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SEVERAL METHODS FOR FUZZY CONDITIONAL INFERENCES

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L.A. Zadeh and E.H. Mamdani have proposed methods for a fuzzy reasoning in which the antecedent involves a fuzzy conditional proposition "If x is A then y is B", with A and B being fuzzy concepts.

This paper points out that the consequences inferred by their methods do not always fit our intuitions, and suggests several new methods which fit our intuitions under several criteria such as modus ponens and modus tollens.

1. FUZZY CONDITIONAL INFERENCES

We shall first consider the following form of inference in which a fuzzy conditional proposition is contained.

Ant 1: If x is A then y is B.
Ant 2: x is A'. (1)

Cons: y is B'.

where x and y are the names of objects, and A, A', B and B' are the labels of fuzzy sets in universes of discourse U, U, V and V, respectively.

An example of this form of inference is as follows:

Ant 1: If a tomato is red then the tomato is ripe.
Ant 2: This tomato is very red.

Cons: This tomato is very ripe.

This form of inference may be viewed as a generalized modus ponens which reduces to a modus ponens when A' = A and B' = B.

Moreover, the following form of inference is also considered which also contains a fuzzy conditional proposition. This inference can be viewed as a generalized modus tollens which reduces to a modus tollens when B' = not B and A' = not A.

Ant 1: If x is A then y is B.
Ant 2: y is B'. (2)

Cons: x is A'.

For these forms of fuzzy conditional inferences, several methods are proposed.

Let A and B be fuzzy sets in U and V, respectively, which are represented as

$$A = \int_U \mu_A(u)/u \;; \qquad B = \int_V \mu_B(v)/v \;, \tag{3}$$

and let x, ∪, ∩, 7 and ⊕ be cartesian product, union, intersection, complement and bounded-sum for fuzzy sets, respectively. Then the following fuzzy relations can be derived from a fuzzy conditional proposition "If x is A then y is B" in Ant 1 of (1) and (2). The fuzzy relations R_m and R_a are proposed by Zadeh [1], R_c is by Mamdani [2], and R_s, R_g, R_{sg} and R_{gg} are new methods proposed here.

$$R_m = (A \times B) \cup (7A \times V). \tag{4}$$

$$R_a = (7A \times V) \oplus (U \times B). \tag{5}$$

$$R_c = A \times B. \tag{6}$$

$$R_s = A \times V \xrightarrow[s]{} U \times B = \int_{U\times V} [\mu_A(u) \xrightarrow[s]{} \mu_B(v)]/(u,v), \tag{7}$$

where

$$\mu_A(u) \xrightarrow[s]{} \mu_B(v) = \begin{cases} 1 & \ldots\ \mu_A(u) \le \mu_B(v), \\ 0 & \ldots\ \mu_A(u) > \mu_B(v). \end{cases}$$

$$R_g = A \times V \xrightarrow[g]{} U \times B = \int_{U\times V} [\mu_A(u) \xrightarrow[g]{} \mu_B(v)]/(u,v), \tag{8}$$

where

$$\mu_A(u) \xrightarrow[g]{} \mu_B(v) = \begin{cases} 1 & \ldots\ \mu_A(u) \le \mu_B(v), \\ \mu_B(v) & \ldots\ \mu_A(u) > \mu_B(v). \end{cases}$$

$$R_{sg} = (A \times V \xrightarrow[s]{} U \times B) \cap (7A \times V \xrightarrow[g]{} U \times 7B). \tag{9}$$

$$R_{gg} = (A \times V \xrightarrow[g]{} U \times B) \cap (7A \times V \xrightarrow[g]{} U \times 7B). \tag{10}$$

Then the consequence B' in Cons of (1) can be deduced from Ant 1 and Ant 2 using the max-min composition "o" of the fuzzy set A' in U and the fuzzy relation obtained above. Thus, we can have

$$B'_m = A' \circ R_m = A' \circ ((A \times B) \cup (7A \times V)), \tag{11}$$

$$B'_a = A' \circ R_a = A' \circ ((7A \times V) \oplus (U \times B)), \tag{12}$$

and so on.

Similarly, the consequence A' in Cons of (2) can be deduced using the composition "o" of the fuzzy set B' in V and the relation given before. Namely, we have

$$A'_m = R_m \circ B' = ((A \times B) \cup (7A \times V)) \circ B', \tag{13}$$

$$A'_a = R_a \circ B' = ((7A \times V) \oplus (U \times B)) \circ B', \tag{14}$$

and so on.

Table I Relations between A'(Ant 2) and B'(Cons) under Ant 1 for the Generalized Modus Ponens in (1)

	A'	B'
Relation I (modus ponens)	A	B
Relation II-1	very A	very B
Relation II-2	very A	B
Relation III-1	more or less A	more or less B
Relation III-2	more or less A	B
Relation IV-1	not A	unknown
Relation IV-2	not A	not B

In the above forms of fuzzy conditional inferences, it seems according to our intuitions that the relations between A' in Ant 2 and B' in Cons for the generalized modus ponens in (1) ought to be satisfied as shown in Table I. Similarly, the relations between B' in Ant 2 and A' in Cons for the generalized modus tollens in (2) ought to be satisfied as in Table II.

Relation I corresponds to modus ponens. Relation II-2 has the consequence different from that of Relation II-1, but if there is not a strong casual relation between "x is A" and "y is B" in Ant 1 (that is, "If x is A then y is B"), the satisfaction of Relation II-2 will be permitted. Relation IV-1 asserts that when x is not A, any information about y can not be deduced from Ant 1. The satisfaction of Relation IV-2 is demanded when the fuzzy proposition "If x is A then y is B" means tacitly the proposition "If x is A then y is B else y is not B". Relation V corresponds to modus tollens. Relation VIII is discussed as in the case of Relation IV.

In Tables I and II, it is noted that very A is defined as A^2, more or less A as $A^{0.5}$, not A as $\neg A$, not very A as $\neg A^2$, not more or less A as $\neg A^{0.5}$, and unknown as V (or U in Table II).

2. COMPARISON BETWEEN FUZZY CONDITIONAL INFERENCE METHODS

In this section we shall make a comparioson between the fuzzy conditional inference methods discussed above and show that Zadeh's methods do not satisfy the relations except Relations IV-1 and VIII-1 and that Mamdani's method does not satisfy the relations except Relations I, II-2, III-2 and VIII-2. New methods given in this paper satisfy almost these relations cited in Tables I and II.

We shall now begin with the maximin method R_m in (4) using some examples.†

[I] The Case of Maximin Method R_m of (4):

Let

$$U = V = 0 + 1 + 2 + \dots + 9 + 10 \tag{15}$$
$$A = 1/0 + 0.8/1 + 0.6/2 + 0.2/3 \tag{16}$$
$$B = 0.2/2 + 0.6/3 + 0.8/4 + 1/5 + 0.8/6 + 0.6/7 + 0.2/8 \tag{17}$$

Then, using R_m of (4) the fuzzy conditional proposition

If x is A then y is B

translates into the matrix form as

$$R_m = (A \times B) \cup (\neg A \times V)$$

$$= \begin{array}{c|ccccccccccc} & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\ \hline 0 & 0 & 0 & .2 & .6 & .8 & 1 & .8 & .6 & .2 & 0 & 0 \\ 1 & .2 & .2 & .2 & .6 & .8 & .8 & .8 & .6 & .2 & .2 & .2 \\ 2 & .4 & .4 & .4 & .6 & .6 & .6 & .6 & .6 & .4 & .4 & .4 \\ 3 & .8 & .8 & .8 & .8 & .8 & .8 & .8 & .8 & .8 & .8 & .8 \\ 4 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 5 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 6 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 7 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 8 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 9 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 10 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{array}$$

† The precise proofs of the satisfaction or failure of each method are omitted because of limitations of space.

Table II Relations between B'(Ant 2) and A'(Cons) under Ant 1 for the Generalized Modus Tollens in (2)

	B'	A'
Relation V (modus tollens)	not B	not A
Relation VI	not very B	not very A
Relation VII	not more or less B	not more or less A
Relation VIII-1	B	unknown
Relation VIII-2	B	A

Thus we can obtain the consequence B'_m in Cons of (1) using (11) and the consequence A'_m in Cons of (2) using (13) as follows when A' in Ant 2 of (1) is A, very A, more or less A or not A, and B' in Ant 2 of (2) is not B, not very B, not more or less B or B, where it is assumed that

$$\text{very } A = 1/0 + 0.64/1 + 0.36/2 + 0.04/3 \tag{18}$$
$$\text{more or less } A = 1/0 + .89/1 + .77/2 + .45/3 \tag{19}$$
$$\text{not } A = .2/1 + .4/2 + .8/3 + 1/(4 + 5 + \dots + 10) \tag{20}$$
$$\text{not very } A = \neg A^2 = .36/1 + .64/2 + .96/3 + 1/(4 + 5 + \dots + 10) \tag{21}$$
$$\text{not more or less } A = \neg A^{0.5} = .11/1 + .23/2 + .55/3 + 1/(4 + 5 + \dots + 10) \tag{22}$$
$$\text{very } B = .04/2 + .36/3 + .64/4 + 1/5 + .64/6 + .36/7 + .04/8 \tag{23}$$
$$\text{more or less } B = .45/2 + .77/3 + .89/4 + 1/5 + .89/6 + .77/7 + .45/8 \tag{24}$$
$$\text{not } B = 1/(0 + 1) + .8/2 + .4/3 + .2/4 + 0/5 + .2/6 + .4/7 + .8/8 + 1/(9 + 10) \tag{25}$$
$$\text{not very } B = 1/(0 + 1) + .96/2 + .64/3 + .36/4 + 0/5 + .36/6 + .64/7 + .96/8 + 1/(9 + 10) \tag{26}$$
$$\text{not more or less } B = 1/(0 + 1) + .55/2 + .23/3 + .11/4 + 0/5 + .11/6 + .23/7 + .55/8 + 1/(9 + 10) \tag{27}$$
$$\text{unknown } (= U, V) = 1/0 + 1/1 + \dots + 1/10 \tag{28}$$

Then the consequence B'_m will be obtained from (11) as

(i) $A \circ R_m$
$= .4/(0+1+2) + .6/3 + .8/4 + 1/5 + .8/6 + .6/7 + .4/(8+9+10)$
$\neq B.$

(ii) very $A \circ R_m$
$= .36/(0+1+2) + .6/3 + .8/4 + 1/5 + .8/6 + .6/7 + .36/(8+9+10)$
$\neq$ very B, B.

(iii) more or less $A \circ R_m$
$= .45/(0+1+2) + .6/3 + .8/4 + 1/5 + .8/6 + .6/7 + .45/(8+9+10)$
$\neq$ more or less B, B.

(iv) not $A \circ R_m$
$= 1/0 + 1/1 + \dots + 1/10$
= unknown.

As for the generalized modus tolle. in (2), A_m' is obtained from (13) as follows.

(v) R_m o not B
= .4/(0+1+2) + .8/3 + 1/(4+5+...+10)
≠ not A.

(vi) R_m o not very B
= .6/(0+1+2) + .8/3 + 1/(4+5+...+10)
≠ not very A.

(vii) R_m o not more or less B
= .23/(0+1) + .4/2 + .8/3 + 1/(4+5+...+10)
≠ not more or less A.

(viii) R_m o B
= 1/0 + .8/1 + .6/2 + .8/3 + 1/(4+5+...+10)
≠ unknown, A.

Hence it is found from this example that Relations except Relation IV-1 are not satisfied in the case of maximin method R_m.

[II] The Case of Arithmetic Method R_a of (5):
In the same way as shown in maximin method R_m of [I], we shall indicate that the Relations except Relations IV-1 and VIII-1 do not hold in the case of arithmetic method R_a of (5) by the use of the same example of (15)-(17).
Let A and B be the fuzzy sets given in (16) and (17), respectively, then the fuzzy conditional proposition translates into the fuzzy relation such as

R_a = (¬A x V) ⊕ (U x B)

	0	1	2	3	4	5	6	7	8	9	10
0	0	0	.2	.6	.8	1	.8	.6	.2	0	0
1	.2	.2	.4	.8	1	1	1	.8	.4	.2	.2
2	.4	.4	.6	1	1	1	1	1	.6	.4	.4
3	.8	.8	1	1	1	1	1	1	1	.8	.8
4	1	1	1	1	1	1	1	1	1	1	1
5	1	1	1	1	1	1	1	1	1	1	1
6	1	1	1	1	1	1	1	1	1	1	1
7	1	1	1	1	1	1	1	1	1	1	1
8	1	1	1	1	1	1	1	1	1	1	1
9	1	1	1	1	1	1	1	1	1	1	1
10	1	1	1	1	1	1	1	1	1	1	1

Hence the consequence B_a' will be obtained as follows using (12).

(i) A o R_a
= .4/(0+1) + .6/2 + .8/(3+4) + 1/5 + .8/(6+7) + .6/8 + .4/(9+10)
≠ B.

(ii) very A o R_a
= .36/(0+1) + .4/2 + .64/3 + .8/4 + 1/5 + .8/6 + .64/7 + .4/8 + .36/(9+10)
≠ very B, B.

(iii) more or less A o R_a
= .45/(0+1) + .6/2 + .8/3 + .89/4 + 1/5 + .89/6 + .8/7 + .6/8 + .45/(9+10)
≠ more or less B, B.

(iv) not A o R_a
= 1/0 + 1/1 + ... + 1/10
= unknown.

In the case of A_a' of (14), we have

(v) R_a o not B
= .4/(0+1) + .6/2 + .8/3 + 1/(4+5+...+10)
≠ not A.

(vi) R_a o not very B
= .6/0 + .64/(1+2) + .96/3 + 1/(4+5+...+10)
≠ not very A.

(vii) R_a o not more or less B
= .23/0 + .4/1 + .55/2 + .8/3 + 1/(4+5+...+10)
≠ not more or less A.

(viii) R_a o B
= 1/0 + 1/1 + ... + 1/10
= unknown.

Thus, this example shows that Relations IV-1 and VIII-1 are satisfied in the case of arithmetic method R_a.

[III] The Case of Mini Method R_c of (6):
Let A and B be fuzzy sets in (16) and (17), respectively, then R_c of (6) becomes

R_c = A x B

	0	1	2	3	4	5	6	7	8	9	10
0	0	0	.2	.6	.8	1	.8	.6	.2	0	0
1	0	0	.2	.6	.8	.8	.8	.6	.2	0	0
2	0	0	.2	.6	.6	.6	.6	.6	.2	0	0
3	0	0	.2	.2	.2	.2	.2	.2	.2	0	0
4	0	0	0	0	0	0	0	0	0	0	0
5	0	0	0	0	0	0	0	0	0	0	0
6	0	0	0	0	0	0	0	0	0	0	0
7	0	0	0	0	0	0	0	0	0	0	0
8	0	0	0	0	0	0	0	0	0	0	0
9	0	0	0	0	0	0	0	0	0	0	0
10	0	0	0	0	0	0	0	0	0	0	0

Then we can obtain the following:

(i) A o R_c
= .2/2 + .6/3 + .8/4 + 1/5 + .8/6 + .6/7 + .2/8
= B.

(ii) very A o R_c
= .2/2 + .6/3 + .8/4 + 1/5 + .8/6 + .6/7 + .2/8
= B.

(iii) more or less A o R_c
= .2/2 + .6/3 + .8/4 + 1/5 + .8/6 + .6/7 + .2/8
= B.

(vi) not A o R_c
= .2/2 + .4/(3+4+...+7) + .2/8
≠ unknown, not B.

(v) R_c o not B
= .4/(0+1+2) + .2/3
≠ not A.

(vi) R_c o not very B
=.6/(0+1+2) + .2/3
≠ not very A.

(vii) R_c = not more or less B
= .23/(0+1+2) + .2/3
≠ not more or less A.

(viii) R_c o B
= 1/0 + .8/1 + .6/2 + .2/3
= A.

This shows that Relations I, II-2, III-2 and VIII-2 are satisfied in the case of min method Rc.

[IV] The Case of R_s of (7):
When A and B are fuzzy sets as in (16) and (17), R_s is derived from (7) as follows

$R_s = A \times V \xrightarrow{s} U \times B$

	0	1	2	3	4	5	6	7	8	9	10
0	0	0	0	0	0	1	0	0	0	0	0
1	0	0	0	0	1	1	1	0	0	0	0
2	0	0	0	1	1	1	1	1	0	0	0
3	0	0	1	1	1	1	1	1	1	0	0
4	1	1	1	1	1	1	1	1	1	1	1
5	1	1	1	1	1	1	1	1	1	1	1
6	1	1	1	1	1	1	1	1	1	1	1
7	1	1	1	1	1	1	1	1	1	1	1
8	1	1	1	1	1	1	1	1	1	1	1
9	1	1	1	1	1	1	1	1	1	1	1
10	1	1	1	1	1	1	1	1	1	1	1

Then we have

(i) A o R_s
= .2/2 + .6/3 + .8/4 + 1/5 + .8/6 + .6/7 + .2/8
= B.

(ii) very A o R_s
= .04/2 + .36/3 + .64/4 + 1/5 + .64/6 + .36/7 + .04/8
= very B.

(iii) more or less A o R_s
= .45/2 + .77/3 + .89/4 + 1/5 + .89/6 + .77/7 + .45/8
= more or less B.

(iv) not A o R_s
= 1/0 + 1/1 + ... + 1/10
= unknown.

(v) R_s o not B
= .2/1 + .4/2 + .8/3 + 1/(4+5+...+10)
= not A.

(vi) R_s o not very B
= .36/1 + .64/2 + .96/3 + 1/(4+5+...+10)
= not very A.

(vii) R_s o not more or less B
= .11/1 + .23/2 + .55/3 + 1/(4+5+...+10)
= not more or less A.

(viii) R_s o B
= 1/0 + 1/1 + ... + 1/10
= unknown.

Thus, this example shows that the method R_s satisfies † Relations I, II-1, III-1, IV-1, V, VI, VII and VIII-1 in Tables I and II.

[V] The Case of R_g of (8):

R_g is obtained from (8), with A and B being the same as (16) and (17).

$$R_g = A \times V \xrightarrow[g]{} U \times B$$

	0	1	2	3	4	5	6	7	8	9	10
0	0	0	.2	.6	.8	1	.8	.6	.2	0	0
1	0	0	.2	.6	1	1	1	.6	.2	0	0
2	0	0	.2	1	1	1	1	1	.2	0	0
3	0	0	1	1	1	1	1	1	1	0	0
4	1	1	1	1	1	1	1	1	1	1	1
5	1	1	1	1	1	1	1	1	1	1	1
6	1	1	1	1	1	1	1	1	1	1	1
7	1	1	1	1	1	1	1	1	1	1	1
8	1	1	1	1	1	1	1	1	1	1	1
9	1	1	1	1	1	1	1	1	1	1	1
10	1	1	1	1	1	1	1	1	1	1	1

Then

(i) A o R_g
= .2/2 + .6/3 + .8/4 + 1/5 + .8/6 + .6/7 + .2/8
= B.

(ii) very A o R_g
= .2/2 + .6/3 + .8/4 + 1/5 + .8/6 + .6/7 + .2/8
= B.

(iii) more or less A o Rg
= .45/2 + .77/3 + .89/4 + 1/5 + .89/6 + .77/7 + .45/8
= more or less B.

(iv) not A o g
= 1/0 + 1/1 + ... + 1/10
= unknown.

(v) Rg o not B
= .4/(0+1+2) + .8/3 + 1/(4+5+...+10)
≠ not A.

(vi) Rg o not very B
= .6/(0+1) + .64/2 + .96/3 + 1/(4+5+...+10)
≠ not very A.

(vii) Rg o not more or less B
= .23/(0+1+2) + .55/3 + 1/(4+5+...+10)
≠ not more or less A.

(viii) Rg o B
= 1/0 + 1/1 + ... + 1/10
= unknown.

Thus it is found that Rg satisfies Relations I, II-2, III-1, IV-1 and VIII-1.

[VI] The Case of Rsg of (9):

$$R_{sg} = (A \times V \xrightarrow[s]{} U \times B) \cap (7A \times V \xrightarrow[g]{} U \times 7B)$$

	0	1	2	3	4	5	6	7	8	9	10
0	0	0	0	0	0	1	0	0	0	0	0
1	0	0	0	0	1	0	1	0	0	0	0
2	0	0	0	1	.2	0	.2	1	0	0	0
3	0	0	1	.4	.2	0	.2	.4	1	0	0
4	1	1	.8	.4	.2	0	.2	.4	.8	1	1
5	1	1	.8	.4	.2	0	.2	.4	.8	1	1
6	1	1	.8	.4	.2	0	.2	.4	.8	1	1
7	1	1	.8	.4	.2	0	.2	.4	.8	1	1
8	1	1	.8	.4	.2	0	.2	.4	.8	1	1
9	1	1	.8	.4	.2	0	.2	.4	.8	1	1
10	1	1	.8	.4	.2	0	.2	.4	.8	1	1

(i) A o Rsg = B.
(ii) very A o Rsg = very B.
(iii) more or less A = more or less B.
(iv) not A o Rsg = not B.
(v) Rsg o not B = not A.
(vi) Rsg o not very B = not very A.
(vii) Rsg o not more or less B = not more or less A.
(viii) Rsg o B
= 1/0 + .8/1 + .6/2 + .4/(3+4+...+10)
≠ A, unknown.

Thus we find that Relations except Relation VIII are satisfied by the method Rsg.

[VII] The Case of Rgg of (10):

$$Rgg = (A \times V \xrightarrow[g]{} U \times B) \cap (7A \times V \xrightarrow[g]{} U \times 7B)$$

	0	1	2	3	4	5	6	7	8	9	10
0	0	0	.2	.6	.8	1	.8	.6	.2	0	0
1	0	0	.2	.6	1	0	1	.6	.2	0	0
2	0	0	.2	1	.2	0	.2	1	.2	0	0
3	0	0	1	.4	.2	0	.2	.4	1	0	0
4	1	1	.8	.4	.2	0	.2	.4	.8	1	1
5	1	1	.8	.4	.2	0	.2	.4	.8	1	1
6	1	1	.8	.4	.2	0	.2	.4	.8	1	1
7	1	1	.8	.4	.2	0	.2	.4	.8	1	1
8	1	1	.8	.4	.2	0	.2	.4	.8	1	1
9	1	1	.8	.4	.2	0	.2	.4	.8	1	1
10	1	1	.8	.4	.2	0	.2	.4	.8	1	1

(i) A o Rgg = B.
(ii) very A o Rgg = B.
(iii) more or less A o Rgg = more or less B.
(iv) not A o Rgg = not B.
(v) Rgg o not B
= .4/(0+1+2) + .8/3 + 1/(4+5+...+10)
≠ not A.

---

† It is assumed in the method R_s that fuzzy sets A and B in (3) satisfy the conditions in the discussion of Relations I-IV:

(i) $\{\mu_A(u) \mid u \in U\} \geq \{\mu_B(v) \mid v \in V\}$,

(ii) $\exists u \in U \;\; \mu_A(u) = 0; \exists u' \in U \;\; \mu_A(u') = 1$,

(iii) $\exists v \in V \;\; \mu_B(v) = 0; \exists v' \in V \;\; \mu_B(v') = 1$.

But in the discussion of Relations V-VIII, we use the condition (i') instead of (i).

(i') $\{\mu_A(u) \mid u \in U\} \subseteq \{\mu_B(v) \mid v \in V\}$.

The same holds for the methods R_g, R_{sg} and R_{gg} discussed later. Thus, if we introduce the following condition (i") satisfying both (i) and (i'), the R_s can satisfy all the Relations I, II-1, III-1, IV-1, V, VI, VII and VIII-1 at the same time. The fuzzy sets A and B in (16) and (17) are shown to satisfy the conditions (i"), (ii) and (iii). Thus, it follows that we have discussed the methods R_m, R_a and R_c by Zadeh and Mamdani under the same coditions.

(i") $\{\mu_A(u) \mid u \in U\} = \{\mu_B(v) \mid v \in V\}$.

Table III Satisfacti of Each Relation in Tables I and II der Each Method

	Ant 2	Cons	R_m	R_a	R_c	R_s	R_g	R_{sg}	R_{gg}
Relation I (modus ponens)	A'	B	X	X	O	O	O	O	O
Relation II-1	very A	very B	X	X	X	O	X	O	X
Relation II-2	very A	B	X	X	O	X	O	X	O
Relation III-1	more or less A	more or less B	X	X	X	O	O	O	O
Relation III-2	more or less A	B	X	X	O	X	X	X	X
Relation IV-1	not A	unknown	O	O	X	O	O	X	X
Relation IV-2	not A	not B	X	X	X	X	X	O	O
Relation V (modus tollens)	not B	not A	X	X	X	O	X	O	X
Relation VI	not very B	not very A	X	X	X	O	X	O	X
Relation VII	not more or less B	not more or less A	X	X	X	O	X	O	X
Relation VIII-1	B	unknown	X	O	X	O	O	X	X
Relation VIII-2	B	A	X	X	O	X	X	X	X

(vi) Rgg o not very B
$= .6/(0+1) + .64/2 + .96/3 + 1/(4+5+...+10)$
$\neq$ not very A.

(vii) Rgg o not more or less B
$= .23/(0+1+2) + .55/3 + 1/(4+5+...+10)$
$\neq$ not more or less A.

(viii) Rgg o B
$= 1/0 + .8/1 + .6/2 + .4/(3+4+...+10)$
$\neq$ A, unknown.

Thus, Rgg satisfies Relations I, II-2, III-1 and IV-2.

The satisfaction (O) or failure (X) of each Relation in Tables I and II under each method is summarized in Table III.

3. PROPERTIES OF EACH METHOD

In this section we shall discuss some interesting properties (syllogism and contrapositive) under each method for fuzzy conditional inference.

Let fuzzy conditional propositions be given as

$$\begin{aligned} P_1 &: \text{ If } x \text{ is } A \text{ then } y \text{ is } B, \\ P_2 &: \text{ If } y \text{ is } B \text{ then } z \text{ is } C, \qquad (29) \\ P_3 &: \text{ If } x \text{ is } A \text{ then } z \text{ is } C, \end{aligned}$$

where A, B and C are fuzzy sets in U, V and W, respectively.

[I] The Case of Maximin Method R_m:

Let

$$R_m(A, B) = (A \times B) \cup (\neg A \times V),$$
$$R_m(B, C) = (B \times C) \cup (\neg B \times W),$$
$$R_m(A, C) = (A \times C) \cup (\neg A \times W),$$

be fuzzy relations which are translated, respectively, from the propositions P_1, P_2 and P_3 using (4). Then the syllogism does not hold. Namely,

$$R_m(A, C) \neq R_m(A, B) \circ R_m(B, C).$$

For example, let fuzzy sets A, B and C be given as

$$A = 1/1 + .8/2 + .6/3 + .4/4 + .2/5, \qquad (30)$$
$$B = .2/4 + .4/5 + .8/6 + 1/7, \qquad (31)$$
$$C = .4/2 + .8/3 + 1/4 + .8/5 + .2/6, \qquad (32)$$

with $U = V = W = 1 + 2 + 3 + ... + 6 + 7$. Then we have

$$R_m(A, B) \circ R_m(B, C)$$

$$= \begin{array}{c} \\ 1\\2\\3\\4\\5\\6\\7 \end{array} \begin{array}{c} \begin{array}{ccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 \end{array} \\ \begin{pmatrix} 0 & 0 & 0 & .2 & .4 & .8 & 1 \\ .2 & .2 & .2 & .2 & .4 & .8 & .8 \\ .4 & .4 & .4 & .4 & .4 & .6 & .6 \\ .6 & .6 & .6 & .6 & .6 & .6 & .6 \\ .8 & .8 & .8 & .8 & .8 & .8 & .8 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix} \end{array} \circ \begin{array}{c} \\ 1\\2\\3\\4\\5\\6\\7 \end{array} \begin{array}{c} \begin{array}{ccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 \end{array} \\ \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ .8 & .8 & .8 & .8 & .8 & .8 & .8 \\ .6 & .6 & .6 & .6 & .6 & .6 & .6 \\ .2 & .4 & .8 & .8 & .8 & .2 & .2 \\ 0 & .4 & .8 & 1 & .8 & .2 & 0 \end{pmatrix} \end{array}$$

$$= \begin{array}{c} \\ 1\\2\\3\\4\\5\\6\\7 \end{array} \begin{array}{c} \begin{array}{ccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 \end{array} \\ \begin{pmatrix} .4 & .4 & .8 & 1 & .8 & .4 & .4 \\ .4 & .4 & .8 & .8 & .8 & .4 & .4 \\ .4 & .4 & .6 & .6 & .6 & .4 & .4 \\ .6 & .6 & .6 & .6 & .6 & .6 & .6 \\ .8 & .8 & .8 & .8 & .8 & .8 & .8 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix} \end{array} \neq \begin{array}{c} \\ 1\\2\\3\\4\\5\\6\\7 \end{array} \begin{array}{c} \begin{array}{ccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 \end{array} \\ \begin{pmatrix} 0 & .4 & .8 & 1 & .8 & .2 & 0 \\ .2 & .4 & .8 & .8 & .8 & .2 & .2 \\ .4 & .4 & .6 & .6 & .6 & .4 & .4 \\ .6 & .6 & .6 & .6 & .6 & .6 & .6 \\ .8 & .8 & .8 & .8 & .8 & .8 & .8 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix} \end{array}$$

$(= R_m(A, C))$.

Thus, $R_m(A, C) \neq R_m(A, B) \circ R_m(B, C)$.

[II] The Case of Arithmetic Method R_a:

Using the same fuzzy sets A, B and C in (30)-(32), we have

$$R_a(A, B) \circ R_a(B, C)$$

$$= \begin{pmatrix} 0 & 0 & 0 & .2 & .4 & .8 & 1 \\ .2 & .2 & .2 & .4 & .6 & 1 & 1 \\ .4 & .4 & .4 & .6 & .8 & 1 & 1 \\ .6 & .6 & .6 & .8 & 1 & 1 & 1 \\ .8 & .8 & .8 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix} \circ \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ .8 & 1 & 1 & 1 & 1 & 1 & .8 \\ .6 & 1 & 1 & 1 & 1 & .8 & .6 \\ .2 & .6 & 1 & 1 & 1 & .4 & .2 \\ 0 & .4 & .8 & 1 & .8 & .2 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} .4 & .6 & .8 & 1 & .8 & .4 & .4 \\ .6 & .6 & 1 & 1 & 1 & .6 & .6 \\ .6 & .8 & 1 & 1 & 1 & .8 & .6 \\ .8 & 1 & 1 & 1 & 1 & .8 & .8 \\ .8 & 1 & 1 & 1 & 1 & 1 & .8 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix} \neq \begin{pmatrix} 0 & .4 & .8 & 1 & .8 & .2 & 0 \\ .2 & .6 & 1 & 1 & 1 & .4 & .2 \\ .4 & .8 & 1 & 1 & 1 & .6 & .4 \\ .6 & 1 & 1 & 1 & 1 & .8 & .6 \\ .8 & 1 & 1 & 1 & 1 & 1 & .8 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix}$$

(= Ra(A, C)).

[III] The Case of Min Method Rc:

Rc(A, B) o Rc(B, C)

$$= \begin{pmatrix} 0 & 0 & 0 & .2 & .4 & .8 & 1 \\ 0 & 0 & 0 & .2 & .4 & .8 & .8 \\ 0 & 0 & 0 & .2 & .4 & .6 & .6 \\ 0 & 0 & 0 & .2 & .4 & .4 & .4 \\ 0 & 0 & 0 & .2 & .2 & .2 & .2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \circ \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & .2 & .2 & .2 & .2 & .2 & 0 \\ 0 & .4 & .4 & .4 & .4 & .2 & 0 \\ 0 & .4 & .8 & .8 & .8 & .2 & 0 \\ 0 & .4 & .8 & 1 & .8 & .2 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & .4 & .8 & 1 & .8 & .2 & 0 \\ 0 & .4 & .8 & .8 & .8 & .2 & 0 \\ 0 & .4 & .6 & .6 & .6 & .2 & 0 \\ 0 & .4 & .4 & .4 & .4 & .2 & 0 \\ 0 & .2 & .2 & .2 & .2 & .2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} = Rc(A, C).$$

[IV] The Case of Rs:†

Rs(A, B) o Rs(B, C)

$$= \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix} \circ \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix} = Rs(A, C).$$

[V] The Case of Rg:

Rg(A, B) o Rg(B, C)

$$= \begin{pmatrix} 0 & 0 & 0 & .2 & .4 & .8 & 1 \\ 0 & 0 & 0 & .2 & .4 & 1 & 1 \\ 0 & 0 & 0 & .2 & .4 & 1 & 1 \\ 0 & 0 & 0 & .2 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix} \circ \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 & 1 & .2 & 0 \\ 0 & .4 & 1 & 1 & 1 & .2 & 0 \\ 0 & .4 & .8 & 1 & .8 & .2 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & .4 & .8 & 1 & .8 & .2 & 0 \\ 0 & .4 & 1 & 1 & 1 & .2 & 0 \\ 0 & .4 & 1 & 1 & 1 & .2 & 0 \\ 0 & 1 & 1 & 1 & 1 & .2 & 0 \\ 0 & 1 & 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix} = Rg(A, C).$$

† It is assumed in the case of Rs, Rg, Rsg and Rgg that fuzzy sets A, B and C in (29) satisfy the following conditions:

$$\{\mu_A(u) \mid u \in U\} \geq \{\mu_B(v) \mid v \in V\} \geq \{\mu_C(w) \mid w \in W\}.$$

$$u \in U \;\; \mu_A(u) = 0; \qquad u' \in U \;\; \mu_A(u') = 1.$$

$$v \in V \;\; \mu_B(v) = 0; \qquad v' \in V \;\; \mu_B(v') = 1.$$

$$w \in W \;\; \mu_C(w) = 0; \qquad w' \in W \;\; \mu_C(w') = 1.$$

Note that fuzzy sets in (30)-(32) satisfy these conditions.

Table IV Satisfaction of Syllogism and Contrapositive

	Rm	Ra	Rc	Rs	Rg	Rsg	Rgg
Syllogism	X	X	O	O	O	O	O
Contrapositive	X	O	X	O	X	X	X

[VI] The Case of Rsg:

Rsg(A, B) o Rsg(B, C)

$$= \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & .2 & 0 \\ 0 & 0 & 0 & 0 & 1 & .2 & 0 \\ 0 & 0 & 0 & 1 & .6 & .2 & 0 \\ 1 & 1 & 1 & .8 & .6 & .2 & 0 \\ 1 & 1 & 1 & .8 & .6 & .2 & 0 \end{pmatrix} \circ \begin{pmatrix} 1 & .6 & .2 & 0 & .2 & .8 & 1 \\ 1 & .6 & .2 & 0 & .2 & .8 & 1 \\ 1 & .6 & .2 & 0 & .2 & .8 & 1 \\ 0 & .6 & .2 & 0 & .2 & 1 & 0 \\ 0 & 1 & .2 & 0 & .2 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix},$$

$$= \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & .2 & 0 & .2 & 0 & 0 \\ 0 & 1 & .2 & 0 & .2 & 0 & 0 \\ 0 & .6 & .2 & 0 & .2 & 1 & 0 \\ 1 & .6 & .2 & 0 & .2 & .8 & 1 \\ 1 & .6 & .2 & 0 & .2 & .8 & 1 \end{pmatrix} = Rsg(A, C).$$

[VII] The Case of Rgg:

Rgg(A, B) o Rgg(B, C)

$$= \begin{pmatrix} 0 & 0 & 0 & .2 & .4 & .8 & 1 \\ 0 & 0 & 0 & .2 & .4 & 1 & 0 \\ 0 & 0 & 0 & .2 & .4 & .2 & 0 \\ 0 & 0 & 0 & .2 & 1 & .2 & 0 \\ 0 & 0 & 0 & 1 & .6 & .2 & 0 \\ 1 & 1 & 1 & .8 & .6 & .2 & 0 \\ 1 & 1 & 1 & .8 & .6 & .2 & 0 \end{pmatrix} \circ \begin{pmatrix} 1 & .6 & .2 & 0 & .2 & .8 & 1 \\ 1 & .6 & .2 & 0 & .2 & .8 & 1 \\ 1 & .6 & .2 & 0 & .2 & .8 & 1 \\ 0 & .6 & .2 & 0 & .2 & 1 & 0 \\ 0 & 1 & .2 & 0 & .2 & .2 & 0 \\ 0 & .4 & 1 & 0 & 1 & .2 & 0 \\ 0 & .4 & .8 & 1 & .8 & .2 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & .4 & .8 & 1 & .8 & .2 & 0 \\ 0 & .4 & 1 & 0 & 1 & .2 & 0 \\ 0 & .4 & .2 & 0 & .2 & .2 & 0 \\ 0 & 1 & .2 & 0 & .2 & .2 & 0 \\ 0 & .6 & .2 & 0 & .2 & 1 & 0 \\ 1 & .6 & .2 & 0 & .2 & .8 & 1 \\ 1 & .6 & .2 & 0 & .2 & .8 & 1 \end{pmatrix} = Rgg(A, C).$$

The satisfaction of syllogism under each method is listed in Table IV.

Finally, we shall investigate the contrapositive of fuzzy conditional proposition under each method.

For a fuzzy conditional proposition P_1:

P_1: If x is A then y is B

and its contrapositive proposition P_2:

P_2: If y is not B then x is not A

we have the following equalities in which Ra(A, B) and Ra(7B, 7A) are obtained from P_1 and P_2, respectively, using (4), and $\tilde{R}a(A, B)$ denotes the converse of Ra(A, B).

$$R_a(7B, 7A) = \tilde{R}_a(A, B),$$

$$R_s(7B, 7A) = \tilde{R}_s(A, B).$$

The other methods can not satisfy the contrapositive, which is shown in Table IV.

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