Abstract !

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Algebraic Product and Algebraic Sum of Fuzzy Sets of Type 2

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Recently L. A. Zadeh proposed the concept of fuzzy sets of type 2 as an extension of ordinary fuzzy sets. A fuzzy set of type 2 can be defined by a fuzzy membership function the grade of which is taken to be a fuzzy set in the unit interval [0, 1] rather than a point in [0, 1].

This paper investigates the algebraic properties of fuzzy grades (or fuzzy sets of type 2) under the operations of algebraic product and algebraic sum which can be defined by using the concept of the extension principle [1].

A _fuzzy set of type 2_ A in a set X is the fuzzy set characterized by a _fuzzy membership function_ as

$$\mu_A : X \longrightarrow [0,1]^{[0,1]} \qquad (1)$$

where the value $\mu_A(x)$ of $x \in X$ is a fuzzy set in the unit interval [0,1] and is called _fuzzy grade_. A fuzzy grade is represented by the following:

$$\mu_A(x) = \int f(u)/u, \quad u \in [0, 1] \qquad (2)$$

where f is a membership function for fuzzy grade $\mu_A(x)$ and is defined as $f: [0,1] \longrightarrow [0,1]$. The symbol $\int$ stands for the union of $f(u)/u$'s.

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The operations of algebraic product ($\cdot$) and algebraic sum ($\dot{+}$) for fuzzy grades are defined as follows: Let μ_A and μ_B be fuzzy grades denoted by†

$$\mu_A = \int f(u)/u$$

$$\mu_B = \int g(w)/w$$

Algebraic Product:

$$\mu_A \cdot \mu_B = \int (f(u) \wedge g(w))/(uw) \qquad (3)$$

Algebraic Sum:

$$\mu_A \dot{+} \mu_B = \int (f(u) \wedge g(w))/(u + w - uw) \qquad (4)$$

Negation:

$$\neg \mu_A = \int f(u)/(1 - u) \qquad (5)$$

where the symbol $\wedge$ stands for min.

Example 1. For the fuzzy grade $\mu_A = \int u/u$, $u \in [0,1]$, as in Fig. 1, we have

$$\mu_A \cdot \mu_A = \int \sqrt{u} \,/\, u \,,$$

$$\mu_A \dot{+} \mu_A = \int 1 - \sqrt{1 - u} \,/\, u \,,$$

$$\neg \mu_A = \int 1 - u \,/\, u \,.$$

A fuzzy grade $\mu_A = \int f(u)/u$ is said to be convex if $f(u_2) \geq f(u_1) \wedge f(u_3)$ for any u_1, u_2, $u_3 \in [0,1]$ such that $u_1 \leq u_2 \leq u_3$. A fuzzy grade μ_A is said to be normal if $\bigvee_u f(u) = 1$. Otherwise it is subnormal.

Example 2. The fuzzy grades in Fig.1 are all normal convex fuzzy grades.

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† We shall abbreviate $\mu_A(x)$ as μ_A for simplicity.

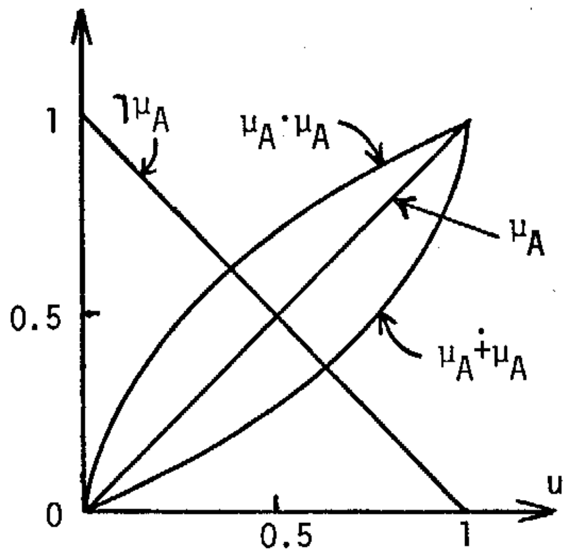

Fig.1 Negation $\neg\mu_A$, algebraic product $\mu_A \cdot \mu_A$, and algebraic sum $\mu_A \dot{+} \mu_A$ of fuzzy grade μ_A .

Now we shall discuss the algebraic properties of fuzzy gardes under the algebraic product $(\cdot)$ and the algebraic sum $(\dot{+})$.

Theorem 1. Let μ_A and μ_B be normal convex fuzzy grades, then $\mu_A \cdot \mu_B$, $\mu_A \dot{+} \mu_B$ and $\neg\mu_A$ are also normal convex.

Theorem 2. Under the operations $\cdot$, $\dot{+}$ and $\neg$, arbitrary fuzzy grades satisfy such laws as commutative laws; associative laws; involution laws; De Morgan's laws and the part of identity laws, i.e., $\mu_A \cdot 1 = \mu_A$, $\mu_A \dot{+} 0 = \mu_A$. The other part of identity laws (that is, $\mu_A \cdot 0 = 0$, $\mu_A \dot{+} 1 = 1$) are satisfied for any normal fuzzy grades, but are not satisfied for subnormal fuzzy grades.

Theorem 3. Normal convex fuzzy grades do not satisfy such laws as idempotent laws; absorption laws; distributive laws; and complement laws. The same holds for arbitrary fuzzy grades.

From the above theorems, we see that normal convex fuzzy grades do not form such algebraic structures as a ring and a lattice. The same is true of arbitrary fuzzy grades.

Finally, the algebraic properties of fuzzy grades under $\cdot$, $\dot{+}$, $\sqcup$ (join) and $\sqcap$ (meet) are also investigated. The operations $\sqcup$ and $\sqcap$ are defined as

Join:

$$\mu_A \sqcup \mu_B = \int (f(u) \wedge g(w)) \,/\, (u \vee w) \qquad (6)$$

Meet:

$$\mu_A \sqcap \mu_B = \int (f(u) \wedge g(w)) \,/\, (u \wedge w) \qquad (7)$$

It is shown that normal convex fuzzy grades form a lattice ordered semigroup under $\cdot$, $\sqcup$, and $\sqcap$ since they satisfy the distributive law

$$\mu_A \cdot (\mu_B \sqcup \mu_C) = (\mu_A \cdot \mu_B) \sqcup (\mu_A \cdot \mu_C)$$

and constitute a lattice under $\sqcup$ and $\sqcap$ [2]. They also form a commutative semiring under $\sqcup$ and $\cdot$.

References

(1) L. A. Zadeh, Fuzzy logic and its application to approximate reasoning, Inform. Processing 74, 591-595, 1974.

(2) M. Mizumoto and K. Tanaka, Some properties of fuzzy sets of type 2, Inform. Control, 31, 4, 312-340, 1976.