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# ALGEBRAIC PROPERTIES OF FUZZY NUMBERS

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Fuzzy number is a fuzzy set in a real line and its operations of +, -, x and ÷ can be defined by using the extension principle. This paper investigates the algebraic properties of fuzzy numbers under the four arithmetic operations of +, -, x and ÷. Furthermore, the ordering of fuzzy numbers is introduced and some properties of fuzzy numbers under join ($\sqcup$) and meet ($\sqcap$) are discussed.

## INTRODUCTION

Recently, L.A. Zadeh proposed the interesting concept of extension principle by which a binary operation defined on X may be extended to fuzzy sets in X, and defined the operations for fuzzy sets of type 2[1,2] and fuzzy numbers[1,3].

In this paper we discuss the algebraic properties of fuzzy numbers, which are fuzzy sets in a real line, under the four arithmetic operations, namely, +, -, x and ÷ which are defined by the extension principle[1]. First, as for the convexity of fuzzy numbers, the fuzzy numbers obtained by applying the operations of +, - and x to convex fuzzy numbers are also convex fuzzy numbers, though the convexity may not be preserved in general if ÷ is applied to convex fuzzy numbers. Second, the convex fuzzy numbers do not form such algebraic structures as ring and field, since the distributive law is not satisfied and there exist no inverse fuzzy numbers under + and x, respectively. Third, the positive convex fuzzy numbers defined over the positive real line, however, satisfy the distributive law and hence form a commutative semiring. And fourth, the ordering of fuzzy numbers is introduced and the properties of fuzzy numbers under the join and the meet combined with the four arithmetic operations are investigated.

## FUZZY NUMBERS

We shall briefly review some of the basic definitions relating to fuzzy numbers and their operations of +, -, x and ÷.

Fuzzy Numbers: A fuzzy number A in a real line R is a fuzzy set characterized by a membership function $\mu_A$ as

$$\mu_A : R \to [0, 1] . \qquad (1)$$

A fuzzy number A may be expressed as

$$A = \int_{x \in R} \mu_A(x)/x \qquad (2)$$

with the understanding that $\mu_A(x) \in [0,1]$ represents the grade of membership of x in A and $\int$ denotes the union of $\mu_A(x)/x$'s.

Example 1. A fuzzy number $\underset{\sim}{2}$ which denotes "about 2" will be given as

$$\underset{\sim}{2} = \int_1^2 x-1/x + \int_2^3 3-x/x \qquad (3)$$

and can be illustrated by the dotted line in Fig. 2, where + stands for the union.

Convex Fuzzy Numbers: A fuzzy number A in R is said to be convex if for any real numbers x, y, z ε R with $x \leq y \leq z$,

$$\mu_A(y) \geq \mu_A(x) \wedge \mu_A(z) \qquad (4)$$

with $\wedge$ standing for min. A fuzzy number A is called normal if the following holds.

$$\max_x \mu_A(x) = 1 . \qquad (5)$$

A fuzzy number which is normal and convex is refered to be as a normal convex fuzzy number.

Example 2. Fuzzy numbers as shown in Fig. 2 are all normal convex fuzzy numbers. Fig.1 gives various kinds of fuzzy numbers, in which it is noted that the fuzzy number $A_2$ is not convex because the support (defined in (9)) of $A_2$ is discrete, that is, $A_2$ does not satisfy (4).

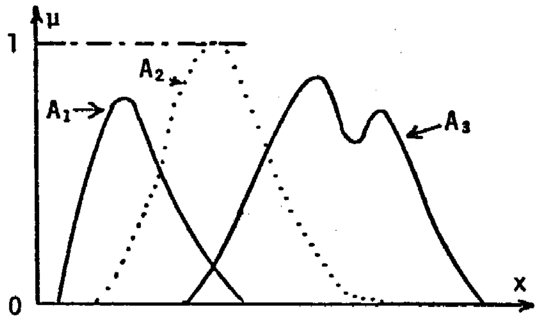


Fig. 1 Various Kinds of Fuzzy Numbers ($A_1$: convex, $A_2$: normal non-convex, $A_3$: non-convex)

Level Sets: The α-level set of a fuzzy number A is a non-fuzzy set denoted by $A_\alpha$ and is defined by

$$A_\alpha = \{x \mid \mu_A(x) \geq \alpha\}, \quad 0 < \alpha \leq 1 . \qquad (6)$$

It is easy to show that

$$\alpha_1 \leq \alpha_2 \Rightarrow A_{\alpha_1} \supseteq A_{\alpha_2} . \qquad (7)$$

If two fuzzy numbers A and B are equal, that is, $\mu_A(x)=\mu_B(x)$ for all x ε R, then we can obtain $A_\alpha = B_\alpha$ for any α, and vice versa. Let fuzzy number A be convex, $A_\alpha$ is a convex set (or an interval) in R, and vice versa. A fuzzy number A may be decomposed into its level sets through the resolution identity[1]

$$A = \int_0^1 \alpha A_\alpha \qquad (8)$$

where $\alpha A_\alpha$ is the product of a scalar α with the set $A_\alpha$ and $\int$ is the union of the $A_\alpha$, with α ranging from 0 to 1.

Support: The support $\Gamma_A$ of a fuzzy number A is defined, as a special case of level set, by the following.

$$\Gamma_A = \{x \mid \mu_A(x) > 0\} . \qquad (9)$$

Extension Principle: Let A and B be fuzzy numbers in R and let * be a binary operation defined in R. Then the operation * can be extended to the fuzzy numbers A and B by the defining relation (the extension principle).

$$A * B = \int_{x,y \in R} (\mu_A(x) \wedge \mu_B(y))/(x * y) \tag{10}$$

where ^ stands for min.

In (10), let the binary operation * be replaced by the ordinary four arithmetic operations of +, -, x and ÷, then the four arithmetic operations over fuzzy numbers will be defined by the following.

Four Arithmetic Operations for Fuzzy Numbers: Assuming that A and B are fuzzy numbers in R, we have by (10)

$$A + B = \int (\mu_A(x) \wedge \mu_B(y))/(x + y) \tag{11}$$

$$A - B = \int (\mu_A(x) \wedge \mu_B(y))/(x - y) \tag{12}$$

$$A \times B = \int (\mu_A(x) \wedge \mu_B(y))/(x \times y) \tag{13}$$

$$A \div B = \int (\mu_A(x) \wedge \mu_B(y))/(x \div y) \tag{14}$$

Although these definitions are useful for any fuzzy numbers and, particularly, for discrete fuzzy numbers, it will be more convenient to convex fuzzy numbers to use the concept of α-level sets of fuzzy numbers.

Let $A_\alpha$ and $B_\alpha$ be α-level sets of convex fuzzy numbers A and B, respectively, then the α-level sets are intervals in R, which are special convex fuzzy numbers whose grades are unity at x belonging to $A_\alpha$ and zero elsewhere. Then let the α-level set of, say, the sum A + B of A and B be denoted by $(A + B)_\alpha$, we can obtain

$$(A + B)_\alpha = A_\alpha + B_\alpha . \tag{15}$$

In other words, the α-level set $(A+B)_\alpha$ is the sum of the α-level sets $A_\alpha$ and $B_\alpha$. Thus, using the resolution identity (8), we can express A+B as

$$A + B = \int_0^1 \alpha(A + B)_\alpha = \int_0^1 \alpha(A_\alpha + B_\alpha) . \tag{16}$$

In a similar fashion, we could obtain A-B, AxB and A÷B as follows.

$$A - B = \int_0^1 \alpha(A_\alpha - B_\alpha) \tag{17}$$

$$A \times B = \int_0^1 \alpha(A_\alpha \times B_\alpha) \tag{18}$$

$$A \div B = \int_0^1 \alpha(A_\alpha \div B_\alpha) \tag{19}$$

Example 3. For the convex fuzzy number $\underset{\sim}{2}$ given by (3), the fuzzy numbers $\underset{\sim}{2}+\underset{\sim}{2}$, $\underset{\sim}{2}-\underset{\sim}{2}$, $\underset{\sim}{2}\times\underset{\sim}{2}$ and $\underset{\sim}{2}\div\underset{\sim}{2}$ are depicted in Fig.2 and are expressed as

$$\underset{\sim}{2} + \underset{\sim}{2} = \int_2^4 \frac{x}{2} - 1/x + \int_4^6 - \frac{x}{2} + 3/x \tag{20}$$

$$\underset{\sim}{2} - \underset{\sim}{2} = \int_{-2}^0 \frac{x}{2} + 1/x + \int_0^2 - \frac{x}{2} + 1/x \tag{21}$$

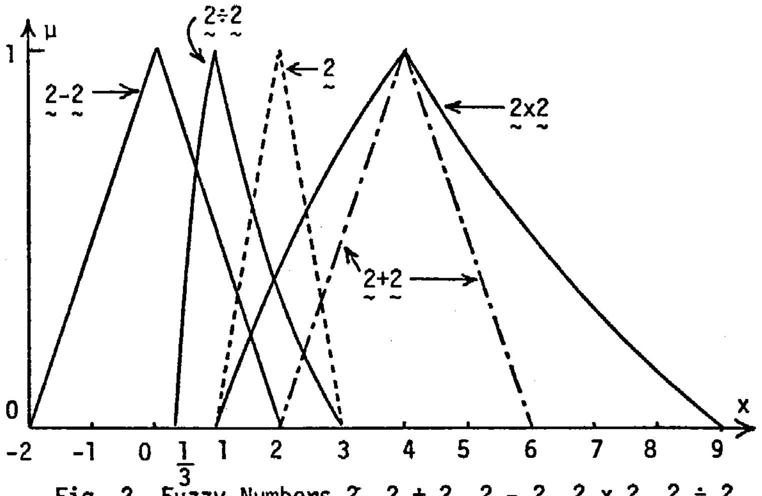


Fig. 2 Fuzzy Numbers $\underset{\sim}{2}$, $\underset{\sim}{2} + \underset{\sim}{2}$, $\underset{\sim}{2} - \underset{\sim}{2}$, $\underset{\sim}{2} \times \underset{\sim}{2}$, $\underset{\sim}{2} \div \underset{\sim}{2}$

$$\underset{\sim}{2} \times \underset{\sim}{2} = \int_1^4 \sqrt{x} - 1/x + \int_4^9 - \sqrt{x} + 3/x \tag{22}$$

$$\underset{\sim}{2} \div \underset{\sim}{2} = \int_{\frac{1}{3}}^1 \frac{-4}{x+1} + 3/x + \int_1^3 \frac{4}{x+1} - 1/x \tag{23}$$

## ALGEBRAIC PROPERTIES OF FUZZY NUMBERS

This section discusses the algebraic properties of fuzzy numbers under the operations of +, -, x and ÷. We shall begin with the convexity of fuzzy numbers under these operations.

[Theorem 1] If A and B are convex fuzzy numbers in a real line R, then A+B, A-B and AxB are also convex fuzzy numbers.

Proof: In general, let $M_1$, $M_2$, $N_1$ and $N_2$ be intervals in R and let $M_1 \subseteq M_2$ and $N_1 \subseteq N_2$, then we can obtain that

$$M_1 + N_1 \subseteq M_2 + N_2 \ ; \quad M_1 \times N_1 \subseteq M_2 \times N_2$$

and that $M_i+N_i$ and $M_i \times N_i$ (i=1,2) also become intervals. For each $0<\alpha\leq 1$, the α-level sets $A_\alpha$ and $B_\alpha$ of convex fuzzy numbers A and B are convex sets (or intervals) in R. Thus, for any $\alpha_1$ and $\alpha_2$ with $0<\alpha_1\leq\alpha_2$, the relations $A_{\alpha_2} \subseteq A_{\alpha_1}$ and $B_{\alpha_2} \subseteq B_{\alpha_1}$ are derived from (7) and hence $A_{\alpha_2}+B_{\alpha_2} \subseteq A_{\alpha_1}+B_{\alpha_1}$ and $A_{\alpha_2} \times B_{\alpha_2} \subseteq A_{\alpha_1} \times B_{\alpha_1}$ are obtained, which lead to $(A+B)_{\alpha_2} \subseteq (A+B)_{\alpha_1}$ and $(A \times B)_{\alpha_2} \subseteq (A \times B)_{\alpha_1}$. And $(A+B)_{\alpha_i}$ and $(A \times B)_{\alpha_i}$ are intervals or convex sets for each $\alpha_i$ (i=1,2). Thus, fuzzy numbers A+B and AxB are shown to be convex fuzzy numbers.

Next, we shall prove the convexity of A-B. Let -B be defined by 0-B, then the membership function of -B will be expressed as

$$\mu_{-B}(x) = \mu_B(-x), \quad x \in R \tag{24}$$

and -B can be easily shown to be convex if B is convex. Thus, A-B is proved to be convex since A-B is represented as A+(-B). Q.E.D.

It should be noted that for the discrete fuzzy numbers, the convexity of A+B, A-B and AxB does not hold even if A and B are in the shape of "convex" like $A_2$ in Fig. 1.

In order to discuss the convexity of fuzzy numbers under ÷, we shall define a special fuzzy number called positive fuzzy number.

Positive Fuzzy Numbers: A fuzzy number A is said to be positive if $0<a_1 \le a_2$ holds for the support $\Gamma_A=[a_1,a_2]$ of A, that is, $\Gamma_A$ is in a positive real line. Similarly, A is called negative if $a_1 \le a_2 < 0$ and zero if $a_1 \le 0 \le a_2$.

Example 4. Fig.2 shows that the fuzzy number $\underset{\sim}{2}-\underset{\sim}{2}$ is a zero fuzzy number and the other fuzzy numbers are all positive.

[Lemma 2] If B is a zero convex fuzzy number, 1/B(=1÷B) is not a convex fuzzy number.

Proof: The fuzzy number 1/B will be defined by the membership function as

$$\mu_{1/B}(x) = \mu_B(\tfrac{1}{x}), \quad x \in R \tag{25}$$

by using (14). Thus, for example, if B is a zero convex fuzzy number depicted in Fig.3 and is expressed by

$$B = \int_{-1}^{1} \tfrac{1}{2}(x+1)/x + \int_{1}^{2} 2 - x/x$$

then the application of (25) to B yields

$$\tfrac{1}{B} = \int_{-\infty}^{-1} \tfrac{1}{2}(\tfrac{1}{x}+1)/x + \int_{\frac{1}{2}}^{1} 2 - \tfrac{1}{x}/x + \int_{1}^{\infty} \tfrac{1}{2}(\tfrac{1}{x}+1)/x$$

and thus 1/B is not a convex fuzzy number (see Fig.3).

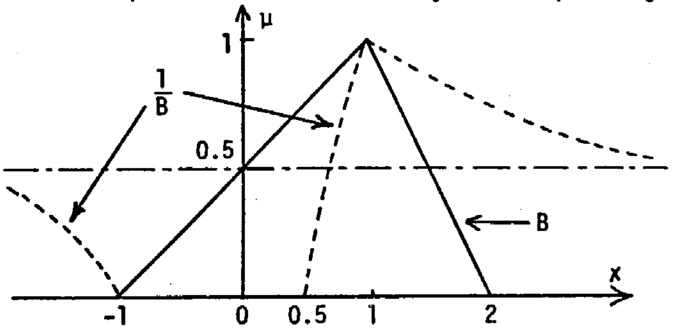


Fig. 3 $\frac{1}{B}$ for zero convex fuzzy number B

[Theorem 3] Let A and B be convex fuzzy numbers, then A÷B is not, in general, a convex fuzzy number.

In this theorem, however, if B is not a zero fuzzy number but a positive (or negative) fuzzy number, the convexity will be reserved.

[Theorem 4] If A is a convex fuzzy number and B is a positive (or negative) convex fuzzy number, then A÷B is a convex fuzzy number.

Proof: It will be sufficient to prove that 1/B is convex if B is positive convex, since A÷B can be represented as Ax(1/B). Let x,y,z be real numbers such that $0<x\le y\le z$, then $0<1/z\le 1/y \le 1/x$ holds. Thus we can have $\mu_B(1/y) \ge \mu_B(1/z) \wedge \mu_B(1/x)$ in virtue of the convexity of B. Using (25) we can write $\mu_{1/B}(y) \ge \mu_{1/B}(z) \wedge \mu_{1/B}(x)$, which leads to the convexity of 1/B. Q.E.D.

The normality of fuzzy numbers can be easily shown by the following.

[Theorem 5] If A and B are normal fuzzy numbers, then A+B, A-B, AxB and A÷B are also normal.

We shall next investigate the algebraic properties of fuzzy numbers under +, -, x and ÷. As is well-known, the family of real numbers forms a field under the ordinary operations + and x. Convex fuzzy numbers, however, do not have their inverses and do not satisfy the distributive law. So the family of convex fuzzy numbers (needless to say, arbitrary fuzzy numbers) does not form such algebraic structures as a ring and a field. On the contrary, positive convex fuzzy numbers defined in a positive real line satisfy the distributive law and thus form a commutative semiring.

[Theorem 6] For any fuzzy numbers A, B and C, we have

$$\left.\begin{aligned}(A+B)+C &= A+(B+C)\\(A\times B)\times C &= A\times(B\times C)\end{aligned}\right\} \tag{26}$$

$$\left.\begin{aligned}A+B &= B+A\\A\times B &= B\times A\end{aligned}\right\} \tag{27}$$

$$\left.\begin{aligned}A+0 &= A\\A\times 1 &= A\end{aligned}\right\} \tag{28}$$

where 0 and 1 are zero and unity, respectively, in the ordinary sense.

[Theorem 7] For any fuzzy number A, there exist no inverse fuzzy numbers A' and A" under + and x, respectively, such that

$$A + A' = 0 \tag{29}$$

$$A \times A'' = 1 \tag{30}$$

Proof: Assume that A is arbitrary fuzzy number and A' satisfies (29) for A, and from (11) it follows that

$$A + A' = \int (\mu_A(x) \wedge \mu_{A'}(y))/(x+y) = 1/0 \tag{31}$$

where the fuzzy number 1/0 means a zero 0.
[a] The case where A is not normal: It is immediately shown that (31) can not be satisfied for any A'.
[b] The case where A is normal: For x and y (=-x) satisfying x+y=0, it is necessary to satisfy from (31)

$$\bigvee_x [\mu_A(x) \wedge \mu_{A'}(-x)] = 1$$

and hence to satisfy

$$\mu_{A'}(-x_0) = \mu_A(x_0) = 1 \tag{32}$$

for some $x_0 \in R$. On the other hand, for x and y with $x+y \ne 0$, we must have by (31)

$$\mu_A(x) \wedge \mu_{A'}(y) = 0\,. \tag{33}$$

As $\mu_{A'}(-x_0)=1$ holds from (32), $\mu_A(x)=0$ must hold for all x such that $x+(-x_0)\ne 0$ in view of (33). This is contrary to the assumption that A is arbitrary normal fuzzy number. Thus, it has been proved that there does not exist an inverse fuzzy number for A under +. The same holds for the case of the operation of x. Q.E.D.

It is noted that if A is reduced to a real number, -A and 1/A are the inverses of A under + and x, respectively.

[Corollary 8] For -A in (24) and 1/A in (25) of a fuzzy number A, we have in general

$$A + (-A) \ne 0 \tag{34}$$

$$A \times (1/A) \ne 1 \tag{35}$$

Proof: This is obvious from $\underset{\sim}{2}-\underset{\sim}{2}$ (=$\underset{\sim}{2}+(-\underset{\sim}{2})$) and $\underset{\sim}{2}\div\underset{\sim}{2}$ (=$\underset{\sim}{2}$x$(1/\underset{\sim}{2})$) in Fig.2.

[Theorem 9] When A, B and C are any fuzzy numbers, the following does not hold in general:

$$A \times (B + C) = (A \times B) + (A \times C) \qquad (36)$$

The same is true for the case where A, B and C are normal convex fuzzy numbers.

Proof: It will be sufficient to show the example of normal convex fuzzy numbers which do not satisfy (36). Now suppose that A, B and C are normal convex fuzzy numbers such that

$$A = \int_2^3 x - 2/x + \int_3^4 4 - x/x \qquad (37)$$

$$B = \int_1^2 1/x \qquad (38)$$

$$C = \int_{-1}^1 \tfrac{1}{2}(x + 1)/x \qquad (39)$$

Then

$$A \times (B + C) = \int_0^6 \frac{-2+\sqrt{4+2x}}{2}/x + \int_6^9 1/x + \int_9^{12} 4-\frac{x}{3}/x$$

$$(A\times B)+(A\times C) = \int_{-2}^{2.5} \frac{5-\sqrt{21-2x}}{2}/x + \int_{2.5}^{6} \frac{-2+\sqrt{4+2x}}{2}/x + \int_6^9 1/x + \int_9^{12} 4 - \frac{x}{3}/x$$

Thus (36) could not be obtained.

From Theorems 7 and 9, we can find (normal) convex fuzzy numbers (needless to say, arbitrary fuzzy numbers) do not satisfy the distributive law and do not have their inverses. Therefore, the family of (normal) convex fuzzy numbers does not form such algebraic structures as a ring and a field.

In the next theorem, however, the distributive law is shown to be satisfied for the positive convex fuzzy numbers.

[Theorem 10] The distributive law of (36) is satisfied for the positive convex fuzzy numbers A, B and C.

Proof. Let α-level sets of positive convex fuzzy numbers A, B and C be $A_\alpha=[a_1,a_2]$, $B_\alpha=[b_1,b_2]$ and $C_\alpha=[c_1,c_2]$, respectively, then each level set is an interval and $0<a_1 \leq a_2$, $0<b_1 \leq b_2$ and $0<c_1 \leq c_2$ hold. Thus, for each $0<\alpha\leq 1$,

$$[A \times (B + C)]_\alpha = A_\alpha \times (B_\alpha + C_\alpha)$$
$$= [a_1, a_2] \times ([b_1, b_2] + [c_1, c_2])$$
$$= [a_1, a_2] \times ([b_1+c_1, b_2+c_2])$$
$$= [a_1(b_1+c_1), a_2(b_2+c_2)] \quad \ldots a_i, b_i, c_i>0 .$$

The right-hand member of (36) will be

$$[(A \times B) + (A \times C)]_\alpha = (A_\alpha \times B_\alpha) + (A_\alpha \times C_\alpha)$$
$$= ([a_1, a_2] \times [b_1, b_2]) + ([a_1, a_2] \times [c_1, c_2])$$
$$= [a_1b_1, a_2b_2] + [a_1c_1, a_2c_2] \quad \ldots a_i, b_i, c_i>0$$
$$= [a_1b_1+a_1c_1, a_2b_2+a_2c_2]$$
$$= [a_1(b_1+c_1), a_2(b_2+c_2)]$$
$$= [A \times (B + C)]_\alpha .$$

Thus, using the resolution identity of (8), we can obtain $A\times(B+C)=(A\times B)+(A\times C)$. Q.E.D.

Note that when α-level set is an empty set φ, then the following holds.

$$A_\alpha + \phi = \phi \ ; \quad A_\alpha \times \phi = \phi \qquad (40)$$

Thus we can obtain the following theorem.

[Theorem 11] The family of positive convex fuzzy numbers forms a commutative semiring with zero 0 and identity 1 under + and x.

Proof: Positive convex fuzzy numbers are closed under + and x (Theorem 1), and associative (26), commutative (27) and distributive (Theorem 10), and have zero 0 and identity 1 (28). Q.E.D.

In Theorem 10, fuzzy numbers A, B and C are assumed to be positive convex, that is, their α-level sets are all positive intervals. It will be found, however, that the following identity (41) can hold even for the case where α-level sets are not positive intervals. Table 1 summarizes this fact.

$$A_\alpha \times (B_\alpha + C_\alpha) = (A_\alpha \times B_\alpha) + (A_\alpha \times C_\alpha) \qquad (41)$$

Therefore, if for each $\alpha\varepsilon(0,1]$, $A_\alpha$, $B_\alpha$ and $C_\alpha$ of convex fuzzy numbers A, B and C satisfy either of the conditions of Table 1, then the distributive law of (36) is shown to be satisfied.

Table 1. The Combination of $A_\alpha$, $B_\alpha$ and $C_\alpha$ Satisfying $A_\alpha\times(B_\alpha+C_\alpha)=(A_\alpha\times B_\alpha)+(A_\alpha\times C_\alpha)$

(+: Positive Interval, 0: Zero Interval, -: Negative Interval)

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| $A_\alpha$ | + | + | + | 0 | 0 | - | - | - |
| $B_\alpha$ | + | 0 | - | + | - | + | 0 | - |
| $C_\alpha$ | + | 0 | - | + | - | + | 0 | - |

Example 5. If A, B and C are all negative convex fuzzy numbers, then their α-level sets, which are negative intervals, satisfy the condition in Table 1. Thus, the negative convex fuzzy numbers are shown to satisfy the distributive law (36). As another example, let A, B and C be convex fuzzy numbers of (37), (38) and (39), respectively, then $(A_\alpha,B_\alpha,C_\alpha)=(+,+,0)$ at $\alpha\leq 0.5$ does not satisfy Table 1. Thus, these convex fuzzy numbers can not satisfy (36) as shown in the proof of Theorem 9. However, if changed A with C, then $(A_\alpha,B_\alpha,C_\alpha)=(0,+,+)$ for $\alpha\leq 0.5$ and $(A_\alpha,B_\alpha,C_\alpha)=(+,+,+)$ for $\alpha>0.5$. Hence, this case satisfies the distributive law of (36).

From this example, it follows that negative convex fuzzy numbers satisfy the distributive law. However, negative convex fuzzy numbers never form a commutative semiring unlike the case of positive convex fuzzy numbers. The reason is that negative convex fuzzy numbers are not closed under x, that is, AxB becomes positive when A and B are negative fuzzy numbers.

## ORDERING OF FUZZY NUMBERS

This section introduces the order relations, join, and meet for fuzzy numbers, and discusses the algebraic properties of fuzzy numbers under these operations combined with the four arithmetic operations.

The ordering, join, and meet of fuzzy numbers can be defined in a similar way as those of fuzzy grades[1,2] which are fuzzy sets in the unit interval [0,1] ($\subsetneq R$).

Join and Meet: Join ($\sqcup$) and meet ($\sqcap$) of fuzzy numbers A and B are defined as follows by using the extension principle (10).

$$A \sqcup B = \int (\mu_A(x) \wedge \mu_B(y))/(x \vee y) \tag{42}$$

$$A \sqcap B = \int (\mu_A(x) \wedge \mu_B(y))/(x \wedge y) \tag{43}$$

where v stands for max and ∧ for min.

Example 6. If $A=\tilde{2}+\tilde{2}$ and $B=\tilde{2}\times\tilde{2}$ are fuzzy numbers in (20) and (22), then

$$A \sqcup B = \int_2^4 \frac{x}{2} - 1/x + \int_4^9 -\sqrt{x} + 3/x \tag{44}$$

$$A \sqcap B = \int_1^4 \sqrt{x} - 1/x + \int_4^6 - \frac{x}{2} + 3/x \tag{45}$$

Order Relations: Two kinds of order relations, namely, $\sqsubseteq$, $\dot{\sqsubseteq}$, for fuzzy numbers can be defined by

$$A \sqsubseteq B \iff A \sqcap B = A \tag{46}$$

$$A \dot{\sqsubseteq} B \iff A \sqcup B = B \tag{47}$$

The following properties of fuzzy numbers under join ($\sqcup$), meet ($\sqcap$) and order relations ($\sqsubseteq$, $\dot{\sqsubseteq}$) are obtained (cf. Ref. [2]).

[Theorem 12] Arbitrary fuzzy numbers satisfy the idempotent laws, the commutative laws and the associative laws under the operations of join ($\sqcup$) and meet ($\sqcap$). Thus, they constitute a partially ordered set under the order relation ($\dot{\sqsubseteq}$). The same is true for the order relation ($\sqsubseteq$). In general, however, we have $\dot{\sqsubseteq} \neq \sqsubseteq$.

[Theorem 13] Convex fuzzy numbers also satisfy the distributive laws and are closed under $\sqcup$ and $\sqcap$. Thus, they form a commutative semiring, but do not form a lattice because they do not satisfy the absorption laws.

[Theorem 14] Normal convex fuzzy numbers are closed and also satisfy the absorption laws under $\sqcup$ and $\sqcap$. Therefore, they form a distributive lattice under $\sqcup$ and $\sqcap$. Consequently, the order relation becomes coincident with $\dot{\sqsubseteq}$ and hence the order relation for normal convex fuzzy numbers may be defined as

$$A \sqsubseteq B \iff A \sqcap B = A \iff A \sqcup B = B \tag{48}$$

Example 7. Let $A=\tilde{2}\div\tilde{2}$ and $B=\tilde{2}$ be normal convex fuzzy numbers in (23) and (3), respectively, then $\tilde{2}\div\tilde{2} \sqcup \tilde{2} = \tilde{2}$ and $\tilde{2}\div\tilde{2} \sqcap \tilde{2} = \tilde{2}\div\tilde{2}$ are obtained from (46) and (47). Thus, we can have $\tilde{2}\div\tilde{2} \sqsubseteq \tilde{2}$ by (48). But, for $A=\tilde{2}+\tilde{2}$ and $B=\tilde{2}\times\tilde{2}$, it follows from (44) and (45) that $A \sqcap B \neq A$ and $A \sqcup B \neq B$. So $\tilde{2}+\tilde{2}$ and $\tilde{2}\times\tilde{2}$ are incomparable.

Next, we shall obtain the algebraic properties of fuzzy numbers under the operations of $\sqcup$ and $\sqcap$ combined with the operations of +, -, x and ÷.

[Theorem 15] Let A, B and C be convex fuzzy numbers, then the following properties are obtained.

$$A + (B \sqcup C) = (A + B) \sqcup (A + C) \tag{49}$$

$$A + (B \sqcap C) = (A + B) \sqcap (A + C) \tag{50}$$

$$A - (B \sqcup C) = (A - B) \sqcap (A - C) \tag{51}$$

$$A - (B \sqcap C) = (A - B) \sqcup (A - C) \tag{52}$$

$$(B \sqcup C) - A = (B - A) \sqcup (C - A) \tag{53}$$

$$(B \sqcap C) - A = (B - A) \sqcap (C - A) \tag{54}$$

[Theorem 16] If A, B and C are positive convex fuzzy numbers, then we have

$$A \times (B \sqcup C) = (A \times B) \sqcup (A \times C) \tag{55}$$

$$A \times (B \sqcap C) = (A \times B) \sqcap (A \times C) \tag{56}$$

$$A \div (B \sqcup C) = (A \div B) \sqcap (A \div C) \tag{57}$$

$$A \div (B \sqcap C) = (A \div B) \sqcup (A \div C) \tag{58}$$

$$(B \sqcup C) \div A = (B \div A) \sqcup (C \div A) \tag{59}$$

$$(B \sqcap C) \div A = (B \div A) \sqcap (C \div A) \tag{60}$$

It is interesting to note in the above two theorems that, more precisely, if A is convex, then Eqs.(49)-(60) do hold even if B and C are non-convex, and that, conversely, if A is not convex, then they do not hold in general even if B and C are convex.

Example 8. Let A be a positive non-convex fuzzy number and let B and C be positive convex fuzzy numbers such that

$$A = \int_0^{0.5} 1 - 2x/x + \int_{0.5}^1 2x - 1/x$$

$$B = \int_0^{0.5} 2x/x + \int_{0.5}^1 1/x$$

$$C = \int_0^{0.5} 2x/x + \int_{0.5}^1 2(1 - x)/x$$

Then

$$A \times (B \sqcap C) = \int_0^{\frac{3}{16}} \frac{3-\sqrt{1+16x}}{2}/x + \int_{\frac{3}{16}}^{0.5} \frac{\sqrt{1+16x} - 1}{2}/x + \int_{0.5}^1 2(1 - x)/x$$

$$(A \times B) \sqcap (A \times C) = \int_0^{u_0} 1 - 2x/x + \int_{u_0}^{0.5} \frac{\sqrt{1+16x} - 1}{2}/x + \int_{0.5}^1 2(1 - x)/x$$

where $u_0 = \frac{5-\sqrt{17}}{4}$. Thus, (56) is not satisfied.

[Theorem 17] Convex fuzzy numbers form a commutative semiring with zero +∞ and identity 0 under $\sqcap$ (as addition) and + (as multiplication). Similarly, they also form a commutative semiring with zero -∞ and 0 under $\sqcup$ and +. Positive convex fuzzy numbers form a commutative semiring with zero -∞ and identity 1 under $\sqcup$ and x. The same holds under $\sqcap$ and x, where zero is +∞ and identity is 1.

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