ADVANCES IN FUZZY SET THEORY AND APPLICATIONS
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SOME PROPERTIES OF FUZZY NUMBERS

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A fuzzy number is a fuzzy set in the real line and its operations of +, -, x and ÷ can be defined by using the extension principle. This paper investigates the algebraic properties of fuzzy numbers under the four arithmetic operations of +, -, x and ÷. Furthermore, the ordering of fuzzy numbers is introduced and some properties of fuzzy numbers under join ($\sqcup$) and meet ($\sqcap$) are discussed.

INTRODUCTION

Recently, L.A. Zadeh proposed the interesting concept of the extension principle by which a binary operation defined on a set X may be extended to fuzzy sets in X, and defined the operations for fuzzy sets of type 2 [2, 3] and fuzzy numbers [1, 3].

In this paper we discuss the algebraic properties of fuzzy numbers, which are fuzzy sets in the real line, under the four arithmetic operations, namely, +, -, x and ÷ which are defined by the extension principle [3]. First, as for the convexity of fuzzy numbers, the fuzzy numbers obtained by applying the operations of +, - and x to convex fuzzy numbers are also convex fuzzy numbers, though the convexity can not be preserved in general if ÷ is applied to convex fuzzy numbers. Second, the convex fuzzy numbers do not form such algebraic structures as a ring and a field, since the distributive law is not satisfied and there exist no inverse fuzzy numbers under + and x. Third, the positive convex fuzzy numbers defined over the positive real line, however, satisfy the distributive law and hence form a commutative semiring. And fourth, the ordering of fuzzy numbers is introduced and the properties of fuzzy numbers under the join and the meet combined with the four arithmetic operations are investigated.

FUZZY NUMBERS

We shall briefly review some of the basic definitions relating to fuzzy numbers and their operations of +, -, x and ÷.

Fuzzy Numbers: A fuzzy number A in the real line R is a fuzzy set characterized by a membership function μ_A as

$$\mu_A : R \longrightarrow [0, 1]. \tag{1}$$

A fuzzy number A is expressed as

$$A = \int_{x \in R} \mu_A(x)/x , \tag{2}$$

with the understanding that $\mu_A(x) \in [0,1]$ represents the grade of membership of x in A and $\int$ denotes the union of $\mu_A(x)/x$'s.

(Example 1) A fuzzy number $\underset{\sim}{2}$ which denotes "about 2" will be given as

$$\underset{\sim}{2} = \int_1^2 x - 1/x + \int_2^3 3 - x/x \tag{3}$$

and can be illustrated by the dotted line in Fig. 2, where + stands for the union.

Convex Fuzzy Numbers: A fuzzy number A in R is said to be convex if for any real numbers x, y, z ε R with $x \le y \le z$,

$$\mu_A(y) \ge \mu_A(x) \wedge \mu_A(z) \tag{4}$$

with $\wedge$ standing for min. A fuzzy number A is called normal if the following holds.

$$\max_x \mu_A(x) = 1. \tag{5}$$

A fuzzy number which is normal and convex is referred to as a normal convex fuzzy number.

(Example 2) Fuzzy numbers as shown in Fig. 2 are all normal convex fuzzy numbers. Fig.1 gives various kinds of fuzzy numbers, in which it is noted that the fuzzy number A_2 is not convex because the support (defined in (9)) of A_2 is discrete, that is, A_2 does not satisfy (4).

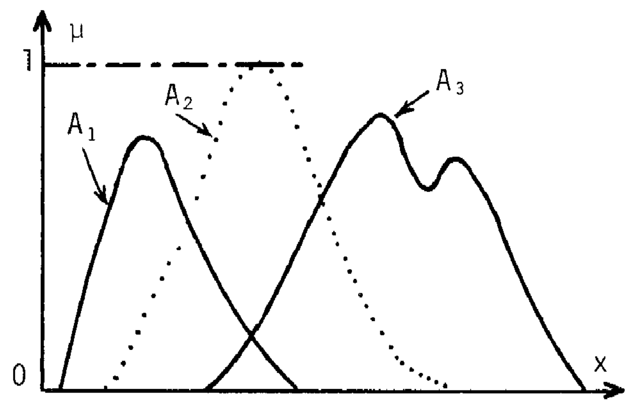


Fig.1 Various Kinds of Fuzzy Numbers (A_1: convex, A_2: normal non-convex, A_3: non-convex).

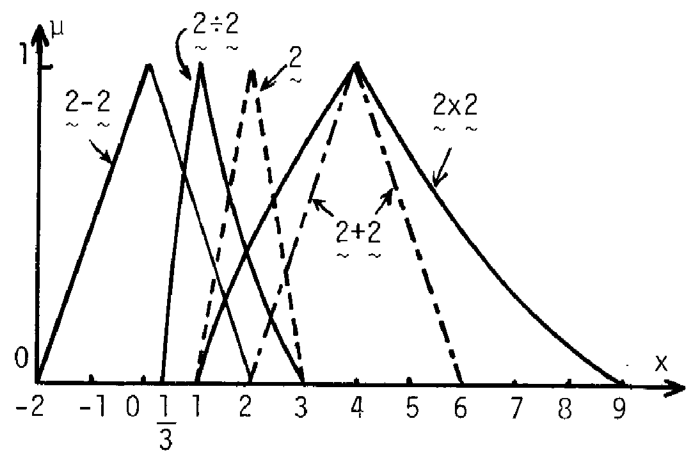


Fig.2 Fuzzy Numbers $\tilde{2}$, $\tilde{2}\tilde{+}\tilde{2}$, $\tilde{2}\tilde{-}\tilde{2}$, $\tilde{2}\tilde{\times}\tilde{2}$, $\tilde{2}\tilde{\div}\tilde{2}$.

Level Sets: An α-level set of a fuzzy number A is a non-fuzzy set denoted by A_α and is defined by

$$A_\alpha = \{ x \mid \mu_A(x) \ge \alpha \}, \quad 0 < \alpha \le 1 . \tag{6}$$

It is easy to show that

$$\alpha_1 \le \alpha_2 \implies A_{\alpha_1} \supseteq A_{\alpha_2} . \tag{7}$$

If two fuzzy numbers A and B are equal, that is, $\mu_A(x) = \mu_B(x)$ for all x ε R, then we can obtain $A_\alpha = B_\alpha$ for any α, and vice versa. Let fuzzy number A be convex, A_α is a convex set (or an interval) in R, and vice versa. A fuzzy number A may be decomposed into its level sets through the resolution identity [3].

$$A = \int_0^1 \alpha A_\alpha , \tag{8}$$

where αA_α is the product of a scalar α with the set A_α and $\int$ is the union of A_α's, with α ranging from 0 to 1.

Support: The support Γ_A of a fuzzy number A is defined, as a special case of level set, by the following.

$$\Gamma_A = \{ x \mid \mu_A(x) > 0 \} . \tag{9}$$

Extension Principle: Let A and B be fuzzy numbers in R and let * be a binary operation defined in R. Then the operation * can be extended to the fuzzy numbers A and B by the defining relation (the extension principle).

$$A * B = \int_{x,y \varepsilon R} (\mu_A(x) \wedge \mu_B(y)) \,/\, (x * y), \tag{10}$$

where $\wedge$ stands for min.

In (10) let the binary operation * be replaced by the ordinary four arithmetic operations of +, -, x and ÷, then the four arithmetic operations over fuzzy numbers are defined by the following.

Four Arithmetic Operations for Fuzzy Numbers: Let A and B be fuzzy numbers in R, we have from (10)

$$A + B = \int (\mu_A(x) \wedge \mu_B(y)) \,/\, (x + y) \,, \tag{11}$$

$$A - B = \int (\mu_A(x) \wedge \mu_B(y)) \,/\, (x - y) \,, \tag{12}$$

$$A \times B = \int (\mu_A(x) \wedge \mu_B(y)) \,/\, (x \times y) \,, \tag{13}$$

$$A \div B = \int (\mu_A(x) \wedge \mu_B(y)) \,/\, (x \div y) \,. \tag{14}$$

The membership functions of these fuzzy numbers are obtained by

$$\mu_{A+B}(a) = \bigvee_{x+y=a} (\mu_A(x) \wedge \mu_B(y))$$

$$= \bigvee_{x} (\mu_A(x) \wedge \mu_B(a - x)) \,, \tag{15}$$

$$\mu_{A-B}(a) = \bigvee_{x} (\mu_A(x) \wedge \mu_B(x - a)) \,, \tag{16}$$

$$\mu_{A\times B}(a) = \bigvee_{x(\neq 0)} (\mu_A(x) \wedge \mu_B(\tfrac{a}{x})) \,, \tag{17}$$

$$\mu_{A\div B}(a) = \bigvee_{x} (\mu_A(x) \wedge \mu_B(\tfrac{x}{a})) \tag{18}$$

$$= \bigvee_{y} (\mu_A(ay) \wedge \mu_B(y)) \,. \tag{19}$$

Although these definitions are useful for any fuzzy numbers, it will be more convenient to convex fuzzy numbers to use the concept of α-level sets of fuzzy numbers.

Let A_α and B_α be α-level sets of convex fuzzy numbers A and B, respectively, then the α-level sets are intervals in R, which are special convex fuzzy numbers whose grades are unity at x belonging to A_α and zero elsewhere. Let the α-level set of, say, the sum A + B of A and B be denoted by $(A + B)_\alpha$, we can obtain

$$(A + B)_\alpha = A_\alpha + B_\alpha . \tag{20}$$

In other words, the α-level set $(A + B)_\alpha$ is the sum of the α-level sets A_α and B_α. Thus, using the resolution identity (8), we can express A + B as

$$A + B = \int_0^1 \alpha(A + B)_\alpha = \int_0^1 \alpha(A_\alpha + B_\alpha) \,. \tag{21}$$

In a similar fashion, we can obtain A - B, A x B and A ÷ B as follows.

$$A - B = \int_0^1 \alpha(A_\alpha - B_\alpha) \,, \tag{22}$$

$$A \times B = \int_0^1 \alpha(A_\alpha \times B_\alpha) \,, \tag{23}$$

$$A \div B = \int_0^1 \alpha(A_\alpha \div B_\alpha) \,. \tag{24}$$

(Example 3) For the convex fuzzy number $\underset{\sim}{2}$ given by (3), the fuzzy numbers $\underset{\sim}{2} + \underset{\sim}{2}$, $\underset{\sim}{2} - \underset{\sim}{2}$, $\underset{\sim}{2} \times \underset{\sim}{2}$ and $\underset{\sim}{2} \div \underset{\sim}{2}$ are depicted in Fig.2 and are expressed as

$$\underset{\sim}{2} + \underset{\sim}{2} = \int_{2}^{4} \frac{x}{2} - 1/x \; + \int_{4}^{6} 3 - \frac{x}{2}/x \;, \tag{25}$$

$$\underset{\sim}{2} - \underset{\sim}{2} = \int_{-2}^{0} \frac{x}{2} + 1/x \; + \int_{0}^{2} 1 - \frac{x}{2}/x \;, \tag{26}$$

$$\underset{\sim}{2} \times \underset{\sim}{2} = \int_{1}^{4} \sqrt{x} - 1/x \; + \int_{4}^{9} 3 - \sqrt{x}/x \;, \tag{27}$$

$$\underset{\sim}{2} \div \underset{\sim}{2} = \int_{\frac{1}{3}}^{1} 3 - \frac{4}{x+1}/x \; + \int_{1}^{3} \frac{4}{x+1} - 1/x \;. \tag{28}$$

ALGEBRAIC PROPERTIES OF FUZZY NUMBERS

This section discusses the algebraic properties of fuzzy numbers under the operations of +, -, x and ÷. We shall begin with the convexity of fuzzy numbers under these operations.

[Theorem 1] If A and B are convex fuzzy numbers in the real line R, then A + B, A - B and A x B are also convex fuzzy numbers.

Proof: In general, let M_1, M_2, N_1 and N_2 be intervals in R and let $M_1 \subseteq M_2$ and $N_1 \subseteq N_2$, then we can obtain that

$$M_1 + N_1 \subseteq M_2 + N_2 \;;\quad M_1 \times N_1 \subseteq M_2 \times N_2$$

and that $M_i + N_i$ and $M_i \times N_i$ (i = 1, 2) also become intervals in R. For each $0 < \alpha \leq 1$, the α-level sets A_α and B_α of convex fuzzy numbers A and B are convex sets (or intervals) in R. Thus, for any α_1 and α_2 with $0 < \alpha_1 \leq \alpha_2 \leq 1$, the relations $A_{\alpha_2} \subseteq A_{\alpha_1}$ and $B_{\alpha_2} \subseteq B_{\alpha_1}$ are derived from (7) and hence $A_{\alpha_2} + B_{\alpha_2} \subseteq A_{\alpha_1} + B_{\alpha_1}$ and $A_{\alpha_2} \times B_{\alpha_2} \subseteq A_{\alpha_1} \times B_{\alpha_1}$ are obtained, which leads to $(A + B)_{\alpha_2} \subseteq (A + B)_{\alpha_1}$ and $(A \times B)_{\alpha_2} \subseteq (A \times B)_{\alpha_1}$. Furthermore, $(A + B)_{\alpha_i}$ and $(A \times B)_{\alpha_i}$ are intervals (or convex sets) for each α_i (i = 1, 2). Thus, fuzzy numbers A + B and A x B are shown to be convex fuzzy numbers.

Next, we shall prove the convexity of A - B. Let - B be defined by 0 - B, then the membership function of - B will be expressed as

$$\mu_{-B}(x) = \mu_B(-x), \quad x \in R \tag{29}$$

and - B can be easily shown to be convex if B is convex. Thus, A - B is proved to be convex since A - B is represented as A + (-B). Q.E.D.

It should be noted that for discrete fuzzy numbers, the convexity of A + B, A - B and A x B does not hold in general even if A and B are in the shape of "convex" like A_2 in Fig.1.

In order to discuss the convexity of fuzzy numbers under ÷, we shall define a special fuzzy number called positive, negative or zero fuzzy number.

Positive Fuzzy Numbers: A fuzzy number A is said to be positive if $0 < a_1 \leq a_2$ holds for the support $\Gamma_A = [a_1, a_2]$ of A, that is, Γ_A is in the positive real line. Similarly, A is called negative if $a_1 \leq a_2 < 0$ and zero if $a_1 \leq 0 \leq a_2$.

(Example 4) Fig.2 shows that the fuzzy number $\underset{\sim}{2} - \underset{\sim}{2}$ is a zero fuzzy number and the other fuzzy numbers are all positive.

[Lemma 2] If B is a zero convex fuzzy number, then $\frac{1}{B}$ (= 1 ÷ B) is not a convex fuzzy number.

Proof: The fuzzy number $\frac{1}{B}$ will be defined by the membership function

$$\mu_{\frac{1}{B}}(x) = \mu_B(\frac{1}{x}) , \quad x \in R \tag{30}$$

by using (14). Thus, for example, if B is a zero convex fuzzy number depicted in Fig.3 and is expressed by

$$B = \int_{-1}^{1} \frac{x+1}{2}/x + \int_{1}^{2} 2 - x/x ,$$

then the application of (30) to B yields

$$\frac{1}{B} = \int_{-\infty}^{-1} \frac{1}{2}(\frac{1}{x} + 1)/x + \int_{\frac{1}{2}}^{1} 2 - \frac{1}{x}/x + \int_{1}^{\infty} \frac{1}{2}(\frac{1}{x} + 1)/x$$

and thus $\frac{1}{B}$ is not a convex fuzzy number (see Fig.3). Q.E.D.

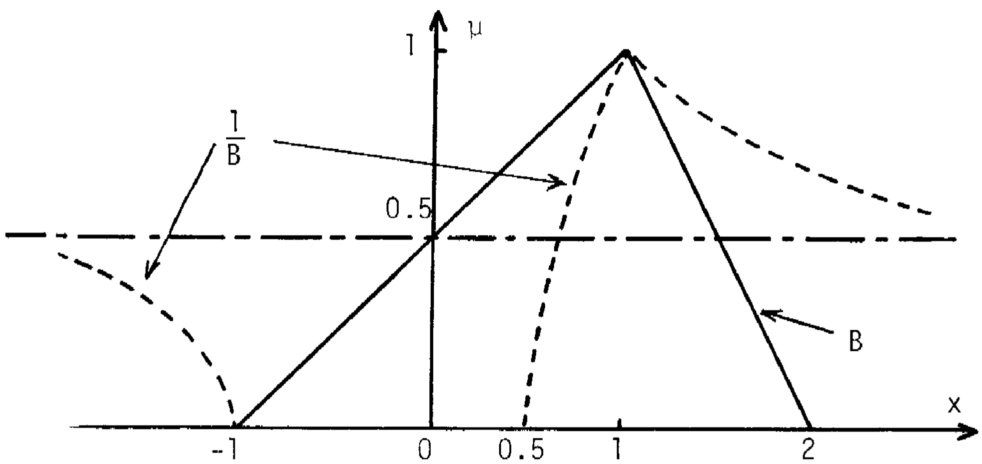


Fig.3 $\frac{1}{B}$ for the Zero Convex Fuzzy Number B.

[Theorem 3] Let A and B be convex fuzzy numbers, then A ÷ B is not, in general, a convex fuzzy number.

In this theorem, however, if B is not a zero fuzzy number but a positive (or negative) fuzzy number, the convexity will be reserved.

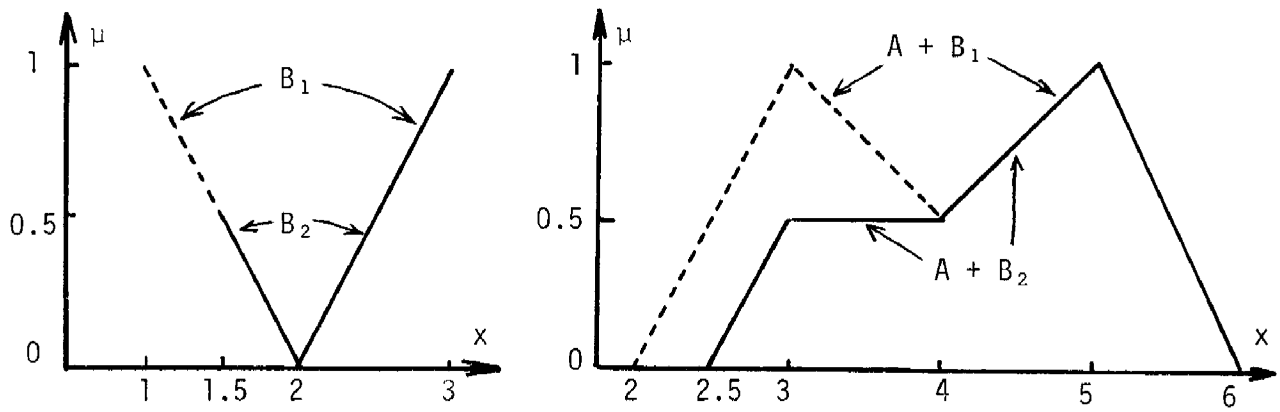
[Theorem 4] If A is a convex fuzzy number and B is a positive (or negative) convex fuzzy number, then A ÷ B is a convex fuzzy number.

Proof: It will be sufficient to prove that $\frac{1}{B}$ is convex if B is positive convex, since A ÷ B can be represented as A x $(\frac{1}{B})$. Let x, y, z be real numbers such that $0 < x \leq y \leq z$, then $0 < \frac{1}{z} \leq \frac{1}{y} \leq \frac{1}{x}$ holds. Thus, we can have $\mu_B(\frac{1}{y}) \geq \mu_B(\frac{1}{z}) \wedge \mu_B(\frac{1}{x})$ in virtue of the convexity of B. Using (30) we can write $\mu_{\frac{1}{B}}(y) \geq \mu_{\frac{1}{B}}(z) \wedge \mu_{\frac{1}{B}}(x)$, which leads to the convexity of $\frac{1}{B}$.

The normality of fuzzy numbers can be easily shown by the following.

[Theorem 5] If A and B are normal fuzzy numbers, then A + B, A - B, A x B and A ÷ B are also normal.

Note. For two fuzzy numbers A and B, if the one is convex and the other is non-convex, then the execution results of A and B under +, -, x and ÷ may be convex or non-convex. We shall show this by the example.



(a) Non-convex fuzzy numbers B_1, B_2. (b) $A+B_1$(non-convex) and $A+B_2$(convex).

Fig.4 Diagram of Example 5.

(Example 5) Let A be the convex fuzzy number $\tilde{2}$ given by (3) and B_1 be a non-convex fuzzy number such as (see Fig.4)

$$B_1 = \int_1^2 2 - x/x \quad + \quad \int_2^3 x - 2/x \ .$$

Then we have $A + B_1$ as

$$A + B_1 = \int_2^3 x - 2/x \quad + \quad \int_3^4 \frac{5 - x}{2}/x \quad + \quad \int_4^5 \frac{x - 3}{2}/x \quad + \quad \int_5^6 6 - x/x$$

which indicates that $A + B_1$ is non-convex (see Fig.4). On the contrary, let B_2 be also a non-convex fuzzy number such as

$$B_2 = \int_{1.5}^2 2 - x/x \quad + \quad \int_2^3 x - 2/x \ ,$$

then $A + B_2$ is given by

$$A + B_2 = \int_{2.5}^3 x - 2.5/x \quad + \quad \int_3^4 0.5/x \quad + \quad \int_4^5 \frac{x - 3}{2}/x \quad + \quad \int_5^6 6 - x/x \ .$$

This shows that $A + B_2$ is convex.

We shall next investigate the algebraic properties of fuzzy numbers under +, −, x and ÷. As is well-known, the family of real numbers forms a field under the ordinary operations + and x. Convex fuzzy numbers, however, are shown not to have their inverses and not to satisfy the distributive law. So the family of convex fuzzy numbers (needless to say, arbitrary fuzzy numbers) does not form the algebraic structures such as a ring and a field. On the contrary, positive convex fuzzy numbers defined in the positive real line satisfy the distributive law and thus they form a commutative semiring with zero and unity.

[Theorem 6] For any fuzzy numbers A, B and C, we have

$$\left.\begin{aligned}(A + B) + C &= A + (B + C)\\ (A \times B) \times C &= A \times (B \times C)\end{aligned}\right\} \text{(associative laws)} \qquad (31)$$

$$\left.\begin{aligned}A + B &= B + A\\ A \times B &= B \times A\end{aligned}\right\} \text{(commutative laws)} \qquad (32)$$

$$\left.\begin{array}{l} A + 0 = A \\ A \times 1 = A \end{array}\right\} \qquad \text{(identity laws)} \tag{33}$$

where 0 and 1 are zero and unity, respectively, in the ordinary sense.

[Theorem 7] For any fuzzy number A, there exist no inverse fuzzy numbers A' and A" under + and x, respectively, such that

$$A + A' = 0 , \tag{34}$$

$$A \times A'' = 1 . \tag{35}$$

Proof: Assume that A is arbitrary fuzzy number and A' satisfies (34) for A. It follows from (11) that

$$A + A' = \int (\mu_A(x) \wedge \mu_{A'}(y)) \;/\; (x + y) = 1/0, \tag{36}$$

where 1/0 is a fuzzy number which means a zero 0 in the ordinary sense.

[a] The case where A is not normal: It is immediately shown that (36) can not be satisfied for any A'.

[b] The case where A is normal: For x and y (= -x) satisfying x + y = 0, it is necessary to satisfy from (36)

$$\bigvee_x [\mu_A(x) \wedge \mu_{A'}(-x)] = 1$$

and hence to satisfy

$$\mu_{A'}(-x_o) = \mu_A(x_o) = 1 \tag{37}$$

for some $x_o \in R$. On the other hand, for x and y with $x + y \neq 0$, we must have by (36)

$$\mu_A(x) \wedge \mu_{A'}(y) = 0 . \tag{38}$$

Since $\mu_{A'}(-x_o) = 1$ holds from (37), $\mu_A(x) = 0$ must hold for all x such that $x + (-x_o) \neq 0$ in view of (38). This is contrary to the assumption that A is arbitrary normal fuzzy number. Thus, it has been proved that there does not exist an inverse fuzzy number for A under +. The same holds for the case of the operation x. Q.E.D.

It is noted that if A is reduced to a real number, -A and $\frac{1}{A}$ are the inverses of A under + and x, respectively.

[Corollary 8] For -A in (29) and $\frac{1}{A}$ in (30) of a fuzzy number A, we have in general

$$A + (-A) \neq 0 , \tag{39}$$

$$A \times (\tfrac{1}{A}) \neq 1 . \tag{40}$$

Proof: This is obvious from $\underset{\sim}{2} - \underset{\sim}{2}$ $(= \underset{\sim}{2} + (\underset{\sim}{-2}))$ and $\underset{\sim}{2} \div \underset{\sim}{2}$ $(= \underset{\sim}{2} \times (\underset{\sim}{\tfrac{1}{2}}))$ in Fig.2.

[Theorem 9] When A, B and C are any fuzzy numbers, the following does not hold in general.

$$A \times (B + C) = (A \times B) + (A \times C) \qquad \text{(distributive law)} \tag{41}$$

The same is true for the case where A, B and C are normal convex fuzzy numbers.

Proof: It will be sufficient to show the example of normal convex fuzzy numbers which do not satisfy (41). Now, suppose that A, B and C are normal convex fuzzy numbers such that

$$A = \int_2^3 x - 2/x \;+\; \int_3^4 4 - x/x , \tag{42}$$

$$B = \int_{1}^{2} 1/x \ , \tag{43}$$

$$C = \int_{-1}^{1} \frac{1}{2}(x + 1)/x \ . \tag{44}$$

Then

$$A \times (B + C) = \int_{0}^{6} \frac{\sqrt{4 + 2x} - 2}{2}/x \ + \int_{6}^{9} 1/x \ + \int_{9}^{12} 4 - \frac{x}{3}/x \ ,$$

$$(A \times B) + (A \times C) = \int_{-2}^{2\cdot 5} \frac{5 - \sqrt{21 - 2x}}{2}/x + \int_{2\cdot 5}^{6} \frac{\sqrt{4 + 2x} - 2}{2}/x + \int_{6}^{9} 1/x + \int_{9}^{12} 4 - \frac{x}{3}/x \ .$$

Thus the distributive law (41) does not hold for the normal convex fuzzy numbers A, B and C.
Q.E.D.

From Theorems 7 and 9, we can find that (normal) convex fuzzy numbers (needless to say, arbitrary fuzzy numbers) do not satisfy the distributive law and do not have their inverses. Therefore, the family of (normal) convex fuzzy numbers does not form such algebraic structures as a ring and a field.

In the next theorem, however, the distributive law is shown to be satisfied for the positive convex fuzzy numbers.

[Theorem 10] The distributive law of (41) is satisfied for the positive convex fuzzy numbers A, B and C.

Proof: Let α-level sets of positive convex fuzzy numbers A, B and C be $A_\alpha = [a_1, a_2]$, $B_\alpha = [b_1, b_2]$ and $C_\alpha = [c_1, c_2]$, respectively, then each level set is an interval in R and $0 < a_1 \le a_2$, $0 < b_1 \le b_2$ and $0 < c_1 \le c_2$ hold. Thus, for each $0 < \alpha \le 1$,

$$\begin{aligned}[A \times (B + C)]_\alpha &= A_\alpha \times (B_\alpha + C_\alpha) \\ &= [a_1, a_2] \times ([b_1, b_2] + [c_1, c_2]) \\ &= [a_1, a_2] \times [b_1+c_1, b_2+c_2] \\ &= [a_1(b_1+c_1), a_2(b_2+c_2)] \quad \ldots a_i, b_i, c_i > 0 \ .\end{aligned}$$

The right hand member of (41) will be

$$\begin{aligned}[(A \times B) + (A \times C)]_\alpha &= (A_\alpha \times B_\alpha) + (A_\alpha \times C_\alpha) \\ &= ([a_1, a_2] \times [b_1, b_2]) + ([a_1, a_2] + [c_1, c_2]) \\ &= [a_1b_1, a_2b_2] + [a_1c_1, a_2c_2] \quad \ldots a_i, b_i, c_i > 0 \\ &= [a_1b_1+a_1c_1, a_2b_2+a_2c_2] \\ &= [a_1(b_1+c_1), a_2(b_2+c_2)] \\ &= [A \times (B + C)]_\alpha \ .\end{aligned}$$

Thus, using the resolution identity of (8), we can obtain A x (B + C) = (A x B) + (A x C).
Q.E.D.

Note that when α-level set is an empty set Φ, the following holds.

$$A_\alpha + \Phi = \Phi \ ; \qquad A_\alpha \times \Phi = \Phi \ .$$

[Theorem 11] The family of positive convex fuzzy numbers forms a commutative semiring with zero 0 and unity 1† under + and x.

Proof: Positive convex fuzzy numbers are closed under + and x (Theorem 1), and associative (31), commutative (32) and distributive (Theorem 10), and have zero 0 and unity 1 (33) under + and x. Q.E.D.

In Theorem 10, fuzzy numbers A, B and C are assumed to be positive convex, that is, their α-level sets are all positive intervals. It will be, however, found that the following identity (45) can hold even for the case where α-level sets are not positive intervals. Table I summarizes this fact, where the symbols +, 0 and - mean positive, zero and negative intervals, respectively. As an illustration,

$$A_\alpha \times (B_\alpha + C_\alpha) = (A_\alpha \times B_\alpha) + (A_\alpha \times C_\alpha). \quad (45)$$

let $A_\alpha = [-a_1, -a_2]$ be a negative interval and $B_\alpha = [b_1, b_2]$ and $C_\alpha = [c_1, c_2]$ be positive intervals, then we can have

$$A_\alpha \times (B_\alpha + C_\alpha) = [-a_1, -a_2] \times [b_1+c_1, b_2+c_2]$$
$$= [-a_1(b_2+c_2), -a_2(b_1+c_1)],$$

$$(A_\alpha \times B_\alpha) + (A_\alpha \times C_\alpha) = [-a_1b_2, -a_2b_1] + [-a_1c_2, -a_2c_1]$$
$$= [-a_1(b_2+c_2), -a_2(b_1+c_1)] = A_\alpha \times (B_\alpha + C_\alpha),$$

which indicates the satisfaction of (45).

Table I. The Combination of A_α, B_α and C_α Satisfying $A_\alpha x(B_\alpha + C_\alpha) = (A_\alpha x B_\alpha) + (A_\alpha x C_\alpha)$ (+: positive interval, 0: zero interval, -: negative interval)

A_α	+	+	+	0	0	-	-	-
B_α	+	0	-	+	-	+	0	-
C_α	+	0	-	+	-	+	0	-

† The algebraic system R = <R; +, x> with addition + and multiplication x is called a commutative semiring with zero 0 and unity 1 if it satisfies these laws:

(i) Closure property:

$$a, b \in R \implies a + b, a \times b \in R .$$

(ii) Associative laws:

$$(a + b) + c = a + (b + c); \quad (a \times b) \times c = a \times (b \times c) .$$

(iii) Commutative laws:

$$a + b = b + a; \quad a \times b = b \times a .$$

(iv) Distributive law:

$$a \times (b + c) = (a \times b) + (a \times c) .$$

(v) Existence of identities: There exist a zero 0 and a unity 1 such that

$$a + 0 = a; \quad a \times 1 = a .$$

If this system R also satisfies the following, R is a field.

(vi) Existence of inverses: There exist a' and a'' for each a such that

$$a + a' = 0; \quad a \times a'' = 1.$$

Therefore, if for each $\alpha \in (0, 1]$, A_α, B_α and C_α of convex fuzzy numbers A, B and C satisfy either of the conditions of Table I, then the distributive law of (41) is shown to be satisfied.

(Example 6) If A, B and C are all negative convex fuzzy numbers, then their α-level sets, which are negative intervals, satisfy the condition in Table I. Thus, the negative convex fuzzy numbers are shown to satisfy the distributive law (41). As another example, let A, B and C be convex fuzzy numbers of (42), (43) and (44), respectively, then $(A_\alpha, B_\alpha, C_\alpha) = (+, +, 0)$ at $\alpha \leq 0.5$ does not satisfy the condition of Table I. Thus, these convex fuzzy numbers A, B and C can not satisfy the distributive law (41) as shown in the proof of Theorem 9. However, if changed A with C, then $(A_\alpha, B_\alpha, C_\alpha) = (0, +, +)$ at $\alpha \leq 0.5$ and $(A_\alpha, B_\alpha, C_\alpha) = (+, +, +)$ at $\alpha > 0.5$. Hence, this case satisfies the distributive law (41).

From this example it follows that negative convex fuzzy numbers satisfy the distributive law. However, negative convex fuzzy numbers never form a commutative semiring unlike the case of positive convex fuzzy numbers. The reason is that negative convex fuzzy numbers are not closed under x, that is, A x B becomes positive when A and B are negative fuzzy numbers. Table II and III show which intervals $A_\alpha + B_\alpha$ and $A_\alpha \times B_\alpha$ can take.

Table II Intervals of $A_\alpha + B_\alpha$

A_α \ B_α	-	0	+
-	-	-, 0	0, ±
0	-, 0	0	0, +
+	0, ±	0, +	+

Table III Intervals of $A_\alpha \times B_\alpha$

A_α \ B_α	-	0	+
-	+	0	-
0	0	0	0
+	-	0	+

ORDERING OF FUZZY NUMBERS

This section introduces order relations, join and meet for fuzzy numbers, and discusses the algebraic properties of fuzzy numbers under these operations combined with the four arithmetic operations +, -, x and ÷.

The ordering, join and meet of fuzzy numbers can be defined in a similar way as those of fuzzy grades [2, 3] which are fuzzy sets in the unit interval [0, 1].

Join and Meet: Join ($\sqcup$) and meet ($\sqcap$) of fuzzy numbers A and B are defined as follows by using the extension principle (10).

$$A \sqcup B = \int (\mu_A(x) \wedge \mu_B(y)) \, / \, (x \vee y) \, , \qquad (46)$$

$$A \sqcap B = \int (\mu_A(x) \wedge \mu_B(y)) \, / \, (x \wedge y) \, , \qquad (47)$$

where $\vee$ stands for max and $\wedge$ for min.

(Example 7) If $A = \underset{\sim}{2} + \underset{\sim}{2}$ and $B = \underset{\sim}{2} \times \underset{\sim}{2}$ are fuzzy numbers given in (25) and (27), respectively, then

$$A \sqcup B = \int_2^4 \frac{x}{2} - 1/x \quad + \quad \int_4^9 3 - \sqrt{x}/x \, , \qquad (48)$$

$$A \sqcap B = \int_{1}^{4} \sqrt{x} - 1/x \quad + \quad \int_{4}^{6} 3 - \tfrac{x}{2}/x \ . \tag{49}$$

Order Relations: Two kinds of order relations, namely, $\underset{\cdot}{\subseteq}$ and $\overset{\cdot}{\subseteq}$, for fuzzy numbers A and B can be defined by

$$A \underset{\cdot}{\subseteq} B \iff A \sqcap B = A \ , \tag{50}$$

$$A \overset{\cdot}{\subseteq} B \iff A \sqcup B = B \ . \tag{51}$$

The following properties of fuzzy numbers under join ($\sqcup$), meet ($\sqcap$) and order relations ($\underset{\cdot}{\subseteq}$, $\overset{\cdot}{\subseteq}$) are obtained (cf. [2]).

[Theorem 12] Arbitrary fuzzy numbers satisfy idempotent laws, commutative laws and associative laws under the operations of join ($\sqcup$) and meet ($\sqcap$). Thus, they constitute a partially ordered set under the order relation ($\overset{\cdot}{\subseteq}$). The same is true of the order relation ($\underset{\cdot}{\subseteq}$). In general, however, we have $\overset{\cdot}{\subseteq} \neq \underset{\cdot}{\subseteq}$.

[Theorem 13] Convex fuzzy numbers also satisfy distributive laws and are closed under $\sqcup$ and $\sqcap$. Thus, they form a commutative semiring under $\sqcup$ and $\sqcap$, but do not form a lattice because they do not satisfy absorption laws under $\sqcup$ and $\sqcap$.

[Theorem 14] Normal convex fuzzy numbers are closed and also satisfy absorption laws under $\sqcup$ and $\sqcap$. Therefore, they form a distributive lattice under $\sqcup$ and $\sqcap$. Consequently, order relation $\underset{\cdot}{\subseteq}$ becomes coincident with $\overset{\cdot}{\subseteq}$ and hence the order relation for normal convex fuzzy numbers can be defined as

$$A \subseteq B \iff A \sqcap B = A \iff A \sqcup B = B \tag{52}$$

(Example 8) Let $A = \underset{\sim}{2} \div \underset{\sim}{2}$ and $B = \underset{\sim}{2}$ be normal convex fuzzy numbers in (28) and (3), respectively, then $(\underset{\sim}{2} \div \underset{\sim}{2}) \sqcup \underset{\sim}{2} = \underset{\sim}{2}$ and $(\underset{\sim}{2} \div \underset{\sim}{2}) \sqcap \underset{\sim}{2} = (\underset{\sim}{2} \div \underset{\sim}{2})$ are obtained from (50) and (51). Thus, we can have $(\underset{\sim}{2} \div \underset{\sim}{2}) \subseteq \underset{\sim}{2}$ by (52). But, for $A = \underset{\sim}{2} + \underset{\sim}{2}$ and $B = \underset{\sim}{2} \times \underset{\sim}{2}$, it follows from (48) and (49) that $A \sqcap B \neq A$ and $A \sqcup B \neq B$. So $\underset{\sim}{2} + \underset{\sim}{2}$ and $\underset{\sim}{2} \times \underset{\sim}{2}$ are incomparable.

Next we shall obtain the algebraic properties of fuzzy numbers under the operations of $\sqcup$ and $\sqcap$ combined with the operations of +, -, x and ÷.

[Theorem 15] Let A, B and C be convex fuzzy numbers, then the following properties are obtained.

$$A + (B \sqcup C) = (A + B) \sqcup (A + C) \tag{53}$$

$$A + (B \sqcap C) = (A + B) \sqcap (A + C) \tag{54}$$

$$A - (B \sqcup C) = (A - B) \sqcap (A - C) \tag{55}$$

$$A - (B \sqcap C) = (A - B) \sqcup (A - C) \tag{56}$$

$$(B \sqcup C) - A = (B - A) \sqcup (C - A) \tag{57}$$

$$(B \sqcap C) - A = (B - A) \sqcap (C - A) \tag{58}$$

[Theorem 16] If A, B and C are positive convex fuzzy numbers, then we have

$$A \times (B \sqcup C) = (A \times B) \sqcup (A \times C) \tag{59}$$

$$A \times (B \sqcap C) = (A \times B) \sqcap (A \times C) \tag{60}$$

$$A \div (B \sqcup C) = (A \div B) \sqcap (A \div C) \tag{61}$$

$$A \div (B \sqcap C) = (A \div B) \sqcup (A \div C) \tag{62}$$

$$(B \sqcup C) \div A = (B \div A) \sqcup (C \div A) \tag{63}$$

$$(B \sqcap C) \div A = (B \div A) \sqcap (C \div A) \tag{64}$$

It is interesting to note in the above two theorems that if A is convex, then Eqs. (53) - (64) do hold even if B and C are non-convex and that, conversely, if A is non-convex, they do not hold in general even if B and C are convex.

(Example 9) Let A be a positive non-convex fuzzy number and let B and C be positive convex fuzzy numbers such that

$$A = \int_0^{0.5} 1 - 2x/x + \int_{0.5}^{1} 2x - 1/x \ ,$$

$$B = \int_0^{0.5} 2x/x + \int_{0.5}^{1} 1/x \ ,$$

$$C = \int_0^{0.5} 2x/x + \int_{0.5}^{1} 2(1-x)/x \ .$$

Then

$$A \times (B \sqcap C) = \int_0^{\frac{3}{16}} \frac{3 - \sqrt{1+16x}}{2}\Big/x + \int_{\frac{3}{16}}^{0.5} \frac{\sqrt{1+16x} - 1}{2}\Big/x + \int_{0.5}^{1} 2(1-x)/x \ ,$$

$$(A \times B) \sqcap (A \times C) = \int_0^{u_0} 1 - 2x/x + \int_{u_0}^{0.5} \frac{\sqrt{1+16x} - 1}{2}\Big/x + \int_{0.5}^{1} 2(1-x)/x \ ,$$

where $u_0 = \dfrac{5 - \sqrt{17}}{4}$. Thus (60) is not satisfied when A is non-convex.

[Theorem 17] Convex fuzzy numbers form a commutative semiring with zero $+\infty$ and unity 0 under $\sqcap$ (as addition) and + (as multiplication). Similarly, they also form a commutative semiring with zero $-\infty$ and unity 0 under $\sqcup$ and +. Positive convex fuzzy numbers form a commutative semiring with zero $+\infty$ and unity 1 under $\sqcap$ and x. The same holds under $\sqcup$ and x, where zero is $-\infty$ and unity is 1.

CONCLUSION

In this paper we have investigated the algebraic properties of fuzzy numbers in the real line under the four arithmetic operations and under join and meet.

The concept of fuzzy numbers will find interesting applications to such fields as decision making and system science, where imprecise data in the real line are usually treated.

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