} [\mu_R(u, w) \wedge \mu_S(w, v)]/(u, v).$$

If R is a unary fuzzy relation (that is, a fuzzy set) over W defined by

$$R = \int_W \mu_R(w)/w,$$

then the composition of R and S is defined as

$$R \circ S = \int_V \bigvee_{w \in W} [\mu_R(w) \wedge \mu_S(w, v)]/v.$$

3. Fuzzy conditional inference

In this chapter, we discuss the inference of the form:

$$\begin{array}{l} \text{Ant 1: If } x \text{ is } A \text{ then } y \text{ is } B \\ \text{Ant 2: } x \text{ is } A'. \\ \hline \text{Cons: } y \text{ is } B'. \end{array} \qquad (1)$$

where A, A', B and B' are fuzzy concepts. An example of this form of inference is the following.

If a tomato is red then the tomato is ripe.
This tomato is very red.
____
This tomato is very ripe.

From now on, we call this form of inference as 'fuzzy conditional inference'.

In the fuzzy conditional inference, if $A = A'$ and A and B are non-fuzzy, then (1) reduces to the classical modus-ponens of two-valued logic. But, if A, A' and B are fuzzy and $A \neq A'$, then this form of inference can not be made by classical modus-ponens of two-valued logic.

In order to make an inference in which fuzzy concepts are contained in the antecedents, Zadeh [4, 5] formalized an inference rule named 'compositional rule of inference', and he suggested methods for fuzzy conditional inference by applying this inference rule.

The consequences inferred by his methods, however, do not always fit our intuitions, so we suggest improved methods which fit our intuitions under several criteria.

3.1. *Compositional rule of inference*

In our daily life we often make such an inference that, from the relationship between some objects x and y and the information about x, we deduce some information about y as a consequence. From this point of view, Zadeh suggested an inference rule named compositional rule of inference (CRI for short).

Compositional rule of inference. If we translate the antecedent of the inference which represents the relationship between some objects x and y into a suitable fuzzy relation R and the antecedent which represents information about x into a suitable unary fuzzy relation (that is, fuzzy set) A, then the consequence can be obtained by the composition of A and R. This rule may be expressed in symbols as

$$\begin{array}{l} x \text{ is } A. \\ \underline{x \text{ and } y \text{ are } R.} \\ y \text{ is } A \circ R. \end{array} \tag{2}$$

where the symbol $\circ$ represents the composition of fuzzy relations.

3.2. *Well-known methods for fuzzy conditional inference*

In this section we present Zadeh's methods [4, 5] and Mamdani's method [1] for fuzzy conditional inference whose form is

$$\begin{array}{l} \text{Ant 1: If } x \text{ is } A \text{ then } y \text{ is } B. \\ \underline{\text{Ant 2: } x \text{ is } A'.\qquad\qquad} \\ \text{Cons: } y \text{ is } B'. \end{array} \tag{3}$$

where A and A' are fuzzy concepts represented as fuzzy sets in the universe of discourse U, and B is a fuzzy concept represented as fuzzy set in the universe of discourse V. Moreover, B' is a consequence represented as fuzzy set in V.

In order to get a consequence B' by applying CRI, the antecedents Ant 1 and Ant 2 must be translated, respectively, into a binary fuzzy relation which is expressed as $R(A_1(x), A_2(y))$ and a unary fuzzy relation which is expressed as $R(A_1(x))$. $A_1(x)$ and $A_2(y)$ are implied attributes of x and y which take values in the universes of discourse U and V, respectively. $R(A_1(x))$ is given by

$$R(A_1(x)) = A' \tag{4}$$

and for $R(A_1(x), A_2(y))$, the following three definitions were suggested by Zadeh and Mamdani (Definitions ① and ② are by Zadeh [4, 5] and Definition ③ is by Mamdani [1]).

Definition 1 (Maximin rule of conditional proposition).

$$R_m(A_1(x), A_2(y)) = (A \times B) \cup (\neg A \times V) \tag{5}$$

where $\times$, $\cup$ and $\neg$ denote Cartesian product, union and complement, respectively.

Definition 2 (Arithmetic rule of conditional proposition).

$$R_a(A_1(x), A_2(y)) = (\neg A \times V) \oplus (U \times B) \tag{6}$$

where $\oplus$ denotes bounded-sum.

Definition 3 (Mini operation rule of conditional proposition).

$$R_c(A_1(x), A_2(y)) = A \times B \tag{7}$$

In the above definitions of $R(A_1(x), A_2(y))$, the consequence $R(A_2(y))$ (that is, B' in Cons of (3)) is obtained by applying CRI, i.e.,

$$R(A_2(y)) = A' \circ [(A \times B) \cup (\neg A \times U)] \tag{8}$$

$$R(A_2(y)) = A' \circ [(\neg A \times V) \oplus (U \times B)] \tag{9}$$

or

$$R(A_2(y)) = A' \circ (A \times B) \tag{10}$$

3.3. *Some criteria for fuzzy conditional inference*

In this section, in order to evaluate the methods by Zadeh and Mamdani and to lay the foundation for formalizing improved methods, we shall consider what relations between antecedents and consequence are required in fuzzy conditional inference and then set up several criteria.

In the fuzzy conditional inference

Ant 1: If x is A then y is B.
Ant 2: x is A'.
Cons: y is B'.

where A, B and A' are fuzzy concepts represented by fuzzy sets, it seems, according to our intuitions, that the satisfaction of the following criteria may be required.

Criterion I

Ant 1: If x is A then y is B.
Ant 2: x is A. (11)
Cons: y is B.

This criterion may be a quite natural demand.

Criterion II-1

Ant 1: If x is A then y is B.
Ant 2: x is *very* A. (12)
Cons: y is *very* B.

This criterion also seems to be a natural one. This will be consented from the following example.

If a tomato is red then the tomato is ripe.
This tomato is very red.
This tomato is very ripe.

Criterion II-2

Ant 1: If x is A then y is B.
Ant 2: x is *very* A. (13)
Cons: y is B.

This criterion has the consequence different from that of Criterion II-1. But when in the Ant 1 there is not a strong casual relation between 'x is A' and 'y is B', the satisfaction of Criterion II-2 will be permitted.

Criterion III

$$\frac{\begin{array}{l}\text{Ant 1: If } x \text{ is } A \text{ then } y \text{ is } B.\\ \text{Ant 2: } x \text{ is } \textit{more or less } A.\end{array}}{\text{Cons: } y \text{ is } \textit{more or less } B.} \tag{14}$$

This criterion may also be a natural one.

Criterion IV-1

$$\frac{\begin{array}{l}\text{Ant 1: If } x \text{ is } A \text{ then } y \text{ is } B.\\ \text{Ant 2: } x \text{ is not } A.\end{array}}{\text{Cons: } y \text{ is } \textbf{unknown.}} \tag{15}$$

This criterion asserts that when x is **not** A, any information about y can not be deduced from Ant 1. Thus, this criterion may be thought to be quite natural, since the fuzzy conditional proposition 'If x is A then y is B' does not make any assertion when x is **not** A.

Criterion IV-2

$$\frac{\begin{array}{l}\text{Ant 1: If } x \text{ is } A \text{ then } y \text{ is } B.\\ \text{Ant 2: } x \text{ is } \textbf{not } A.\end{array}}{\text{Cons: } y \text{ is } \textbf{not } B.} \tag{16}$$

The satisfaction of this criterion is demanded when the fuzzy conditional proposition:

If x is A then y is B

means tacitly

If x is A then y is B else y is **not** B.

Though this criterion may not be accepted in ordinary logic, in daily life we often encounter the situation in which this criterion can hold.

From the above criteria we can classify the fuzzy conditional inferences into the following four types:

Type 1: Criteria I, II-1, III, IV-1 are satisfied
Type 2: Criteria I, II-2, III, IV-1 are satisfied
Type 3: Criteria I, II-1, III, IV-2 are satisfied
Type 4: Criteria I, II-2, III, IV-2 are satisfied (17)

3.4. *Reconsideration of the methods by Zadeh and Mamdani*

As previously stated, for the following form of inference:

$$\frac{\begin{array}{l}\text{Ant 1: If } x \text{ is } A \text{ then } y \text{ is } B.\\ \text{Ant 2: } x \text{ is } A'.\end{array}}{\text{Cons: } y \text{ is } B'.} \tag{18}$$

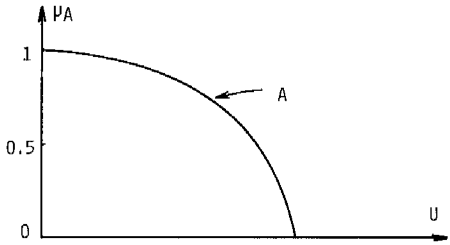

Fig. 1. Membership function $\mu_A(u)$ of fuzzy set A in U

Zadeh suggested that the B' in Cons is obtained by

$$B'_m = A' \circ [(A \times B) \cup (\neg A \times V)] \tag{19}$$

$$B'_a = A' \circ [(\neg A \times V) \oplus (U \times B)] \tag{20}$$

and Mamdani suggested that

$$B'_c = A' \circ (A \times B). \tag{21}$$

We shall show that Zadeh's methods do not satisfy the criteria stated in Section 3.3 except Criterion IV-1 and that Mamdani's method does not satisfy except Criteria I and II-2.

Now, we shall show what will B'_m, B'_a and B'_c be when A' in (19)–(21) is equal to A, **very** A ($= A^2$), **more or less** A ($= A^{0.5}$) or **not** A ($= \neg A$), where fuzzy sets A in U and B in V are given as in Fig. 1 and Fig. 2, respectively.

3.4.1. *The case of maximin rule*

Let A' be A, then B'_m becomes as follows using (19):

$$\begin{aligned} B'_m &= A \circ [(A \times B) \cup (\neg A \times V)] \\ &= \int_U \mu_A(u)/u \circ \int_{U \times V} (\mu_A(u) \wedge \mu_B(v)) \vee (1 - \mu_A(u))/(u, v) \\ &= \int_V \bigvee_{u \in U} [\mu_A(u) \wedge ((\mu_A(u) \wedge \mu_B(v)) \vee (1 - \mu_A(u)))]/v. \end{aligned} \tag{22}$$

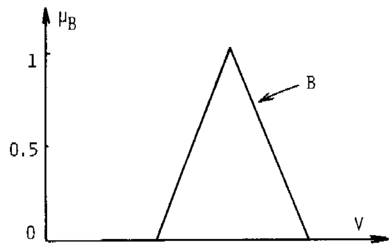

Fig. 2. Membership function $\mu_B(v)$ of fuzzy set B in V

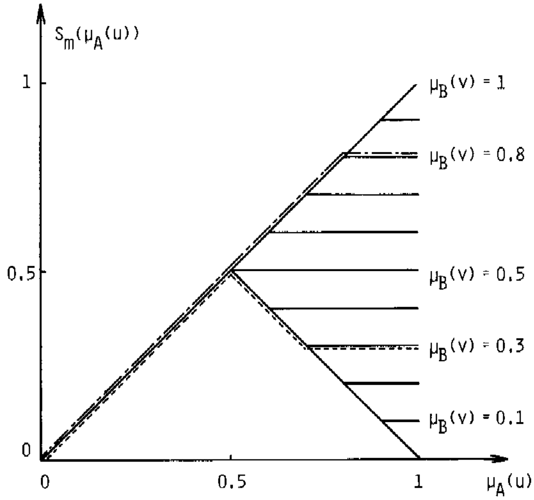

Fig. 3. $S_m(\mu_A(u))$ of (23)

Now, let

$$S_m(\mu_A(u)) = \mu_A(u) \wedge ((\mu_A(u) \wedge \mu_B(v)) \vee (1 - \mu_A(u))). \tag{23}$$

The value $S_m(\mu_A(u))$ with a parameter $\mu_B(v)$ is shown in Fig. 3. In Fig. 3, if $\mu_B(v) = 0.3$, $S_m(\mu_A(u))$ is shown by - - - - -, and if $\mu_B(v) = 0.8$, $S_m(\mu_A(u))$ becomes what is shown by –·–·—, and so on. In Fig. 1, $\mu_A(u)$ takes all values in the unit interval [0, 1] according to u varying all over U. Thus from Fig. 3 we have

$$\bigvee_{u \in U} S_m(\mu_A(u)) = \begin{cases} \mu_B(v) & \text{-----} \quad \mu_B(v) \geqslant 0.5 \\ 0.5 & \text{-----} \quad \mu_B(v) \leqslant 0.5 \end{cases} \tag{24}$$

Therefore, from (22) and (24) we have

$$B'_m = \int_V \bigvee_{u \in U} S_m(\mu_A(u))/v, \tag{25}$$

and the membership function of B'_m is shown in Fig. 4.

From Fig. 4, $B'_m \neq B$ is obtained, and thus it is shown that Criterion I is not satisfied.

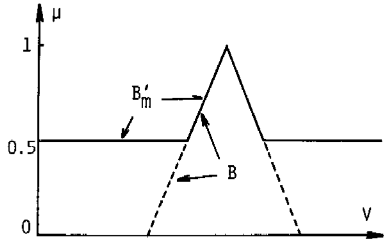

Fig. 4. Membership function of B'_m when $A' = A$

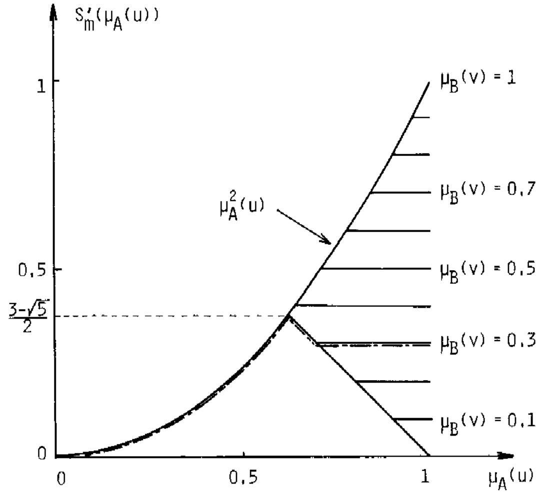

Fig. 5. $S'_m(\mu_A(u))$ of (27)

Second, suppose $A' = \textbf{very}\ A (= A^2)$, then

$$B'_m = A^2 \circ [(A \times B) \cup (\neg A \times V)]$$

$$= \int_V \mu_A^2(u)/u \circ \int_{U \times V} (\mu_A(u) \wedge \mu_B(v)) \vee (1 - \mu_A(u))/(u, v)$$

$$= \int_V \bigvee_{u \in U} \mu_A^2(u) \wedge [(\mu_A(u) \wedge \mu_B(v)) \vee (1 - \mu_A(u))]/v. \tag{26}$$

Now, let

$$S'_m(\mu_A(u)) = \mu_A^2(u) \wedge [(\mu_A(u) \wedge \mu_B(v)) \vee (1 - \mu_A(u))]. \tag{27}$$

The value $S'_m(\mu_A(u))$ with a parameter $\mu_B(v)$ is depicted as in Fig. 5. Thus,

$$\bigvee_{u \in U} S'_m(\mu_A(u)) = \begin{cases} \mu_B(v) \text{ ----- } \mu_B(v) \geqslant \dfrac{3-\sqrt{5}}{2} \\ \dfrac{3-\sqrt{5}}{2} \text{ ----- } \mu_B(v) \leqslant \dfrac{3-\sqrt{5}}{2}. \end{cases} \tag{28}$$

Therefore,

$$B'_m = \int_V \bigvee_{u \in U} S'_m(\mu_A(u))/v. \tag{29}$$

and the membership function of B'_m is shown in Fig. 6. Hence from Fig. 6, we can see

$$B'_m \neq \textbf{very}\ B$$

$$B'_m \neq B.$$

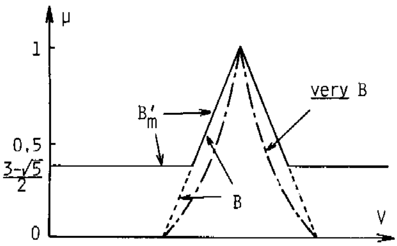

Fig. 6. Membership function of B'_m when $A' =$ **very** A

This shows that both Criterion II-1 and Criterion II-2 are not satisfied.

In a similar way, when $A' =$ **more or less** A $(= A^{0.5})$, the membership function of B'_m will be as in Fig. 7. From Fig. 7, $B'_m \neq$ **more or less** B is obtained and thus Criterion III is not satisfied.

Finally we shall show that Criterion IV-1 is satisfied when $A' =$ **not** A. Let $A' =$ **not** A $(= \neg A)$, then

$$B'_m = (\neg A) \circ [(A \times B) \cup (\neg A \times V)]$$

$$= \int_U 1 - \mu_A(u)/u \circ \int_{U \times V} (\mu_A(u) \wedge \mu_B(v)) \vee (1 - \mu_A(u))/(u, v)$$

$$= \int_V \bigvee_{u \in U} (1 - \mu_A(u)) \wedge [(\mu_A(u) \wedge \mu_B(v)) \vee (1 - \mu_A(u))]/v \tag{30}$$

Now, from Fig. 1, there exists $u \in U$ which makes $\mu_A(u) = 0$, so that

$$(30) = \int_V 1 \wedge [(0 \wedge \mu_B(v)) \vee 1]/v$$

$$= \int_V 1/v$$

= **unknown.**

This shows that Criterion IV-1 is satisfied.

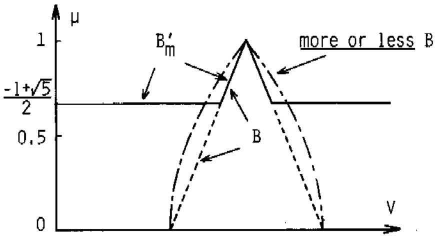

Fig. 7. Membership function of B'_m when $A' =$ **more or less** A

Since Criterion IV-1 is inconsistent with Criterion IV-2, it is clear that Criterion IV-2 is not satisfied.

3.4.2. *The case of arithmetic rule*

Suppose that $A' = A^\alpha$ ($\alpha > 0$), then the consequence B'_a is obtained as follows:

$$B'_a = A^\alpha \circ [(\neg A \times V) \oplus (U \times B)]$$

$$= \int_U \mu_A^\alpha(u)/u \circ \int_{U\times V} 1 \wedge (1 - \mu_A(u) + \mu_B(v))/(u, v)$$

$$= \int_V \bigvee_{u \in U} [\mu_A^\alpha(u) \wedge (1 \wedge (1 - \mu_A(u) + \mu_B(v)))]/v. \tag{31}$$

Now, let

$$S_a(\mu_A(u), \alpha) = \mu_A^\alpha(u) \wedge (1 \wedge (1 - \mu_A(u) + \mu_B(v))).$$

From Fig. 1, $\mu_A(u)$ takes all values in [0, 1] according to u varying over U. Then from Fig. 8, when $\alpha = 1$,

$$\bigvee_{u\in U} S_a(\mu_A(u), 1) = \bigvee_{u \in U} \mu_A(u) \wedge (1 \wedge (1 - \mu_A(u) + \mu_B(v)))$$
$$= \frac{1 + \mu_B(v)}{2}.$$

Hence

$$B'_a = \int_V \bigvee_{u \in U} S_a(\mu_A(u), 1)/v$$

$$= \int_V \frac{1 + \mu_B(v)}{2} \Big/ v \tag{32}$$

and the membership function of B'_a, that is, $(1 + \mu_B(v))/2$, is shown in Fig. 9. From Fig. 9, $B'_a \neq B$ and so it is shown that Criterion I is not satisfied.

When $\alpha = 2$, that is, $A' =$ **very** A $(= A^2)$, we have

$$\bigvee_{u\in U} S_a(\mu_A(u), 2) = \bigvee_{u \in U} \mu_A^2(u) \wedge (1 \wedge (1 - \mu_A(u) + \mu_B(v)))$$
$$= \frac{3 + 2\mu_B(v) - \sqrt{5 + 4\mu_B(v)}}{2}.$$

Thus

$$B'_a = \int_V \frac{3 + 2\mu_B(v) - \sqrt{5 + 4\mu_B(v)}}{2} \Big/ v. \tag{33}$$

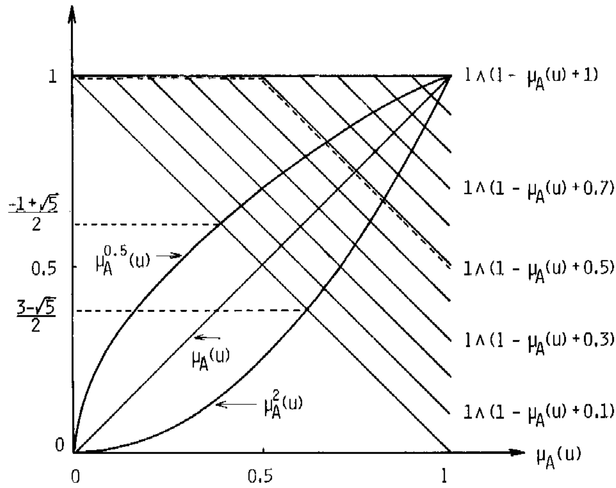

Fig. 8. $\mu_A(u)$, $\mu_A^2(u)$, $\mu_A^{0.5}(u)$ and $1\wedge(1-\mu_A(u)+\mu_B(v))$

Hence

$$B'_a \neq \int_V \mu_B^2(v)/v = \textbf{very } B \tag{34}$$

$$B'_a \neq \int_V \mu_B(v)/v = B. \tag{35}$$

The membership functions of B'_a, **very** B and B are shown in Fig. 10. These show that Criteria II-1 and II-2 are not satisfied.

When $\alpha = 0.5$, that is, $A' =$ **more or less** A $(=A^{0.5})$,

$$\bigvee_{u\in U} S_a(\mu_A(u), 0.5) = \bigvee_{u\in U} \mu_A^{0.5}(u)\wedge(1\wedge(1-\mu_A(u)+\mu_B(v)))$$

$$= \frac{-1+\sqrt{5+4\mu_B(v)}}{2}.$$

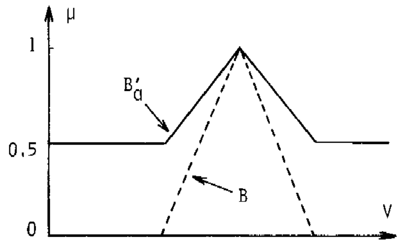

Fig. 9. Membership function of B'_a when $A' = A$

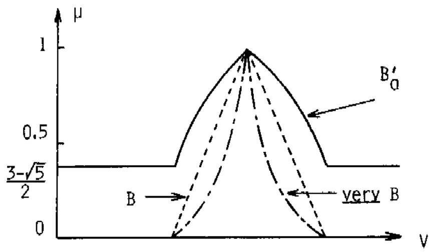

Fig. 10. Membership functions of fuzzy sets B'_a (when A' = **very** A), B and **very** B

Thus

$$B'_a = \int_V \frac{-1+\sqrt{5+4\mu_B(v)}}{2} \Big/ v. \tag{36}$$

$$\neq \int_V \mu_B^{0.5}(v)/v$$

$$(= \textbf{more or less } B). \tag{37}$$

This shows that Criterion III is not satisfied. The membership function of B'_a when $A' = A^{0.5}$ (=**more or less** A) is shown in Fig. 11.

Finally we shall show that Criterion IV-1 is satisfied.

Suppose A' = **not** A $(= \neg A)$, then

$$B'_a = (\neg A) \circ [(\neg A \times V) \oplus (U \times B)]$$

$$= \int_V \bigvee_{u \in U} (1-\mu_A(u)) \wedge [1 \wedge (1-\mu_A(u)+\mu_B(v))]/v$$

$$= \int_V 1 \wedge [1 \wedge (1+\mu_B(v))]/v$$

$$= \int_V 1/v$$

$$= \textbf{unknown.} \tag{38}$$

This shows that Criterion IV-1 is satisfied. Note that this criterion cannot be satisfied if $\mu_A(u) > 0$ for all $u \in U$.

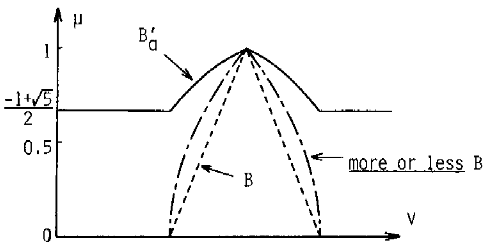

Fig. 11. Membership functions of B'_a (when A' = **more or less** A), B and **more or less** B

3.4.3. *The case of mini operation rule*

First, suppose $A' = A^\alpha$, then

$$B'_c = A^\alpha \circ (A \times B)$$
$$= \int_U \mu_A^\alpha(u)/u \circ \int_{U\times V} \mu_A(u) \wedge \mu_B(v)/(u, v)$$
$$= \int_V \bigvee_{u\in U} \mu_A^\alpha(u) \wedge (\mu_A(u) \wedge \mu_B(v))/v. \tag{39}$$

From Fig. 1, there exists $u \in U$ which makes $\mu_A(u) = 1$. Thus

$$(39) = \int_V 1 \wedge (1 \wedge \mu_B(v))/v$$
$$= \int_V \mu_B(v)/v$$
$$= B. \tag{40}$$

This shows that Criteria I and II-2 are satisfied, but Criteria II-1 and III are not satisfied.

Second, let $A' =$ **not** A, then

$$B'_c = (\neg A) \circ (A \times B)$$
$$= \int_U 1 - \mu_A(u)/u \circ \int_{U\times V} \mu_A(u) \wedge \mu_B(v)/(u, v)$$
$$= \int_V \bigvee_{u\in U} [(1 - \mu_A(u)) \wedge \mu_A(u) \wedge \mu_B(v)]/v$$
$$= \begin{cases} \int_V 0.5/v \text{ ----- } \mu_B(v) \geq 0.5 \\ \int_V \mu_B(v)/v \text{ ----- } \mu_B(v) \leq 0.5. \end{cases} \tag{41}$$

This shows that Criteria IV-1 and IV-2 are not satisfied.

It is interesting to note that when $A' =$ **unknown** ($= U$), we have

$$B'_c = \int_U 1/u \circ \int_{U\times V} \mu_A(u) \wedge \mu_B(v)/(u, v)$$
$$= \int_V \bigvee_{u\in U} [1 \wedge \mu_A(u) \wedge \mu_B(v)]/v$$
$$= \int_V \mu_B(v)/v = B.$$

This consequence can not be accepted according to our intuitions.

Above discussions show that using the methods (Zadeh's methods and Mamdani's method), almost all criteria stated in Section 3.3 can not be satisfied and it may be clear that consequences inferred by these methods do not always fit our intuitions.

4. Formalization of improved methods

In this chapter, we present new methods for each type of fuzzy conditional inference which satisfy criteria stated in Section 3.3. The difference between the new and previous methods are the definition of fuzzy relations translated from fuzzy conditional proposition.

4.1. *Basic consideration*

By the arithmetic rule of conditional proposition defined by Zadeh, the fuzzy conditional proposition

$$P = \text{If } x \text{ is } A \text{ then } y \text{ is } B$$

is translated into the fuzzy relation:

$$\begin{aligned} R_a(A_1(x), A_2(y)) &= (\neg A \times V) \oplus (U \times B) \\ &= \int_{U \times V} 1 \wedge (1 - \mu_A(u) + \mu_B(v))/(u, v). \end{aligned}$$

In this fuzzy relation, the membership function

$$1 \wedge (1 - \mu_A(u) + \mu_B(v))$$

is what is obtained by Łukasiewicz's definition of material implication in $Ł_{\aleph}$ logic, that is,

$$v(P \to Q) = 1 \wedge (1 - v(P) + v(Q))$$

where $v(P \to Q)$, $v(P)$ and $v(Q)$ denote the truth values of propositions $P \to Q$, P and Q, respectively. Thus this definition may be viewed as an adaptation of material implication in $Ł_{\aleph}$ logic to fuzzy conditional proposition. By paying an attention to this fact, $R_a(A_1(x), A_2(y))$ may be represented as follows.

$$\begin{aligned} R_a(A_1(x), A_2(y)) &= (\neg A \times V) \oplus (U \times B) \\ &= \int_{U \times V} 1 \wedge (1 - \mu_A(u) + \mu_B(v))/(u, v) \\ &= \int_{U \times V} \mu_A(u) \underset{Ł}{\longrightarrow} \mu_B(v)/(u, v) \\ &= (A \times V) \underset{Ł}{\longrightarrow} (U \times B) \end{aligned} \qquad (42)$$

where $\to_Ł$ denotes the material implication in $Ł_{\aleph}$ logic.

In this section, we discuss what properties are required to the implication in order to satisfy Criterion I stated in Section 3.3 in the case where we adapt the implication of other many-valued logic systems to the definition of fuzzy relations.

For the fuzzy conditional proposition P: If x is A then y is B, let fuzzy sets A of U and B of V be given as

$$A = \int_U \mu_A(u)/u$$

$$B = \int_V \mu_B(v)/v.$$

In general, fuzzy conditional proposition P may be translated into a binary fuzzy relation $R(A_1(x), A_2(y))$ by adopting an implication of a many-valued logic system, i.e.,

$$\begin{aligned} R(A_1(x), A_2(y)) &= A \times V \to U \times B \\ &= \int_{U \times V} \mu_A(u) \to \mu_B(v)/(u, v) \end{aligned} \tag{43}$$

where the value of $\mu_A(u) \to \mu_B(v)$ is defined according to an employed logic system.

Now suppose $R(A_1(x)) = A$, then we can obtain the consequence $R(A_2(y))$ by applying CRI to $R(A_1(x))$ and $R(A_1(x), A_2(y))$. Thus,

$$\begin{aligned} R(A_2(y)) &= A \circ R(A_1(x), A_2(y)) \\ &= \int_U \mu_A(u)/u \circ \int_{U \times V} \mu_A(u) \to \mu_B(v)/(u, v) \\ &= \int_V \bigvee_{u \in U} [\mu_A(u) \wedge (\mu_A(u) \to \mu_B(v))]/v. \end{aligned} \tag{44}$$

In order that Criterion I is satisfied, that is, $R(A_2(y)) = B$, the equality

$$\bigvee_{u \in U} [\mu_A(u) \wedge (\mu_A(u) \to \mu_B(v))] = \mu_B(v) \tag{45}$$

must be satisfied for arbitrary v in V and in order that the equality (45) is satisfied, it is necessary that the inequality

$$\mu_A(u) \wedge (\mu_A(u) \to \mu_B(v)) \leq \mu_B(v)$$

holds for arbitrary u in U and v in V. Therefore, for the purpose that the righthand of (44) is equal to B, we must employ as the implication of (43) an implication of the logic system in which at least the inequality

$$v[P \wedge (P \rightarrow Q)] \leq v(Q) \tag{46}$$

is satisfied for arbitrary propositions P and Q.

Note that (46) is not satisfied in $L_{\aleph}$ (42).

The logic system of 'standard sequence $S_{\aleph}$' in which the implication is defined as

$$v(P \rightarrow Q) = \begin{cases} 1 & \text{-----} \quad v(P) \leq v(Q) \\ 0 & \text{-----} \quad v(P) > v(Q) \end{cases}$$

and the logic system of 'Gödelian sequence $G_{\aleph}$' in which implication is defined as

$$v(P \rightarrow Q) = \begin{cases} 1 & \text{-----} \quad v(P) \leq v(Q) \\ v(Q) & \text{-----} \quad v(P) > v(Q) \end{cases}$$

satisfy the inequality (46) (see [2]).

On the bases of the above discussion we suggest improved methods in the next section.

4.2 *Formalization of improved methods*

In this section we define new fuzzy relations for each type of fuzzy conditional inferences in (17) and show that if we apply CRI to these fuzzy relations, criteria stated in Section 3.3 are satisfied.

4.2.1 *Method for fuzzy conditional inference of Type* 1

The fuzzy conditional inference of Type 1 is the one whose form is

$$\begin{array}{l} \text{Ant 1: If } x \text{ is } A \text{ then } y \text{ is } B. \\ \text{Ant 2: } x \text{ is } A'. \\ \hline \text{Cons : } y \text{ is } B'. \end{array} \tag{47}$$

where A, B and A' are fuzzy concepts represented as fuzzy sets in universes of discourse U, V and U, respectively, and which requires the satisfaction of Criteria I, II-1, III and IV-1.

For this inference, if Ant 2 translates into a unary fuzzy relation

$$R(A_1(x)) = A' \tag{48}$$

and Ant 1 translates into a binary fuzzy relation $R_s(A_1(x), A_2(y))$, which will be defined in (52), then the consequence $R(A_2(y))$ is obtained by applying CRI, i.e.,

$$R(A_2(y)) = R(A_1(x)) \circ R_s(A_1(x), A_2(y)). \tag{49}$$

This $R(A_2(y))$ is equal to B' in (47).

Now we shall give the definition of $R_s(A_1(x), A_2(y))$.

Definition 4.1. Let fuzzy sets A in U and B in V be

$$A = \int_U \mu_A(u)/u \tag{50}$$

$$B = \int_V \mu_B(v)/v \tag{51}$$

then $R_s(A_1(x), A_2(y))$ is defined by the following.

$$R_s(A_1(x), A_2(y)) = A \times V \xrightarrow[s]{} U \times B$$

$$= \int_{U \times V} \mu_A(u)/(u, v) \xrightarrow[s]{} \int_{U \times V} \mu_B(v)/(u, v)$$

$$= \int_{U \times V} \mu_A(u) \xrightarrow[s]{} \mu_B(v)/(u, v) \tag{52}$$

where

$$\mu_A(u) \xrightarrow[s]{} \mu_B(v) = \begin{cases} 1 \text{ ----- } & \mu_A(u) \leq \mu_B(v) \\ 0 \text{ ----- } & \mu_A(u) > \mu_B(v). \end{cases} \tag{53}$$

This definition is based on the implication in $S_\aleph$ logic system, i.e.,

$$v(P \to Q) = \begin{cases} 1 \text{ ----- } & v(P) \leq v(Q) \\ 0 \text{ ----- } & v(P) > v(Q) \end{cases} \tag{54}$$

Next, we shall show that using this method, Criteria I, II-1, III and IV-1 are satisfied under the following conditions (55), (56) and (57).[2] Let fuzzy sets A in U and B in V be given as (50) and (51), respectively, and the following condition be assumed

$$\{\mu_A(u) \mid u \in U\} \supseteq \{\mu_B(v) \mid v \in V\} \tag{55}$$

$$\exists u \in U \quad \mu_A(u) = 0, \qquad \exists u' \in U \quad \mu_A(u') = 1 \tag{56}$$

$$\exists v \in V \quad \mu_B(v) = 0, \qquad \exists v' \in U \quad \mu_B(v') = 1 \tag{57}$$

and, as a general case, suppose $R(A_1(x)) = A^\alpha$ $(\alpha > 0)$, then (49) will be

$$R(A_2(y)) = R(A_1(x)) \circ R_s(A_1(x), A_2(y))$$

$$= A^\alpha \circ (A \times V \xrightarrow[s]{} U \times B)$$

$$= \int_U \mu_A^\alpha(u)/u \circ \int_{U \times V} \mu_A(u) \xrightarrow[s]{} \mu_B(v)/(u, v)$$

$$= \int_V \bigvee_{u \in U} \mu_A^\alpha(u) \wedge (\mu_A(u) \xrightarrow[s]{} \mu_B(v))/v. \tag{58}$$

Here, for each v in V, we can obtain two subsets U_1 and U_2 of U which satisfy the following condition.

$$U_1 \cup U_2 = U, \qquad U_1 \cap U_2 = \emptyset \tag{59}$$

$$\forall u \in U_1 \quad \mu_A(u) \leq \mu_B(v) \tag{60}$$

$$\forall u \in U_2 \quad \mu_A(u) > \mu_B(v) \tag{61}$$

[2] It is noted that we have discussed the methods by Zadeh and Mamdani under the same conditions.

Then

$$(58) = \int_V \bigvee_{u \in U_1} \mu_A^\alpha(u)/v \qquad \text{from (60)}$$

$$= \int_V \mu_B^\alpha(v)/v \qquad \text{from (55) and (60)}$$

$$= B^\alpha.$$

This shows that when $\alpha = 1$ ($A' = A$), $\alpha = 2$ ($A' = A^2$) and $\alpha = 0.5$ ($A' = A^{0.5}$), Criteria I, II-1 and III are satisfied, respectively.

Next, suppose $R(A_1(x)) = \textbf{not } A$, then (49) becomes

$$R(A_2(y)) = R(A_1(x)) \circ R_s(A_1(x), A_2(y))$$

$$= (\neg A) \circ (A \times V \xrightarrow[s]{} U \times B)$$

$$= \int_U 1 - \mu_A(u)/u \circ \int_{U \times V} \mu_A(u) \xrightarrow[s]{} \mu_B(v)/(u, v)$$

$$= \int_V \bigvee_{u \in U} [(1 - \mu_A(u)) \wedge (\mu_A(u) \xrightarrow[s]{} \mu_B(v))]/v. \tag{62}$$

From the assumption (56) there exists u in U which makes $\mu_A(u) = 0$. Therefore

$$\bigvee_{u \in U} [(1 - \mu_A(u)) \wedge (\mu_A(u) \xrightarrow[s]{} \mu_B(v))] = 1.$$

Thus,

$$(62) = \int_V 1/v$$

$$= \textbf{unknown}. \tag{63}$$

This shows that Criterion IV-1 is satisfied.

4.2.2. *Method for the fuzzy conditional inference of Type* 2

Fuzzy conditional inference of Type 2 is the one the form of which is the same as (47) and which requires the satisfaction of Criteria I, II-2, III and IV-1.

For this inference, Ant 1 translates into binary fuzzy relation $R_g(A_1(x), A_2(y))$, which is defined below, and the consequence $R(A_2(y))$ is obtained in the similar way of the case of Type 1, i.e.,

$$R(A_2(y)) = R(A_1(x)) \circ R_g(A_1(x), A_2(y))$$

$$= A' \circ R_g(A_1(x), A_2(y)). \tag{64}$$

The definition of $R_g(A_1(x), A_2(y))$ is given by the following.

Definition 4.2. Let fuzzy sets A in U and B in V be the same as (50) and (51), respectively, then $R_g(A_1(x), A_2(y))$ is defined as

$$R_g(A_1(x), A_2(y)) = A \times V \underset{g}{\longrightarrow} U \times B$$

$$= \int_{U \times V} \mu_A(u) \underset{g}{\longrightarrow} \mu_B(v)/(u, v), \tag{65}$$

where

$$\mu_A(u) \underset{g}{\longrightarrow} \mu_B(v) = \begin{cases} 1 & \text{-----} \quad \mu_A(u) \leq \mu_B(v) \\ \mu_B(v) & \text{-----} \quad \mu_A(u) > \mu_B(v). \end{cases} \tag{66}$$

This definition is viewed as an adoption of Gödel's definition of the implication in $G_{\aleph}$ logic system, i.e.,

$$v(P \to Q) = \begin{cases} 1 & \text{-----} \quad v(P) \leq v(Q) \\ v(Q) & \text{-----} \quad v(P) > v(Q). \end{cases}$$

We shall show that using this method, Criteria I, II-2, III and IV-I are satisfied. Let the conditions (55), (56) and (57) be satisfied and suppose $R(A_1(x)) = A^\alpha$ $(\alpha > 0)$. Then we have (64) as

$$R(A_2(y)) = R(A_1(x)) \circ R_g(A_1(x), A_2(y))$$

$$= A^\alpha \circ (A \times V \underset{g}{\longrightarrow} U \times B)$$

$$= \int_U \mu_A(u)/u \circ \int_{U \times V} \mu_A(u) \underset{g}{\longrightarrow} \mu_B(v)/(u, v)$$

$$= \int_V \bigvee_{u \in U} \mu_A^\alpha(u) \wedge (\mu_A(u) \underset{g}{\longrightarrow} \mu_B(v))/v. \tag{67}$$

Here, for each v in V, we can obtain two sets U_1 and U_2 which satisfy the following conditions.

$$U_1 \cup U_2 = U, \qquad U_1 \cap U_2 = \emptyset \tag{68}$$

$$\forall u \in U_1 \quad \mu_A(u) \leq \mu_B(v) \tag{69}$$

$$\forall u \in U_2 \quad \mu_A(u) > \mu_B(v). \tag{70}$$

Therefore

$$(67) = \int_V \Big(\bigvee_{u \in U_1} \mu_A^\alpha(u)\Big) \vee \Big(\bigvee_{u \in U_2} \mu_A^\alpha(u) \wedge \mu_B(v)\Big)/v$$

$$= \int_V \mu_B^\alpha(v) \vee \Big[\Big(\bigvee_{u \in U_2} \mu_A^\alpha(u)\Big) \wedge \mu_B(v)\Big]/v$$

$$= \int_V \mu_B^\alpha(v) \vee (1 \wedge \mu_B(v))/v$$

$$= \begin{cases} \int_V \mu_B^\alpha(v)/v = B^\alpha & \text{-----} \quad \alpha \leq 1 \\ \int_V \mu_B(v)/v = B & \text{-----} \quad \alpha > 1. \end{cases}$$

This shows that when $\alpha=1$, $\alpha=2$ and $\alpha=0.5$, Criteria I, II-2 and III are satisfied, respectively.

Furthermore, the satisfaction of Criterion IV-1 can be shown in the similar way in the case of Type 1.

4.2.3. *Method for fuzzy conditional inference of Type 3*

Fuzzy conditional inference of Type 3 is the one whose form is the same as (47) and which demands the satisfaction of Criteria I, II-1, III and IV-2.

For this inference, Ant 1 translates into a binary fuzzy relation $R_{sg}(A_1(x), A_2(y))$, which is defined below, and then the consequence $R(A_2(y))$ is obtained by

$$\begin{aligned} R(A_2(y)) &= R(A_1(x)) \circ R_{sg}(A_1(x), A_2(y)) \\ &= A' \circ R_{sg}(A_1(x), A_2(y)). \end{aligned} \tag{71}$$

$R_{sg}(A_1(x), A_2(y))$ is obtained by the following.

Definition 4.3. Let fuzzy sets A in U and B in V be the same as (50) and (51), respectively, then $R_{sg}(A_1(x), A_2(y))$ is defined as follows.

$$R_{sg}(A_1(x), A_2(y)) = (A \times V \underset{s}{\longrightarrow} U \times B) \cap (\neg A \times V \underset{g}{\longrightarrow} U \times \neg B) \tag{72}$$

We shall show that using this method, Criteria I, II-1, III and IV-2 are satisfied under the conditions (55), (56) and (57).

Suppose $R(A_1(x)) = A^\alpha$ ($\alpha > 0$), then (71) becomes

$$\begin{aligned} &R(A_2(y)) \\ &\quad = R(A_1(x)) \circ R_{sg}(A_1(x), A_2(y)) \\ &\quad = A^\alpha \circ [(A \times V \underset{s}{\longrightarrow} U \times B) \cap (\neg A \times V \underset{g}{\longrightarrow} U \times \neg B)] \\ &\quad = \int_U \mu_A^\alpha(u)/u \circ \int_{U \times V} (\mu_A(u) \underset{s}{\longrightarrow} \mu_B(v)) \\ &\qquad\qquad \wedge [(1-\mu_A(u)) \underset{g}{\longrightarrow} (1-\mu_B(v))]/(u, v) \\ &\quad = \int_V \bigvee_{u \in U} [\mu_A^\alpha(u) \wedge [(\mu_A(u) \underset{s}{\longrightarrow} \mu_B(v)) \\ &\qquad\qquad \wedge [(1-\mu_A(u)) \underset{g}{\longrightarrow} (1-\mu_B(v))]]]/v. \end{aligned} \tag{73}$$

Here, for each v in V there exist three sets U_1, U_2 and U_3 which satisfy the following conditions:

$$U_1 \cup U_2 \cup U_3 = U \tag{74}$$

$$U_i \cap U_j = \emptyset \qquad i, j \in \{1, 2, 3\}, \quad i \neq j \tag{75}$$

$$\forall u \in U_1 \quad \mu_A(u) < \mu_B(v) \tag{76}$$

$$\forall u \in U_2 \quad \mu_A(u) = \mu_B(v) \tag{77}$$

$$\forall u \in U_3 \quad \mu_A(u) > \mu_B(v). \tag{78}$$

Then

$$(73) = \int_V \left[\bigvee_{u\in U_1} \mu_A^\alpha(u) \wedge (1-\mu_B(v))\right] \vee \left[\bigvee_{u\in U_2} \mu_A^\alpha(u)\right] \Big/ v$$

$$= \int_V \bigvee_{u\in U_2} \mu_A^\alpha(u)/v$$

$$= \int_V \mu_B^\alpha(v)/v$$

$$= B^\alpha.$$

This shows that Criteria I, II-1 and III are satisfied.

Next, suppose $R(A_1(x)) = \textbf{not } A$, then the consequence $R(A_2(y))$ is obtained by the following.

$$R(A_2(y))$$

$$= R(A_1(x)) \circ R_{sg}(A_1(x), A_2(y))$$

$$= (\neg A) \circ [(A \times V \xrightarrow[s]{} U \times B) \cap (\neg A \times V \xrightarrow[g]{} U \times \neg B)]$$

$$= \int_U 1-\mu_A(u)/u \circ \int_{U\times V} (\mu_A(u) \xrightarrow[s]{} \mu_B(v))$$

$$\wedge [(1-\mu_A(u)) \xrightarrow[g]{} (1-\mu_B(v))]/(u, v)$$

$$= \int_V \bigvee_{u\in U} [(1-\mu_A(u)) \wedge (\mu_A(u) \xrightarrow[s]{} \mu_B(v))$$

$$\wedge [(1-\mu_A(u)) \xrightarrow[g]{} (1-\mu_B(v))]]/v$$

$$= \int_V \left[\bigvee_{u\in U_1} (1-\mu_A(u)) \wedge (1-\mu_B(v))\right] \vee \left[\bigvee_{u\in U_2} 1-\mu_A(u)\right] \Big/ v$$

$$= \int_V (1-\mu_B(v)) \vee (1-\mu_B(v))/v$$

$$= \int_V (1-\mu_B(v))/v$$

$$= \neg B$$

$$= \textbf{not } B.$$

This indicates that Criterion IV-1 is satisfied.

4.2.4. *Method for fuzzy conditional inference of Type* 4

Fuzzy conditional inference of Type 4 is the one the form of which is the same as (47) and which demands the satisfaction of Criteria I, II-2, III and IV-2.

For this inference, Ant 1 translates into a binary fuzzy relation $R_{gg}(A_1(x), A_2(y))$, which is defined below, and the consequence $R(A_2(y))$ is obtained in the similar way of the case of Type 1, i.e.,

$$R(A_2(y)) = R(A_1(x)) \circ R_{gg}(A_1(x), A_2(y)) \tag{79}$$

Now we shall give the definition of $R_{gg}(A_1(x), A_2(y))$.

Definition 4.4. Let fuzzy sets A in U and B in V be as in (50) and (51), then $R_{gg}(A_1(x), A_2(y))$ is defined by the following.

$$R_{gg}(A_1(x), A_2(y)) = (A \times V \underset{g}{\longrightarrow} U \times B) \cap (\neg A \times V \underset{g}{\longrightarrow} U \times \neg B) \tag{80}$$

Since the satisfaction of the Criteria I, II-2, III and IV-2 can be shown in the similar way of Type 3, we shall omit the proof.

We shall next give an example of using the method of fuzzy conditional inference of Type 1.

Example 1. Let

$$U = V = 0+1+2+3+4+5+6+7+8+9+10$$

$$A = \text{small} = 1/0 + 0.8/1 + 0.6/2 + 0.4/3 + 0.2/4$$

$$B = \text{middle} = 0.2/2 + 0.4/3 + 0.8/4 + 1/5 + 0.8/6 + 0.4/7 + 0.2/8$$

Then the fuzzy conditional proposition

If x is small then y is middle

translates into

$$R_s(A_1(x), A_2(y)) = \text{small} \times V \underset{s}{\longrightarrow} U \times \text{middle}$$

$$= \begin{array}{c} \\ 0\\1\\2\\3\\4\\5\\6\\7\\8\\9\\10 \end{array} \begin{array}{c} \begin{array}{ccccccccccc} 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \end{array} \\ \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix} \end{array}$$

(1) Let $R(A_1(x)) = \text{small}$, then

$$R(A_2(y)) = \text{small} \circ R_s(A_1(x), A_2(y))$$
$$= 0.2/2 + 0.4/3 + 0.8/4 + 1/5 + 0.8/6 + 0.4/7 + 0.2/8$$
$$= \text{middle}.$$

(2) When $R(A_1(x)) = \textbf{very}\ \text{small}$,

$$R(A_2(y)) = (\textbf{very}\ \text{small}) \circ R_s(A_1(x), A_2(y))$$
$$= (\text{small})^2 \circ R_s(A_1(x), A_2(y))$$
$$= 0.04/2 + 0.16/3 + 0.64/4 + 1/5 + 0.64/6 + 0.16/7 + 0.04/8$$
$$= (\text{middle})^2$$
$$= \textbf{very}\ \text{middle}.$$

(3) If $R(A_1(x)) = \textbf{not}\ \text{small}$, then

$$R(A_2(y)) = (\textbf{not}\ \text{small}) \circ R_s(A_1(x), A_2(y))$$
$$= 0 + 1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9 + 10$$
$$= V$$
$$= \textbf{unknown}.$$

Stated in English, these inferences may be expressed as follows.

(1) If x is small then y is middle.
x is small.
———————————
y is middle.

(2) If x is small then y is middle.
x is very small.
———————————
y is very middle.

(3) If x is small then y is middle.
x is not small.
———————————
y is unknown.

4.3. *Some properties of R_s and R_g*

In this section, we describe some interesting properties of fuzzy relations R_s defined by (52) and R_g defined by (65). Note that the fuzzy relations R_m and R_a defined by Zadeh does not have these properties the fuzzy relation R_a has only Property 2, and the fuzzy relation R_c defined by Mamdani has only the following Property 1.

Property 1. Let fuzzy conditional propositions P_1, P_2 and P_3 be given as

P_1 = If x is A then y is B

P_2 = If y is B then z is C

P_3 = If x is A then z is C

where A, B and C are fuzzy concepts represented as the following fuzzy sets,

$$A=\int_U \mu_A(u)/u \qquad B=\int_V \mu_B(v)/v \qquad C=\int_W \mu_C(w)/w.$$

Let $R_s(A_1(x), A_2(y))$, $R_s(A_2(y), A_3(z))$ and $R_s(A_1(x), A_3(z))$ be fuzzy relations which are translated from P_1, P_2 and P_3 using (52), respectively, and $R_g(A_1(x), A_2(y))$, $R_g(A_2(y), A_3(z))$ and $R_g(A_1(x), A_3(z))$ be fuzzy relations translated from P_1, P_2 and P_3 using (65), respectively. Then, under the following conditions, i.e.,

$$\{\mu_A(u) \mid u\in U\}\supseteq\{\mu_B(v)\mid v\in V\}\supseteq\{\mu_C(w)\mid w\in W\} \tag{81}$$

$$\begin{aligned}&\exists u\in U \quad \mu_A(u)=0, \qquad \exists u'\in U \quad \mu_A(u')=1\\ &\exists v\in V \quad \mu_B(v)=0, \qquad \exists v'\in V \quad \mu_B(v')=1\\ &\exists w\in W \quad \mu_C(w)=0, \qquad \exists w'\in W \quad \mu_C(w')=1\end{aligned} \tag{82}$$

the following equalities are satisfied.

$$R_s(A_1(x), A_3(z))=R_s(A_1(x), A_2(y))\circ R_s(A_2(y), A_3(z)) \tag{83}$$

$$R_g(A_1(x), A_3(z))=R_g(A_1(x), A_2(y))\circ R_g(A_2(y), A_3(z)) \tag{84}$$

This property may be illustrated as

$$\begin{array}{l} P_1=\text{If } x \text{ is } A \text{ then } y \text{ is } B \leftrightarrow R_s(A_1(x), A_2(y))\\ P_2=\text{If } y \text{ is } B \text{ then } z \text{ is } C \leftrightarrow R_s(A_2(y), A_3(z))\\ \hline P_3=\text{If } x \text{ is } A \text{ then } z \text{ is } C \leftrightarrow R_s(A_1(x), A_2(y))\circ R_s(A_2(y), A_3(z))\end{array}$$

Proof. We shall prove the equality (83).

$$\begin{aligned}&R_s(A_1(x), A_2(y))\circ R_s(A_2(y), A_3(z))\\ &\quad=\int_{U\times V} \mu_A(u)\underset{s}{\longrightarrow}\mu_B(v)/(u,v)\circ\int_{V\times W}\mu_B(v)\underset{s}{\longrightarrow}\mu_C(w)/(v,w)\\ &\quad=\int_{U\times W}\bigvee_{v\in V}[(\mu_A(u)\underset{s}{\longrightarrow}\mu_B(v))\wedge(\mu_B(v)\underset{s}{\longrightarrow}\mu_C(w))]/(u,w).\end{aligned} \tag{85}$$

Now, let

$$S(v)=(\mu_A(u)\underset{s}{\longrightarrow}\mu_B(v))\wedge(\mu_B(v)\underset{s}{\longrightarrow}\mu_C(w))$$

then

$$S(v)=\begin{cases}1 \text{ ---- } \mu_A(u)\le\mu_B(v)\le\mu_C(w)\\ 0 \text{ ---- otherwise.}\end{cases}$$

From the assumptions (81) and (82), whenever the inequality

$$\mu_A(u)\le\mu_C(w)$$

holds, there exists some v in V which satisfies the inequality

$$\mu_A(u) \leq \mu_B(v) \leq \mu_C(w).$$

Therefore,

$$\bigvee_{v \in V} S(v) = \begin{cases} 1 \text{----} \mu_A(u) \leq \mu_B(w) \\ 0 \text{----} \mu_A(u) > \mu_B(w) \end{cases} \tag{86}$$

and hence

$$\begin{aligned} (85) &= \int_{U \times W} \bigvee_{v \in V} S(v)/(u, w) \\ &= \int_{U \times W} \mu_A(u) \underset{s}{\longrightarrow} \mu_C(w)/(u, w) \\ &= R_s(A_1(x), A_3(z)). \end{aligned}$$

This completes the proof of the equality (83).

The similar way is applicable to the proof of the equality (84).

Example 2. Let fuzzy conditional propositions P_1, P_2 and P_3 be

$P_1 =$ If x is A then y is B.

$P_2 =$ If y is B then z is C.

$P_3 =$ If x is A then z is C.

and fuzzy sets A in U, B in V and C in W be given as

$$A = 1/1 + 0.8/2 + 0.6/3 + 0.4/4 + 0.2/5$$
$$B = 0.2/4 + 0.4/5 + 0.8/6 + 1/7$$
$$C = 0.4/2 + 0.8/3 + 1/4 + 0.8/5 + 0.2/6$$

where

$$U = V = W = 1 + 2 + 3 + 4 + 5 + 6 + 7.$$

Then, $R_g(A_1(x), A_2(y))$, $R_g(A_2(y), A_3(z))$ and $R_g(A_1(x), A_3(z))$ which are translated from P_1, P_2 and P_3 using (65) are obtained as follows.

$$R_g(A_1(x), A_2(y)) = A \times V \underset{g}{\longrightarrow} U \times B = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \end{array} \begin{array}{c} \begin{array}{ccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 \end{array} \\ \begin{pmatrix} 0 & 0 & 0 & 0.2 & 0.4 & 0.8 & 1 \\ 0 & 0 & 0 & 0.2 & 0.4 & 1 & 1 \\ 0 & 0 & 0 & 0.2 & 0.4 & 1 & 1 \\ 0 & 0 & 0 & 0.2 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix} \end{array}$$

$$R_g(A_2(y), A_3(z)) = B \times W \xrightarrow[g]{} V \times C = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \end{array}\begin{array}{c} \begin{array}{ccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 \end{array} \\ \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 & 1 & 0.2 & 0 \\ 0 & 0.4 & 1 & 1 & 1 & 0.2 & 0 \\ 0 & 0.4 & 0.8 & 1 & 0.8 & 0.2 & 0 \end{pmatrix} \end{array}$$

$$R_g(A_1(x), A_3(z)) = A \times W \xrightarrow[g]{} U \times C = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \end{array}\begin{array}{c} \begin{array}{ccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 \end{array} \\ \begin{pmatrix} 0 & 0.4 & 0.8 & 1 & 0.8 & 0.2 & 0 \\ 0 & 0.4 & 1 & 1 & 1 & 0.2 & 0 \\ 0 & 0.4 & 1 & 1 & 1 & 0.2 & 0 \\ 0 & 1 & 1 & 1 & 1 & 0.2 & 0 \\ 0 & 1 & 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix} \end{array}$$

Then, the composition of $R_g(A_1(x), A_2(y))$ and $R_g(A_2(y), A_3(z))$ leads to

$$R_g(A_1(x), A_2(y)) \circ R_g(A_2(y), A_3(z))$$

$$= \begin{pmatrix} 0 & 0 & 0 & 0.2 & 0.4 & 0.8 & 1 \\ 0 & 0 & 0 & 0.2 & 0.4 & 1 & 1 \\ 0 & 0 & 0 & 0.2 & 0.4 & 1 & 1 \\ 0 & 0 & 0 & 0.2 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 & 1 & 0.2 & 0 \\ 0 & 0.4 & 1 & 1 & 1 & 0.2 & 0 \\ 0 & 0.4 & 0.8 & 1 & 0.8 & 0.2 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0.4 & 0.8 & 1 & 0.8 & 0.2 & 0 \\ 0 & 0.4 & 1 & 1 & 1 & 0.2 & 0 \\ 0 & 0.4 & 1 & 1 & 1 & 0.2 & 0 \\ 0 & 1 & 1 & 1 & 1 & 0.2 & 0 \\ 0 & 1 & 1 & 1 & 1 & 0.2 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix}$$

$$= R_g(A_1(x), A_3(z))$$

This shows the satisfaction of the equality (84).

Property 2. For the fuzzy conditional proposition P_1

$P_1 =$ If x is A then y is B

and its contrapositive proposition P_2, that is,

$P_2 =$ If y is **not** B then x is **not** A

where A and B are fuzzy sets given as

$$A = \int_U \mu_A(u)/u \qquad B = \int_V \mu_B(v)/v$$

let $R_s(A_1(x), A_2(y))$ and $R_s(A_2(y), A_1(x))$ be fuzzy relations which are translated from P_1 and P_2 using (52), respectively. Then the following equation is satisfied.

$$R_s(A_2(y), A_1(x)) = \tilde{R}_s(A_1(x), A_2(y)) \tag{87}$$

where $\tilde{R}_s(A_1(x), A_2(y))$ denotes the converse relation of $R_s(A_1(x), A_2(y))$. This property can be expressed in symbol as

$$\begin{array}{lcl} \text{If } x \text{ is } A \text{ then } y \text{ is } B \longrightarrow & & R_s(A_1(x), A_2(y)) \\ \Updownarrow \text{contrapositive proposition} & & \Updownarrow \text{converse relation} \\ \text{If } y \text{ is } \textbf{not } B \text{ then } x \text{ is } \textbf{not } A \longrightarrow & & R_s(A_2(y), A_1(x)) \end{array}$$

Note that R_g of (65) does not satisfy this property.

Proof

$$R_s(A_1(x), A_2(y)) = A \times V \underset{s}{\longrightarrow} U \times B$$

$$= \int_{U\times V} \mu_A(u) \underset{s}{\longrightarrow} \mu_B(v)/(u, v)$$

hence

$$\tilde{R}_s(A_1(x), A_2(y)) = \int_{V\times U} \mu_A(u) \underset{s}{\longrightarrow} \mu_B(v)/(v, u)$$

$$= \int_{V\times U} (1-\mu_B(v)) \underset{s}{\longrightarrow} (1-\mu_A(u))/(v, u)$$

$$= \neg B \times U \underset{s}{\longrightarrow} V \times \neg A$$

$$= R_s(A_2(y), A_1(x)).$$

This completes the proof of the equation (87).

Example 3. Let a fuzzy conditional proposition P_1 and its contradictive proposition P_2 be given as

$P_1 =$ If x is A then y is B.

$P_2 =$ If y is **not** B then x is **not** A.

and let fuzzy sets A in U and B in V be given as

$$A = 1/1 + 0.8/2 + 0.5/3 + 0.3/4$$
$$B = 0.2/10 + 0.4/11 + 0.7/12 + 1/13$$

where

$$U = 1 + 2 + 3 + 4 + 5$$
$$V = 8 + 9 + 10 + 11 + 12 + 13.$$

Then $R_s(A_1(x), A_2(y))$ and $R_s(A_2(y), A_1(x))$ are

$$R_s(A_1(x), A_2(y)) = A \times V \underset{s}{\longrightarrow} U \times B = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array} \begin{array}{c} \begin{array}{cccccc} 8 & 9 & 10 & 11 & 12 & 13 \end{array} \\ \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix} \end{array}$$

$$R_s(A_2(y), A_1(x)) = \neg B \times U \underset{s}{\longrightarrow} V \times \neg A = \begin{array}{c} \\ 8 \\ 9 \\ 10 \\ 11 \\ 12 \\ 13 \end{array} \begin{array}{c} \begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \end{array} \\ \begin{pmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{pmatrix} \end{array}.$$

Now,

$$\tilde{R}_s(A_1(x), A_2(y)) = \begin{array}{c} \\ 8 \\ 9 \\ 10 \\ 11 \\ 12 \\ 13 \end{array} \begin{array}{c} \begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \end{array} \\ \begin{pmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{pmatrix} \end{array} = R_s(A_2(y), A_1(x))$$

This shows that the equality (87) is satisfied.

5. Concluding remarks

In this paper we pointed out that the methods by Zadeh and Mamdani for the fuzzy conditional inference do not give the consequences which fits our intuitions, and gave improved methods which fit our intuitions under several criteria.
The inference form treated here, however, is only the following:

If x is A then y is B.
x is A'.
―――――――――――
y is B'.

Therefore, the formalization of inference methods for the more complicated form of inference, such as

$$\frac{\begin{array}{l}\text{If } x \text{ is } A \text{ then } y \text{ is } B \text{ else } y \text{ is } C.\\ x \text{ is } A'.\end{array}}{y \text{ is } B'.}$$

or

$$\frac{\begin{array}{l}\text{If } x \text{ is } A_1 \text{ and } A_2 \text{ and} \cdots \text{and } A_n \text{ then } y \text{ is } B.\\ x \text{ is } A_1' \text{ and} \cdots \text{and } A_n'.\end{array}}{y \text{ is } B'.}$$

would be the future subjects.

References

[1] E.H. Mamdani, Application of fuzzy logic to approximate reasoning using linguistic systems, IEEE Trans. Comput., C-26 (1977) 1182–1191.
[2] N. Rescher, Many Valued Logic (McGraw-Hill, New York, 1969).
[3] L.A. Zadeh, Fuzzy sets, Information and Control 8 (1965) 338–353.
[4] L.A. Zadeh, Calculus of fuzzy restrictions, in: L.A. Zadeh and K. Tanaka et al., Eds., Fuzzy Sets and Their Applications to Cognitive and Decision Processes (Academic Press, New York, 1975) 1–39.
[5] L.A. Zadeh, The concept of a linguistic variable and its application to approximate reasoning I, II, III, Information Sci., 8 (1975) 199–251; 8 (1975) 301–357; 9 (1975) 43–80.