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Some Properties of Fuzzy Sets of Type 2

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The concept of fuzzy sets of type 2 has been defined by L. A. Zadeh as an extension of ordinary fuzzy sets. The fuzzy set of type 2 can be characterized by a fuzzy membership function the grade (or fuzzy grade) of which is a fuzzy set in the unit interval [0, 1] rather than a point in [0, 1].

This paper investigates the algebraic structures of fuzzy grades under the operations of join $\sqcup$, meet $\sqcap$, and negation $\neg$ which are defined by using the extension principle, and shows that convex fuzzy grades form a commutative semiring and normal convex fuzzy grades form a distributive lattice under $\sqcup$ and $\sqcap$. Moreover, the algebraic properties of fuzzy grades under the operations $\bar{\sqcup}$ and $\bar{\sqcap}$ which are slightly different from $\sqcup$ and $\sqcap$, respectively, are briefly discussed.

1. INTRODUCTION

Since Zadeh (1965) formulated the concept of fuzzy sets which can deal with ill-defined objects, a number of researchers are engaged in the studies on fuzzy sets and their applications to automata, languages, pattern recognitions, decision making, logic, control, and so on. Against these many applications of fuzzy sets, the theoretical considerations on fuzzy sets theory are also earnestly studied by some fuzzy theorists such as Goguen (1967), Brown (1971), and DeLuca and Termini (1972). In addition to these studies, Zadeh (1973, 1974) recently proposed the concept of fuzzy sets of type 2 as an extension of fuzzy sets. Using the concept of fuzzy sets of type 2, fuzzy linguistic logic (Zadeh, 1973, 1974) and fuzzy–fuzzy automata and grammars (Mizumoto and Tanaka, 1974) are formulated.

A fuzzy set of type 2 is defined by a fuzzy membership function, the grade (that is, fuzzy grade) of which is a fuzzy set in the unit interval [0, 1] rather than a point in [0, 1]. In the definition of ordinary fuzzy sets, the range of membership function is [0, 1] and the operations to the grades of fuzzy sets are "max" and "min." As is well known, the interval [0, 1]

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forms a linear ordered set or a distributive lattice under max and min. Thus, ordinary fuzzy sets form a distributive lattice (Zadeh, 1965).

In this paper we investigate the algebraic structures of fuzzy grades (in other words, fuzzy sets of type 2) under the operations of join $\sqcup$, meet $\sqcap$, and negation $\neg$ for fuzzy grades. Main results are: Convex fuzzy grades form a commutative semiring; normal convex fuzzy grades form a distributive lattice.

Furthermore, the algebraic properties of fuzzy grades under the operations of join $\sqcup$ and meet $\sqcup$, which are defined differently than $\sqcup$ and $\sqcap$, are briefly discussed.

2. Fuzzy Sets of Type 2

We shall briefly review some of the basic definitions relating to ordinary fuzzy sets for the purpose of fuzzy sets of type 2, which are discussed later.

Fuzzy Sets. A *fuzzy set* A in a set X is characterized by a membership function μ_A which takes the values in the interval [0, 1], i.e.,

$$\mu_A: X \to [0, 1]. \tag{1}$$

The value of μ_A at x, $\mu_A(x)$, represents the *grade of membership* (*grade*, for short) of x in A and is a point in [0, 1].

A fuzzy set A is represented as follows.

$$\begin{aligned} A &= \mu_A(x_1)/x_1 + \mu_A(x_2)/x_2 + \cdots + \mu_A(x_n)/x_n \\ &= \sum_i \mu_A(x_i)/x_i, \qquad x_i \in X, \end{aligned} \tag{2}$$

where the operation $+$ stands for logical sum (or).

The operations of fuzzy sets are defined as follows.

Containment.

$$A \subseteq B \Leftrightarrow \mu_A(x) \leqslant \mu_B(x), \qquad \forall x \in X; \tag{3}$$

Union.

$$A \cup B \Leftrightarrow \mu_{A \cup B}(x) = \mu_A(x) \vee \mu_B(x); \tag{4}$$

Intersection.

$$A \cap B \Leftrightarrow \mu_{A \cap B}(x) = \mu_A(x) \wedge \mu_B(x); \tag{5}$$

Complement.

$$\bar{A} \Leftrightarrow \mu_{\bar{A}}(x) = 1 - \mu_A(x); \tag{6}$$

where $\vee$ stands for max and $\wedge$ stands for min.

The grades for ordinary fuzzy sets are easily proved to satisfy the following properties.

$$\mu_A \leqslant \mu_A \quad (\textit{reflexive law}). \tag{7}$$

$$\mu_A \leqslant \mu_B, \quad \mu_B \leqslant \mu_A \Rightarrow \mu_A = \mu_B \quad (\textit{antisymmetric law}). \tag{8}$$

$$\mu_A \leqslant \mu_B, \quad \mu_B \leqslant \mu_C \Rightarrow \mu_A \leqslant \mu_C \quad (\textit{transitive law}). \tag{9}$$

$$\left.\begin{aligned} \mu_A \vee \mu_A &= \mu_A \\ \mu_A \wedge \mu_A &= \mu_A \end{aligned}\right\} \quad (\textit{idempotent laws}). \tag{10}$$

$$\left.\begin{aligned} \mu_A \vee \mu_B &= \mu_B \vee \mu_A \\ \mu_A \wedge \mu_B &= \mu_B \wedge \mu_A \end{aligned}\right\} \quad (\textit{commutative laws}). \tag{11}$$

$$\left.\begin{aligned} (\mu_A \vee \mu_B) \vee \mu_C &= \mu_A \vee (\mu_B \vee \mu_C) \\ (\mu_A \wedge \mu_B) \wedge \mu_C &= \mu_A \wedge (\mu_B \wedge \mu_C) \end{aligned}\right\} \quad (\textit{associative laws}). \tag{12}$$

$$\left.\begin{aligned} \mu_A \wedge (\mu_A \vee \mu_B) &= \mu_A \\ \mu_A \vee (\mu_A \wedge \mu_B) &= \mu_A \end{aligned}\right\} \quad (\textit{absorption laws}). \tag{13}$$

$$\left.\begin{aligned} \mu_A \wedge (\mu_B \vee \mu_C) &= (\mu_A \wedge \mu_B) \vee (\mu_A \wedge \mu_C) \\ \mu_A \vee (\mu_B \wedge \mu_C) &= (\mu_A \vee \mu_B) \wedge (\mu_A \vee \mu_C) \end{aligned}\right\} \quad (\textit{distributive laws}). \tag{14}$$

$$\mu_{\bar{\bar{A}}} = \mu_A \quad (\textit{involution law}). \tag{15}$$

$$\left.\begin{aligned} \overline{\mu_A \vee \mu_B} &= \mu_{\bar{A}} \wedge \mu_{\bar{B}} \\ \mu_A \wedge \mu_B &= \mu_{\bar{A}} \vee \mu_{\bar{B}} \end{aligned}\right\} \quad (\textit{De Morgan's laws}) \tag{16}$$

$$\mu_A \vee 0 = \mu_A, \quad \mu_A \wedge 1 = \mu_A \quad (\textit{identity laws}) \tag{17}$$

$$\mu_A \vee 1 = 1, \quad \mu_A \wedge 0 = 0 \tag{18}$$

$$\left.\begin{aligned} \mu_A \vee \mu_{\bar{A}} &\neq 1 \\ \mu_A \wedge \mu_{\bar{A}} &\neq 0 \end{aligned}\right\} \quad (\textit{failure of complement laws}). \tag{19}$$

From the above properties of grades for fuzzy sets, grades constitute a distributive lattice under $\vee$, $\wedge$, $^-$, but do not form a Boolean lattice because of the failure of complement laws (19). The same holds for ordinary fuzzy sets.

In the above definition of ordinary fuzzy sets, the grades take the values in the unit interval [0, 1]. In reality, however, we often encounter the situation that the grade itself is frequently ill-defined, as in the statement that the

grade is "high," "low," "about 0.8," "middle," "not high," or "very low." To explain this fact Zadeh (1973, 1974) formulated a fuzzy set of type 2 whose grade is a fuzzy set in the interval [0, 1] rather than a point in [0, 1].

Fuzzy sets of type 2. A *fuzzy set of type* 2 A in a set X is the fuzzy set which is characterized by a *fuzzy membership function* μ_A as

$$\mu_A: X \to [0, 1]^J, \tag{20}$$

with the value $\mu_A(x)$ being called a *fuzzy grade* and being a fuzzy set in [0, 1] (or in the subset J of [0, 1]).

Note. In this paper it is assumed that J is a finite set. However, the algebraic properties of fuzzy grades in J discussed later are satisfied in the case where J is continuous.

Note. Since the grade of fuzzy set of type 2 is a fuzzy set in $J \subseteq [0, 1]$, the ordinary fuzzy set is renamed as a *fuzzy set of type* 1. By analogy with this we can define a *fuzzy set of type* n ($n = 1, 2,...$) by the following:

$$\mu_A : X \to [0, 1]^{J_1 \cdot^{\cdot^{\cdot^{J_{n-1}}}}} \tag{21}$$

where $J_1, J_2,..., J_{n-1}$ are the subsets of [0, 1].

EXAMPLE 1. Suppose that $X = \{\text{Susie, Betty, Helen, Ruth, Pat}\}$ is a set of women and that A is a fuzzy set of type 2 of *beautiful* women in X. Then we may have

$$\begin{aligned} A = \textit{beautiful} = \textit{middle}/\text{Susie} + \textit{not low}/\text{Betty} + \textit{low}/\text{Helen} \\ + \textit{very high}/\text{Ruth} + \textit{high}/\text{Pat}, \end{aligned} \tag{22}$$

where the fuzzy grades labeled *middle*, *low*, *high* are assumed to be fuzzy sets in $J = \{0, 0.1,..., 0.9, 1\} \subseteq [0, 1]$ and, for example, are expressed as follows.

$$\textit{middle} = 0.3/0.3 + 0.7/0.4 + 1/0.5 + 0.7/0.6 + 0.3/0.7. \tag{23}$$

$$\textit{low} = 1/0 + 0.9/0.1 + 0.7/0.2 + 0.4/0.3. \tag{24}$$

$$\textit{high} = 0.4/0.7 + 0.7/0.8 + 0.9/0.9 + 1/1. \tag{25}$$

Moreover, fuzzy grades named *not low* and *very high* are defined from fuzzy grades *low* and *high* by using the concept of linguistic hedges (Zadeh, 1972).[1]

$$\textit{not low} = 0.1/0.1 + 0.3/0.2 + 0.6/0.3 + 1/(0.4 + 0.5 + \cdots + 1). \quad (26)$$

$$\textit{very high} = 0.16/0.7 + 0.49/0.8 + 0.81/0.9 + 1/1. \quad (27)$$

It should be noted that *not low* and ¬ *low*, which is defined later, are a different concept.

The operations of fuzzy sets of type 2 are defined by using the extension principle by Zadeh (1973).[2]

Let $\mu_A(x)$ and $\mu_B(x)$ be two fuzzy grades (that is, fuzzy sets in $J \subseteq [0, 1]$) of fuzzy sets of type 2, A and B, respectively, represented as

$$\begin{aligned}\mu_A(x) &= f(u_1)/u_1 + f(u_2)/u_2 + \cdots + f(u_n)/u_n \\ &= \sum_i f(u_i)/u_i, \qquad u_i \in J,\end{aligned} \quad (28)$$

$$\begin{aligned}\mu_B(x) &= g(w_1)/w_1 + g(w_2)/w_2 + \cdots + g(w_m)/w_m \\ &= \sum_j g(w_j)/w_j, \qquad w_j \in J,\end{aligned} \quad (29)$$

where the functions f and g are membership functions of fuzzy grades (fuzzy sets in $J \subseteq [0, 1]$) $\mu_A(x)$ and $\mu_B(x)$, respectively, and the values $f(u_i)$ and $g(w_j)$ in $[0, 1]$ represent the grades for u_i and w_j in J, respectively.

Thus the operations for fuzzy sets of type 2 are expressed by the following.

Union.

$$\begin{aligned}A \cup B \Leftrightarrow \mu_{A\cup B}(x) &= \mu_A(x) \sqcup \mu_B(x) \\ &= \left(\sum_i f(u_i)/u_i\right) \sqcup \left(\sum_j g(w_j)/w_j\right) \\ &= \sum_{i,j} (f(u_i) \wedge g(w_j))/(u_i \vee w_j).\end{aligned} \quad (30)$$

[1] Generally, let $A = \sum_i \mu_A(x_i)/x_i$ be a fuzzy set, then we have: not $A = \sum_i (1 - \mu_A(x_i))/x_i$, very $A = A^2 = \sum_i \mu_A(x_i)^2/x_i$.

[2] In general, let $A = \sum_i \mu_A(x_i)/x_i$ and $B = \sum_j \mu_B(x_j)/x_j$ be two fuzzy sets in X and let $*$ be a binary operation defined in X. Then the operation $*$ can be extended to fuzzy sets A and B by the defining relation (the *extension principle*).

$$\begin{aligned}A * B &= (\textstyle\sum_i \mu_A(x_i)/x_i) * (\sum_j \mu_B(x_j)/x_j) \\ &= \textstyle\sum_{i,j} (\mu_A(x_i) \wedge \mu_B(x_j))/(x_i * x_j),\end{aligned}$$

where $\wedge$ stands for min.

Intersection.

$$A \cap B \Leftrightarrow \mu_{A\cap B}(x) = \mu_A(x) \sqcap \mu_B(x)$$
$$= \left(\sum_i f(u_i)/u_i\right) \sqcap \left(\sum_j g(w_j)/w_j\right)$$
$$= \sum_{i,j} (f(u_i) \wedge g(w_j))/(u_i \wedge w_j). \tag{31}$$

Complement.

$$\bar{A} \Leftrightarrow \mu_{\bar{A}}(x) = \neg\mu_A(x)$$
$$= \sum_i f(u_i)/(1 - u_i), \tag{32}$$

where $\vee$ and $\wedge$ represent max and min, respectively. We call the operations for fuzzy grades, that is, $\sqcup$ as *join*, $\sqcap$ as *meet*, and $\neg$ as *negation* hereafter.

Remark. The grade, say, $\mu_A(x) = 0.8$ of ordinary fuzzy set can be represented as $\mu_A(x) = 1/0.8$ by using the notation for fuzzy grades. Therefore we can see that the above operations for fuzzy grades are an extension of those of grades for ordinary fuzzy sets.

EXAMPLE 2. Let $J = \{0, 0.1,..., 0.9, 1\}$ and let fuzzy grades $\mu_A(x)$ and $\mu_B(x)$ be given as

$$\mu_A(x) = 0.5/0 + 0.7/0.1 + 0.3/0.2,$$
$$\mu_B(x) = 0.9/0 + 0.6/0.1 + 0.2/0.2.$$

Then we have

$$\mu_A(x) \sqcup \mu_B(x) = (0.5/0 + 0.7/0.1 + 0.3/0.2) \sqcup (0.9/0 + 0.6/0.1 + 0.2/0.2)$$
$$= \frac{0.5 \wedge 0.9}{0 \vee 0} + \frac{0.5 \wedge 0.6}{0 \vee 0.1} + \frac{0.5 \wedge 0.2}{0 \vee 0.2}$$
$$+ \frac{0.7 \wedge 0.9}{0.1 \vee 0} + \frac{0.7 \wedge 0.6}{0.1 \vee 0.1} + \frac{0.7 \wedge 0.2}{0.1 \vee 0.2}$$
$$+ \frac{0.3 \wedge 0.9}{0.2 \vee 0} + \frac{0.3 \wedge 0.6}{0.2 \vee 0.1} + \frac{0.3 \wedge 0.2}{0\,2 \vee 0.2}$$
$$= 0.5/0 + 0.5/0.1 + 0.2/0.2 + 0.7/0.1 + 0.6/0.1 + 0.2/0.2$$
$$+ 0.3/0.2 + 0.3/0.2 + 0.2/0.2$$
$$= 0.5/0 + (0.5 \vee 0.7 \vee 0.6)/0.1$$
$$+ (0.2 \vee 0.2 \vee 0.3 \vee 0.3 \vee 0.2)/0.2$$
$$= 0.5/0 + 0.7/0.1 + 0.3/0.2.$$

Similarly, we have

$$\mu_A(x) \sqcap \mu_B(x) = 0.7/0 + 0.6/0.1 + 0.2/0.2,$$
$$\neg\mu_A(x) = 0.5/1 + 0.7/0.9 + 0.3/0.8.$$

Remark. As fuzzy grades are fuzzy sets in $J \subseteq [0, 1]$, we can obtain $\mu_A(x) \cup \mu_B(x)$, $\mu_A(x) \cap \mu_B(x)$ and $\overline{\mu_A(x)}$ from (4), (5), and (6), respectively. However, it should be noted that the operation of $\cup$ is different from that of $\sqcup$. The same holds for $\cap$ to $\sqcap$, and $^-$ to $\neg$. For example, from the above example 2, it is obtained that

$$\mu_A(x) \cup \mu_B(x) = 0.9/0 + 0.7/0.1 + 0.3/0.2,$$
$$\mu_A(x) \cap \mu_B(x) = 0.5/0 + 0.6/0.1 + 0.2/0.2,$$
$$\overline{\mu_A(x)} = 0.5/0 + 0.3/0.1 + 0.7/0.2 + 1/(0.3 + 0.4 + \cdots + 1).$$

Thus we find that $\mu_A(x) \cup \mu_B(x)$ is not coincident with $\mu_A(x) \sqcup \mu_B(x)$. The same is true for $\mu_A(x) \cap \mu_B(x)$ and $\mu_A(x) \sqcap \mu_B(x)$, and $\overline{\mu_A(x)}$ and $\neg\mu_A(x)$.

3. Algebraic Structures of Fuzzy Grades under $\sqcup$, $\sqcap$, and $\neg$

From the fact that the algebraic structures of fuzzy sets of type 2 under the operations of union $\cup$, intersection $\cap$, and complement $^-$ are dependent on the algebraic structures of fuzzy grades under the operations of join $\sqcup$, meet $\sqcap$, and negation $\neg$, we shall discuss what kinds of algebraic structures fuzzy grades form under $\sqcup$, $\sqcap$, and $\neg$.

It should be noted that the grades of ordinary fuzzy sets form a distributive lattice and that the fuzzy grades of fuzzy sets of type 2 form a distributive lattice under $\cup$, $\cap$, and $^-$ of (4), (5), (6).

We shall start from the following theorem.[3]

Theorem 1. *Under the operations* $\sqcup$, $\sqcap$, $\neg$ *in* (30), (31), (32), *arbitrary fuzzy grades in* J *satisfy such laws as idempotent laws* (10), *commutative laws*

[3] The satisfaction or failure of each law for fuzzy grades will be discussed by making reference to (10)–(19), so the readers should read them by replacing $\vee$ by $\sqcup$, $\wedge$ by $\sqcap$, and $^-$ by $\neg$.

(11), *associative laws* (12), *involution laws* (15), *De Morgan's laws* (16); *but do not satisfy absorption laws* (13), *distributive laws* (14), *identity law* (18), *complement laws* (19). *The part* (17) *of identity laws, however, is satisfied, i.e.,*

$$\mu_A \sqcap 1 = \mu_A \qquad \textit{and} \qquad \mu_A \sqcup 0 = \mu_A \tag{33}$$

are satisfied.[4]

EXAMPLE 3. At first we shall show the examples of fuzzy grades which do not satisfy the absorption laws, the distributive laws, the identity law (18), and the complement laws.

Failure of absorption laws. Let μ_A and μ_B be convex fuzzy grades (which are defined later) as

$$\mu_A = 0.3/0 + 0.4/0.1 + 0.6/0.2 + 0.8/0.3 + 0.9/0.4,$$
$$\mu_B = 0.1/0 + 0.2/0.1 + 0.3/0.2 + 0.4/0.3 + 0.5/0.4,$$

where J is $\{0, 0.1, 0.2, 0.3, 0.4\}$. Then

$$\begin{aligned} \mu_A \sqcap (\mu_A \sqcup \mu_B) &= \mu_A \sqcup (\mu_A \sqcap \mu_B)\ ^{5} \\ &= 0.3/0 + 0.4/0.1 + 0.5/0.2 + 0.5/0.3 + 0.5/0.4 \\ &\neq \mu_A . \end{aligned} \tag{34}$$

Failure of distributive laws.

$$\mu_A = 0.9/0.1 + 0.2/0.2 + 0.1/0.3 + 0.8/0.4,$$
$$\mu_B = 0.4/0.1 + 0.5/0.2 + 0.6/0.3 + 0.3/0.4,$$
$$\mu_C = 0.2/0.1 + 0.3/0.2 + 0.6/0.3 + 0.8/0.4.$$

Then we have

$$\mu_A \sqcap (\mu_B \sqcup \mu_C) = 0.6/0.1 + 0.3/0.2 + 0.6/0.3 + 0.6/0.4,$$
$$(\mu_A \sqcap \mu_B) \sqcup (\mu_A \sqcap \mu_C) = 0.6/0.1 + 0.5/0.2 + 0.6/0.3 + 0.6/0.4.$$

Thus, we see that the distributive law is not satisfied. The same holds for the case in which $\sqcup$ and $\sqcap$ are interchanged.

[4] We shall hereafter abbreviate $\mu_A(x)$ as μ_A for simplicity.

[5] The expression of (34), i.e., $\mu_A \sqcap (\mu_A \sqcup \mu_B) = \mu_A \sqcup (\mu_B \sqcap \mu_B)$, is satisfied for any fuzzy grades in spite of the failure of the absorption laws. See the proof of Theorem 1 denoted later.

Failure of a part of identity laws. Let

$$\mu_A = 0.7/0.1 + 0.5/0.2 + 0.8/0.3.$$

Then, noting that the numbers 1 and 0 are represented as 1/1 and 1/0, respectively, we obtain

$$\mu_A \sqcup 1 = 0.8/1 \neq 1,$$
$$\mu_A \sqcap 0 = 0.8/0 \neq 0.$$

Thus, it is shown that part (18) of the identity laws is not satisfied.

Failure of complement laws. Let

$$\mu_A = 0.8/0.1 + 1/0.2 + 0.5/0.3,$$

then the negation of μ_A is given as

$$\neg\mu_A = 0.8/0.9 + 1/0.8 + 0.5/0.7.$$

Thus we have

$$\mu_A \sqcup (\neg\mu_A) = 0.8/0.9 + 1/0.8 + 0.5/0.7 \neq 1,$$
$$\mu_A \sqcap (\neg\mu_A) = 0.8/0.1 + 1/0.2 + 0.5/0.3 \neq 0.$$

Next we shall prove Theorem 1 saying that the idempotent laws, commutative laws, associative laws, involution laws, De Morgan's laws, another part (33) of identity laws, and (34) are all satisfied for arbitrary fuzzy grades.

Idempotent laws. We shall prove $\mu_A \sqcup \mu_A = \mu_A$. Let μ_A be represented as

$$\mu_A = \sum_i f(u_i)/u_i, \qquad u_i \in J.$$

Then from (30), $\mu_A \sqcup \mu_A$ is given as follows.[6]

$$\mu_A \sqcup \mu_A = \sum_{i,j} f(u_i) \cdot f(u_j)/u_i \vee u_j.$$

Let $h(a)$ be the grade of $a \in J$ in $\mu_A \sqcup \mu_A$, then $h(a)$ will be represented as[7]

$$h(a) = \sum_{u_i \vee u_j = a} f(u_i) \cdot f(u_j). \tag{35}$$

[6] In this and the subsequent proofs, we use $a + b$ or $a \vee b$ for $\max\{a, b\}$, and $a \cdot b$ or $a \wedge b$ for $\min\{a, b\}$. In addition the notation $\sum_i a_i$ represents $a_1 + a_2 + \cdots + a_n$ (or $a_1 \vee a_2 \vee \cdots \vee a_n$). Furthermore, the laws which illustrate the derivation of expressions are laws for ordinary grades (not fuzzy grades).

[7] In other words, $\mu_A \sqcup \mu_A = \sum_{a \in J} h(a)/a$.

By the way, u_i and u_j, which satisfy $u_i \vee u_j = a$, are given as

$$\begin{Bmatrix} u_i = a \\ u_j \leqslant a \end{Bmatrix} \quad \text{or} \quad \begin{Bmatrix} u_i \leqslant a \\ u_j = a \end{Bmatrix}.$$

Thus, (35) will be expressed as

$$\begin{aligned} h(a) &= \sum_{\substack{u_i = a \\ u_j \leqslant a}} f(u_i) \cdot f(u_j) + \sum_{\substack{u_j = a \\ u_i \leqslant a}} f(u_i) \cdot f(u_j) \\ &= f(a) \cdot \sum_{u_j \leqslant a} f(u_j) + f(a) \cdot \sum_{u_i \leqslant a} f(u_i) \\ &= f(a) + f(a) \cdots \text{from absorption law } b \cdot (b + b_1 + \cdots) = b \\ &= f(a). \end{aligned}$$

Therefore we have $\mu_A \sqcup \mu_A = \mu_A$ from $h(a) = f(a)$ for any a in J.
The same holds for $\mu_A \sqcap \mu_A = \mu_A$.

Commutative laws. $\mu_A \sqcup \mu_B = \mu_B \sqcup \mu_A$.
Let μ_A and μ_B be

$$\mu_A = \sum_i f(u_i)/u_i, \qquad \mu_B = \sum_j g(u_j)/u_j; \tag{36}$$

then we have

$$\begin{aligned} \mu_A \sqcup \mu_B &= \left(\sum_i f(u_i)/u_i\right) \sqcup \left(\sum_j g(u_j)/u_j\right) \\ &= \sum_{i,j} f(u_i) \cdot g(u_j)/u_i \vee u_j \\ &= \sum_{i,j} g(u_j) \cdot f(u_i)/u_j \vee u_i \\ &= \left(\sum_j g(u_j)/u_j\right) \sqcup \left(\sum_i f(u_i)/u_i\right) \\ &= \mu_B \sqcup \mu_A. \end{aligned}$$

The same holds for $\mu_A \sqcap \mu_B = \mu_B \sqcap \mu_A$.

Involution law. $\neg\neg\mu_A = \mu_A$. From (32) we have

$$\neg\mu_A = \sum f(u_i)/(1 - u_i).$$

Thus,

$$\neg(\neg\mu_A) = \sum_i f(u_i)/(1 - (1 - u_i))$$

$$= \sum_i f(u_i)/u_i = \mu_A .$$

Associative laws. $\mu_A \cup (\mu_B \cup \mu_C) = (\mu_A \cup \mu_B) \cup \mu_C$. Let μ_A and μ_B be as in (36) and let $\mu_C = \sum_k h(u_k)/u_k$, then we obtain that

$$\mu_A \cup (\mu_B \cup \mu_C) = \left(\sum_i f(u_i)/u_i\right) \cup \left(\sum_{j,k} g(u_j) \cdot h(u_k)/u_j \vee u_k\right)$$

$$= \sum_{i,j,k} f(u_i) \cdot (g(u_j) \cdot h(u_k))/u_i \vee (u_j \vee u_k)$$

$$= \sum_{i,j,k} (f(u_i) \cdot g(u_j)) \cdot h(u_k)/(u_i \vee u_j) \vee u_k$$

$$= \left(\sum_{i,j} f(u_i) \cdot g(u_j)/u_i \vee u_j\right) \cup \left(\sum_k h(u_k)/u_k\right)$$

$$= (\mu_A \cup \mu_B) \cup \mu_C .$$

The same holds for $\mu_A \cap (\mu_B \cap \mu_C) = (\mu_A \cap \mu_B) \cap \mu_C$.

De Morgan's laws. $\neg(\mu_A \cup \mu_B) = (\neg\mu_A) \cap (\neg\mu_B)$.

$$\neg(\mu_A \cup \mu_B) = \neg\left(\sum_{i,j} f(u_i) \cdot g(u_j)/u_i \vee u_j\right)$$

$$= \sum_{i,j} f(u_i) \cdot g(u_j)/1 - (u_i \vee u_j)$$

$$= \sum_{i,j} f(u_i) \cdot g(u_j)/(1 - u_i) \wedge (1 - u_j)$$

$$= \left(\sum_i f(u_i)/(1 - u_i)\right) \cap \left(\sum_j g(u_j)/(1 - u_j)\right)$$

$$= (\neg\mu_A) \cap (\neg\mu_B).$$

The same holds for $\neg(\mu_A \cap \mu_B) = (\neg\mu_A) \cup (\neg\mu_B)$.

Another part of identity laws. $\mu_A \sqcap 1 = \mu_A$.

$$\begin{aligned}\mu_A \sqcap 1 &= \left(\sum_i f(u_i)/u_i\right) \sqcap 1/1 \\ &= \sum_i (f(u_i) \wedge 1)/(u_i \wedge 1) = \sum_i f(u_i)/u_i \\ &= \mu_A .\end{aligned}$$

The same holds for $\mu_A \sqcup 0 = \mu_A$.

Proof of (34). $\mu_A \sqcap (\mu_A \sqcup \mu_B) = \mu_A \sqcup (\mu_A \sqcap \mu_B)$.

$\mu_A \sqcap (\mu_A \sqcup \mu_B)$ is given as follows by omitting the subscripts of u_i, w_j, z_k for simplicity.

$$\mu_A \sqcap (\mu_A \sqcup \mu_B) = \sum_{u,w,z \in J} f(u) \cdot f(w) \cdot g(z)/u \cdot (w \vee z).$$

Let $p(a)$ be the grade of $a \in J$ in $\mu_A \sqcap (\mu_A \sqcup \mu_B)$, then $p(a)$ can be represented as

$$p(a) = \sum_{u \cdot (w \vee z) = a} f(u) \cdot f(w) \cdot g(z). \tag{38}$$

Furthermore, u, w, $z \in J$ which satisfy $u \cdot (w \vee z) = a$ can be divided into four parts, that is,

$$\begin{Bmatrix} u = a \\ w \geqslant a \\ z : \text{free} \end{Bmatrix} \quad \text{or} \quad \begin{Bmatrix} u = a \\ w : \text{free} \\ z \geqslant a \end{Bmatrix} \quad \text{or} \quad \begin{Bmatrix} u \geqslant a \\ w = a \\ z \leqslant a \end{Bmatrix} \quad \text{or} \quad \begin{Bmatrix} u \geqslant a \\ w \leqslant a \\ z = a \end{Bmatrix}. \tag{39}$$

Thus (38) is expressed as follows in view of independency of u, w, and z in each part.

$$\begin{aligned} p(a) = &\sum_{\substack{u=a \\ w \geqslant a \\ z}} f(u) \cdot f(w) \cdot g(z) + \sum_{\substack{u=a \\ w \\ z \geqslant a}} f(u) \cdot f(w) \cdot g(z) \\ &+ \sum_{\substack{u \geqslant a \\ w=a \\ z \leqslant a}} f(u) \cdot f(w) \cdot g(z) + \sum_{\substack{u \geqslant a \\ w \leqslant a \\ z=a}} f(u) \cdot f(w) \cdot g(z) \\ = &f(a) \cdot \sum_{w \geqslant a} f(w) \cdot \sum_z g(z) + f(a) \cdot \sum_w f(w) \cdot \sum_{z \geqslant a} g(z) \\ &+ f(a) \cdot \sum_{u \geqslant a} f(u) \cdot \sum_{z \leqslant a} g(z) + \sum_{u \geqslant a} f(u) \cdot \sum_{w \leqslant a} f(w) \cdot g(a) \end{aligned}$$

$$= f(a) \cdot \sum_{z} g(z) + f(a) \cdot \sum_{z \geqslant a} g(z)$$

(··· from absorption laws)

$$+ f(a) \cdot \sum_{z \leqslant a} g(z) + \sum_{u \geqslant a} f(u) \cdot \sum_{w \leqslant a} f(w) \cdot g(a)$$

$$= f(a) \cdot \sum_{z} g(z) + \sum_{u \geqslant a} f(u) \cdot \sum_{w \leqslant a} f(w) \cdot g(a). \tag{40}$$

In a similar way, $\mu_A \sqcup (\mu_A \sqcap \mu_B)$ is as follows.

$$\mu_A \sqcup (\mu_A \sqcap \mu_B) = \sum_{u,w,z \in J} f(u) \cdot f(w) \cdot g(z)/u \vee (w \cdot z). \tag{41}$$

Let $q(a)$ be the grade of $a \in J$ in $\mu_A \sqcup (\mu_A \sqcap \mu_B)$, then $q(a)$ is

$$q(a) = \sum_{uv(w \cdot z) = a} f(u) \cdot f(w) \cdot g(z). \tag{42}$$

u, w, z in J such that $u \vee (w \cdot z) = a$ are also divided into four parts as

$$\begin{Bmatrix} u = a \\ w \leqslant a \\ z : \text{free} \end{Bmatrix} \quad \text{or} \quad \begin{Bmatrix} u = a \\ w : \text{free} \\ z \leqslant a \end{Bmatrix} \quad \text{or} \quad \begin{Bmatrix} u \leqslant a \\ w = a \\ z \geqslant a \end{Bmatrix} \quad \text{or} \quad \begin{Bmatrix} u \leqslant a \\ w \geqslant a \\ z = a \end{Bmatrix}. \tag{43}$$

Thus, $q(a)$ is as follows.

$$q(a) = f(a) \cdot \sum_{w \leqslant a} f(w) \cdot \sum_{z} g(z) + f(a) \cdot \sum_{w} f(w) \cdot \sum_{z \leqslant a} g(z)$$

$$+ f(a) \cdot \sum_{u \leqslant a} f(u) \cdot \sum_{z \geqslant a} g(z) + \sum_{u \leqslant a} f(u) \cdot \sum_{w \geqslant a} f(w) \cdot g(a)$$

$$= f(a) \cdot \sum_{z} g(z) + f(a) \cdot \sum_{z \leqslant a} g(z)$$

$$+ f(a) \cdot \sum_{z \geqslant a} g(z) + \sum_{u \leqslant a} f(u) \cdot \sum_{w \geqslant a} f(w) \cdot g(a)$$

$$= f(a) \cdot \sum_{z} g(z) + \sum_{u \leqslant a} f(u) \cdot \sum_{w \geqslant a} f(w) \cdot g(a). \tag{44}$$

Therefore, from (40) and (44) we get $p(a) = q(a)$ for any $a \in J$, which implies $\mu_A \sqcap (\mu_A \sqcup \mu_B) = \mu_A \sqcup (\mu_A \sqcap \mu_B)$.

This completes the proof of Theorem 1. ∎

Next we shall define order relations on fuzzy grades and show that arbitrary fuzzy grades form a partially ordered set under the order relation.

THEOREM 2. *If an order relation* $\dot{\sqsubseteq}$ *over arbitrary fuzzy grades in* J *is defined as*

$$\mu_A \dot{\sqsubseteq} \mu_B \Leftrightarrow \mu_A \sqcap \mu_B = \mu_A , \tag{45}$$

then the set of arbitrary fuzzy grades forms a partially ordered set under $\dot{\sqsubseteq}$. *Similarly, let* $\dot{\sqsubseteq}$ *be an order relation given as*

$$\mu_A \dot{\sqsubseteq} \mu_B \Leftrightarrow \mu_A \sqcup \mu_B = \mu_B , \tag{46}$$

then the arbitrary fuzzy grades also form a partially ordered set under $\dot{\sqsubseteq}$. *In general we have* $\dot{\sqsubseteq} \neq \dot{\sqsubseteq}$.

Proof. We shall show that any fuzzy grades in J under $\dot{\sqsubseteq}$ satisfy the reflexive law (7), the antisymmetric law (8), and the transitive law (9). It should be noted that the same holds for $\dot{\sqsubseteq}$ and the inequality of $\dot{\sqsubseteq}$ and $\dot{\sqsubseteq}$ is proved from the failure of the absorption laws of arbitrary fuzzy grades. From the idempotency of fuzzy grades, i.e., $\mu_A \sqcap \mu_A = \mu_A$, we have the reflexive law $\mu_A \dot{\sqsubseteq} \mu_A$. Suppose that $\mu_A \dot{\sqsubseteq} \mu_B$ and $\mu_B \dot{\sqsubseteq} \mu_A$, then from the commutativity of $\sqcap$, the antisymmetric law is obtained, namely,

$$\mu_A = \mu_A \sqcap \mu_B = \mu_B \sqcap \mu_A = \mu_B .$$

Finally, let $\mu_A \dot{\sqsubseteq} \mu_B$ and $\mu_B \dot{\sqsubseteq} \mu_C$, then the transitive law is obtained as follows in view of the associativity of $\sqcap$, i.e.,

$$\mu_A = \mu_A \sqcap \mu_B = \mu_A \sqcap (\mu_B \sqcap \mu_C) = (\mu_A \sqcap \mu_B) \sqcap \mu_C = \mu_A \sqcap \mu_C .$$

Thus we have $\mu_A \dot{\sqsubseteq} \mu_C$.

Hence, arbitrary fuzzy grades in J satisfy the reflexive law, the associative law, and the transitive law under $\dot{\sqsubseteq}$, so the set of arbitrary fuzzy grades in J constitutes a partially ordered set under $\dot{\sqsubseteq}$. ∎

Next we shall define a convex fuzzy grade and a normal fuzzy grade as a special case of fuzzy grades and show that convex fuzzy grades form a commutative semiring and normal convex fuzzy grades form a distributive lattice under $\sqcup$ and $\sqcap$.

Let $J = \{u_1, u_2, ..., u_n\}$ be a subset of $[0, 1]$ which satisfies $u_1 < u_2 <$

$\cdots < u_n$. A fuzzy grade $\mu_A = \sum_i f(u_i)/u_i$ in J is said to be *convex* if for any integers i, k with $i \leqslant k$, the following is satisfied, i.e.,

$$f(u_j) \geqslant \min\{f(u_i), f(u_k)\},^8 \tag{47}$$

where j is any integer which satisfies $i \leqslant j \leqslant k$.

A fuzzy grade μ_A in J is said to be *normal* if

$$\max_i f(u_i) = 1. \tag{48}$$

Otherwise it is *subnormal*. Furthermore, a fuzzy grade which is convex and normal is referred to as a *normal convex* fuzzy grade.

EXAMPLE 4. Various types of fuzzy grades in $J = \{0.1, 0.2, 0.3, 0.4\}$ are listed as follows.

$\mu_A = 0.8/0.1 + 0.3/0.2 + 0.5/0.3 + 0.9/0.4$ (subnormal, nonconvex).
$\mu_A = 0.3/0.1 + 0.6/0.2 + 0.8/0.3 + 0.5/0.4$ (subnormal, convex).
$\mu_A = 0.7/0.1 + 0.2/0.2 + 1/0.3 + 0.3/0.4$ (normal, nonconvex).
$\mu_A = 0.5/0.1 + 0.8/0.2 + 1/0.3 + 0.7/0.4$ (normal, convex).

At first we shall discuss some properties of convex fuzzy grades under $\sqcup$, $\sqcap$, and $\neg$.

THEOREM 3. *If μ_A and μ_B are convex fuzzy grades in J, then $\mu_A \sqcup \mu_B$, $\mu_A \sqcap \mu_B$, and $\neg\mu_A$ are also convex.*

Proof. It is obvious from the definition (32) that $\neg\mu_A$ is convex. We shall show that $\mu_A \sqcup \mu_B$ is convex if μ_A and μ_B are convex. We have the following equations from the assumption that fuzzy grades $\mu_A = \sum_{u\in J} f(u)/u$ and $\mu_B = \sum_{w\in J} g(w)/w$ are convex fuzzy grades in J.

$$f(a_j) \geqslant f(a_i) \cdot f(a_k), \tag{49}$$

$$g(a_j) \geqslant g(a_i) \cdot g(a_k). \tag{50}$$

where a_i, a_j, a_k in J are any numbers such that $a_i \leqslant a_j \leqslant a_k$, and the operation $\cdot$ denotes $\min(\wedge)$.

[8] As we assume that J is a finite ordered set, this definition of convex fuzzy grades (or convex fuzzy sets in J) is a special case of that of convex fuzzy sets by Zadeh (1965).

$\mu_A \sqcup \mu_B$ is given as

$$\mu_A \sqcup \mu_B = \sum_{u,w \in J} f(u) \cdot g(w)/u \vee w. \tag{51}$$

Let $h(a)$ be the grade of $a \in J$ in fuzzy grade $\mu_A \sqcup \mu_B$, then $h(a)$ is

$$h(a) = \sum_{u \vee w = a} f(u) \cdot g(w). \tag{52}$$

u and w which satisfy $u \vee w = a$ are divided into two classes as

$$\{u = a,\ w \leqslant a\} \qquad \text{or} \qquad \{u \leqslant a,\ w = a\}.$$

Thus $h(a)$ is rewritten as

$$h(a) = f(a) \cdot \sum_{w \leqslant a} g(w) + g(a) \cdot \sum_{u \leqslant a} f(u). \tag{53}$$

Therefore, substituting a_j, a_i, a_k into a of $h(a)$, we have three expressions such that

$$h(a_j) = f(a_j) \cdot \sum_{w \leqslant a_j} g(w) + g(a_j) \cdot \sum_{u \leqslant a_j} f(u), \tag{54}$$

$$h(a_i) = f(a_i) \cdot \sum_{w \leqslant a_i} g(w) + g(a_i) \cdot \sum_{u \leqslant a_i} f(u), \tag{55}$$

$$h(a_k) = f(a_k) \cdot \sum_{w \leqslant a_k} g(w) + g(a_k) \cdot \sum_{u \leqslant a_k} f(u). \tag{56}$$

The goal of the proof is to show the following inequality under the assumption $a_i \leqslant a_j \leqslant a_k$.

$$h(a_j) \geqslant h(a_i) \wedge h(a_k). \tag{57}$$

From (55) and (56) we obtain

$$\begin{aligned} h(a_i) \wedge h(a_k) = {} & f(a_i) \cdot f(a_k) \cdot \sum_{w \leqslant a_i} g(w) \cdot \sum_{w \leqslant a_k} g(w) \cdots \text{Ⓐ} \\ & + g(a_i) \cdot g(a_k) \cdot \sum_{u \leqslant a_i} f(u) \cdot \sum_{u \leqslant a_k} f(u) \cdots \text{Ⓑ} \\ & + f(a_i) \cdot \sum_{u \leqslant a_k} f(u) \cdot g(a_k) \cdot \sum_{w \leqslant a_i} g(w) \cdots \text{Ⓒ} \\ & + f(a_k) \cdot \sum_{u \leqslant a_i} f(u) \cdot g(a_i) \cdot \sum_{w \leqslant a_k} g(w) \cdots \text{Ⓓ} \\ = {} & \text{Ⓐ} + \text{Ⓑ} + \text{Ⓒ} + \text{Ⓓ}. \end{aligned}$$

Firstly, from (49) and the assumption $a_i \leqslant a_k$, Ⓐ is as follows.

$$Ⓐ \leqslant f(a_j) \cdot \sum_{w \leqslant a_i} g(w).$$

Similarly, Ⓑ is

$$Ⓑ \leqslant g(a_j) \cdot \sum_{u \leqslant a_i} f(u).$$

Ⓒ is given as follows from the absorption law and the convexity of g (that is, μ_B).[9]

$$Ⓒ = f(a_i) \cdot g(a_k) \cdot \sum_{w \leqslant a_i} g(w) \leqslant f(a_i) \cdot g(a_j).$$

Similarly,

$$Ⓓ \leqslant f(a_j) \cdot g(a_i).$$

Summing up the right-hand sides of the four inequalities obtained above, the following expression is obtained.

$$\begin{aligned} h(a_i) \wedge h(a_k) &= Ⓐ + Ⓑ + Ⓒ + Ⓓ \\ &\leqslant f(a_j) \cdot \sum_{w \leqslant a_i} g(w) + g(a_j) \cdot \sum_{u \leqslant a_i} f(u) + f(a_i) \cdot g(a_j) \\ &\quad + f(a_j) \cdot g(a_i) \\ &= f(a_j) \cdot \sum_{w \leqslant a_i} g(w) + g(a_j) \cdot \sum_{u \leqslant a_i} f(u) \\ &\leqslant f(a_j) \cdot \sum_{w \leqslant a_j} g(w) + g(a_j) \cdot \sum_{u \leqslant a_j} f(u) \cdots \text{from } a_i \leqslant a_j \\ &= h(a_j). \end{aligned}$$

Thus we have shown that $h(a_j) \geqslant h(a_i) \wedge h(a_k)$.

The same holds for the convexity of $\mu_A \sqcap \mu_B$. ∎

THEOREM 4. *For convex fuzzy grades in J, the distributive laws are satisfied, namely,*

$$\mu_A \sqcap (\mu_B \sqcup \mu_C) = (\mu_A \sqcap \mu_B) \sqcup (\mu_A \sqcap \mu_C), \tag{58}$$

$$\mu_A \sqcup (\mu_B \sqcap \mu_C) = (\mu_A \sqcup \mu_B) \sqcap (\mu_A \sqcup \mu_C). \tag{59}$$

[9] If $a_i > a_{i-1} > a_{i-2} > \cdots$ and $a_i \leqslant a_j \leqslant a_k$, then $g(a_j) \geqslant (g(a_i) + g(a_{i-1}) + g(a_{i-2}) + \cdots) \cdot g(a_k)$ is derived from the definition of convexity of (50).

Proof. We shall prove that the distributive law $(\mu_A \sqcap \mu_B) \sqcup (\mu_A \sqcap \mu_C) = \mu_A \sqcap (\mu_B \sqcup \mu_C)$ is satisfied if μ_A, μ_B, μ_C are convex fuzzy grades in J.

Let convex fuzzy grades μ_A, μ_B, and μ_C be

$$\mu_A = \sum_{u \in J} f(u)/u,$$

$$\mu_B = \sum_{w \in J} g(w)/w,$$

$$\mu_C = \sum_{z \in J} h(z)/z.$$

From the definition of $(\mu_A \sqcap \mu_B) \sqcup (\mu_A \sqcap \mu_C)$, we obtain

$$(\mu_A \sqcap \mu_B) \sqcup (\mu_A \sqcap \mu_C) = \sum f(u) \cdot g(w) \cdot f(u') \cdot h(z)/(u \cdot w \vee u' \cdot z). \quad (60)$$

Let $p(a)$ be the grade of $a \in J$ in fuzzy grade $(\mu_A \sqcap \mu_B) \sqcup (\mu_A \sqcap \mu_C)$, then $p(a)$ is as follows.

$$p(a) = \sum_{u \cdot w \vee u' \cdot z = a} f(u) \cdot g(w) \cdot f(u') \cdot h(z). \quad (61)$$

Thus, u, w, u', z which satisfy $u \cdot w \vee u' \cdot z = a$ is

$$\{u \cdot w = a,\ u' \cdot z \leqslant a\} \quad \text{or} \quad \{u' \cdot z = a,\ u \cdot w \leqslant a\}$$

and, more precisely, can be divided into eight classes such as

$$\begin{Bmatrix} u = a \\ w \geqslant a \\ u' \leqslant a \\ z : \text{free} \end{Bmatrix} \text{ or } \begin{Bmatrix} u = a \\ w \geqslant a \\ u' : \text{free} \\ z \leqslant a \end{Bmatrix} \text{ or } \begin{Bmatrix} u \geqslant a \\ w = a \\ u' \leqslant a \\ z : \text{free} \end{Bmatrix} \text{ or } \begin{Bmatrix} u \geqslant a \\ w = a \\ u' : \text{free} \\ z \leqslant a \end{Bmatrix}$$

$$\text{or } \begin{Bmatrix} u \leqslant a \\ w : \text{free} \\ u' = a \\ z \geqslant a \end{Bmatrix} \text{ or } \begin{Bmatrix} u : \text{free} \\ w \leqslant a \\ u' = a \\ z \geqslant a \end{Bmatrix} \text{ or } \begin{Bmatrix} u \leqslant a \\ w : \text{free} \\ u' \geqslant a \\ z = a \end{Bmatrix} \text{ or } \begin{Bmatrix} u : \text{free} \\ w \leqslant a \\ u' \geqslant a \\ z = a \end{Bmatrix}.$$

Therefore, noting that u, w, u', z of each class are mutually independent, we have from (61) that

$$p(a) = \sum_{u \cdot w \vee u' \cdot z = a} f(u) \cdot g(w) \cdot f(u') \cdot h(z)$$

$$= ① + ② + ③ + \boxed{4} + ⑤ + ⑥ + ⑦ + \boxed{8},$$

where ①, ②,..., $\boxed{8}$ are given by the following.

$$① = \sum_{\substack{u \leqslant a \\ w \geqslant a \\ u' \leqslant a \\ z}} f(u) \cdot g(w) \cdot f(u') \cdot h(z)$$

$$= f(a) \cdot \sum_{w \geqslant a} g(w) \cdot \sum_{u' \leqslant a} f(u') \cdot \sum_{z} h(z)$$

$$= f(a) \cdot \sum_{w \geqslant a} g(w) \cdot \sum_{z} h(z) \cdots \text{from the absorption law.}$$

$$② = f(a) \cdot \sum_{w \geqslant a} g(w) \cdot \sum_{u'} f(u') \cdot \sum_{z \leqslant a} h(z)$$

$$= f(a) \cdot \sum_{w \geqslant a} g(w) \cdot \sum_{z \leqslant a} h(z) \cdots \text{from the absorption law.}$$

$$⑨ = ① + ② = f(a) \cdot \sum_{w \geqslant a} g(w) \cdot \sum_{z} h(z).$$

$$③ = g(a) \cdot \sum_{z} h(z) \cdot \sum_{u \geqslant a} f(u) \cdot \sum_{u' \leqslant a} f(u')$$

$$= g(a) \cdot \sum_{z} h(z) \cdot \left[f(a) \cdot \sum_{u' \leqslant a} f(u') + \sum_{u > a} f(u) \cdot \sum_{u' \leqslant a} f(u') \right]$$

$$= g(a) \cdot \sum_{z} h(z) \cdot f(a) \cdots \text{from the absorption law and the convexity of } f.[10]$$

$$\boxed{10} = ⑨ + ③$$

$$= f(a) \cdot \sum_{w \geqslant a} g(w) \cdot \sum_{z} h(z) + f(a) \cdot g(a) \cdot \sum_{z} h(z)$$

$$= f(a) \cdot \sum_{w \geqslant a} g(w) \cdot \sum_{z} h(z).$$

$$\boxed{4} = g(a) \cdot \sum_{z \leqslant a} h(z) \cdot \sum_{u \geqslant a} f(u) \cdot \sum_{u'} f(u')$$

$$= g(a) \cdot \sum_{z \leqslant a} h(z) \cdot \sum_{u \geqslant a} f(u).$$

[10] From the convexity of f (that is, μ_A) we have $\sum_{u > a} f(u) \cdot \sum_{u' \leqslant a} f(u') \leqslant f(a)$.

Moreover,

$$⑤ = \sum_{u \leqslant a} f(u) \cdot \sum_{w} g(w) \cdot f(a) \cdot \sum_{z \geqslant a} h(z)$$

$$= f(a) \cdot \sum_{w} g(w) \cdot \sum_{z \geqslant a} h(z) \cdots \text{from the absorption law.}$$

$$⑥ = \sum_{u} f(u) \cdot \sum_{w \leqslant a} g(w) \cdot f(a) \cdot \sum_{z \geqslant a} h(z)$$

$$= f(a) \cdot \sum_{w \leqslant a} g(w) \cdot \sum_{z \geqslant a} h(z) \cdots \text{from the absorption law.}$$

$$⑪ = ⑤ + ⑥ = f(a) \cdot \sum_{w} g(w) \cdot \sum_{z \geqslant a} h(z).$$

$$⑦ = \sum_{u \leqslant a} f(u) \cdot \sum_{w} g(w) \cdot \sum_{u' \geqslant a} f(u') \cdot h(a)$$

$$= f(a) \cdot \sum_{w} g(w) \cdot h(a) \cdots \text{from the absorption law and the convexity of } f.$$

$$\boxed{12} = ⑪ + ⑦$$

$$= f(a) \cdot \sum_{w} g(w) \cdot \sum_{z \geqslant a} h(z) + f(a) \cdot \sum_{w} g(w) \cdot h(a)$$

$$= f(a) \cdot \sum_{w} g(w) \cdot \sum_{z \geqslant a} h(z).$$

$$\boxed{8} = \sum_{u} f(u) \cdot \sum_{w \leqslant a} g(w) \cdot \sum_{u' \geqslant a} f(u') \cdot h(a)$$

$$= \sum_{u' \geqslant a} f(u') \cdot \sum_{w \leqslant a} g(w) \cdot h(a).$$

Hence, $p(a)$ is

$$p(a) = ① + ② + ③ + \boxed{4} + ⑤ + ⑥ + ⑦ + \boxed{8}$$

$$= \boxed{10} + \boxed{4} + \boxed{12} + \boxed{8}$$

$$= f(a) \cdot \sum_{w \geqslant a} g(w) \cdot \sum_{z} h(z) + g(a) \cdot \sum_{z \leqslant a} h(z) \cdot \sum_{u \geqslant a} f(u)$$

$$+ f(a) \cdot \sum_{w} g(w) \cdot \sum_{z \geqslant a} h(z) + \sum_{u \geqslant a} f(u) \cdot \sum_{w \leqslant a} g(w) \cdot h(a).^{11}$$

On the other hand, $\mu_A \sqcap (\mu_B \sqcup \mu_C)$ is given by

$$\mu_A \sqcap (\mu_B \sqcup \mu_C) = \sum f(u) \cdot g(w) \cdot h(z)/u \cdot (w \vee z).$$

Thus, the grade $q(a)$ of $a \in J$ in $\mu_A \sqcap (\mu_B \sqcup \mu_C)$ is as follows.

$$q(a) = \sum_{u \cdot (w \vee z) = a} f(u) \cdot g(w) \cdot h(z).$$

u, w, z with $u \cdot (w \vee z) = a$ are

$$\begin{Bmatrix} u = a \\ w \geqslant a \\ z : \text{free} \end{Bmatrix} \quad \text{or} \quad \begin{Bmatrix} u = a \\ w : \text{free} \\ z \geqslant a \end{Bmatrix} \quad \text{or} \quad \begin{Bmatrix} u \geqslant a \\ w = a \\ z \leqslant a \end{Bmatrix} \quad \text{or} \quad \begin{Bmatrix} u \geqslant a \\ w \leqslant a \\ z = a \end{Bmatrix}.$$

Thus, $q(a)$ is

$$q(a) = f(a) \cdot \sum_{w \geqslant a} g(w) \cdot \sum_{z} h(z) + f(a) \cdot \sum_{w} g(w) \cdot \sum_{z \geqslant a} h(z)$$

$$+ g(a) \cdot \sum_{u \geqslant a} f(u) \cdot \sum_{z \leqslant a} h(z) + \sum_{u \geqslant a} f(u) \cdot \sum_{w \leqslant a} g(w) \cdot h(a)$$

$$= p(a).$$

This completes the proof of $(\mu_A \sqcap \mu_B) \sqcup (\mu_A \sqcap \mu_C) = \mu_A \sqcap (\mu_B \sqcup \mu_C)$. A similar method is applicable to the proof of $(\mu_A \sqcup \mu_B) \sqcap (\mu_A \sqcup \mu_C) = \mu_A \sqcup (\mu_B \sqcap \mu_C)$. ∎

Note. It is noted that the convexity of fuzzy grades μ_B and μ_C was not used in the process of the above proof in spite of the use of the convexity of μ_A (or f). Therefore, we can say that if μ_A is convex fuzzy grade, then the distributive laws can be satisfied even if μ_B and μ_C are not convex fuzzy grades.

THEOREM 5. *Convex fuzzy grades in J form a semiring (more precisely, a commutative semiring with identities) under $\sqcup$ and $\sqcap$.*

Proof. Convex fuzzy grades in J are distributive with respect to $\sqcup$ and $\sqcap$

[11] $\sum_{u \geqslant a} f(u) = \sum_{u' \geqslant a} f(u')$.

(Theorem 4) and form a commutative semigroup with identity under $\sqcup$ and $\sqcap$, respectively (Theorem 1 and 3), where the identity is $1/0$ $(=0)$ under $\sqcup$ and $1/1$ $(=1)$ under $\sqcap$, respectively (see (33)). This concludes the proof. ∎

Note that the convex fuzzy grade $\phi = \sum_{u \in J} 0/u$ is regarded as zero element under $\sqcup$ and $\sqcap$, respectively. That is to say,

$$\mu_A \sqcup \phi = \phi, \qquad \mu_A \sqcap \phi = \phi. \tag{62}$$

Remark. As convex fuzzy grades do not satisfy the absorption laws in general (see Example 3), they cannot constitute a lattice (more precisely, a distributive lattice).

Next we shall discuss the properties of normal fuzzy grades under $\sqcup$ and $\sqcap$.

THEOREM 6. *If μ_A and μ_B are normal fuzzy grades in J, then $\mu_A \sqcup \mu_B$, $\mu_A \sqcap \mu_B$, $\neg\mu_A$ are also normal.*

Proof. It is obvious from the definition of normal fuzzy grades (48). ∎

THEOREM 7. *For normal fuzzy grades, part* (18) *of the identity law, i.e.,*

$$\mu_A \sqcup 1 = 1, \qquad \mu_A \sqcap 0 = 0 \tag{63}$$

is satisfied.

Proof. Obvious. ∎

Finally, we shall investigate some properties of normal convex fuzzy grades and show that they form a distributive lattice under $\sqcup$ and $\sqcap$.

THEOREM 8. *If μ_A and μ_B are normal convex fuzzy grades in J, then $\mu_A \sqcup \mu_B$, $\mu_A \sqcap \mu_B$, $\neg\mu_A$ are also normal convex.*

Proof. It is immediate from Theorem 3 and 6. ∎

THEOREM 9. *For normal convex fuzzy grades in J, the absorption laws, i.e.,*

$$\mu_A \sqcap (\mu_A \sqcup \mu_B) = \mu_A, \qquad \mu_A \sqcup (\mu_A \sqcap \mu_B) = \mu_A \tag{64}$$

are satisfied.

Proof. We shall prove the absorption law $\mu_A \sqcap (\mu_A \sqcup \mu_B) = \mu_A$ under the assumption that μ_A and μ_B are normal convex fuzzy grades. The process

of the proof of this absorption law is, however, similar to that of the proof of Theorem 1 (see proof of (34)). Thus, the grade $p(a)$ at $a \in J$ in $\mu_A \sqcap (\mu_A \sqcup \mu_B)$ is given as in (40). Hence, using the convexity and the normality of μ_A and μ_B, we have $p(a)$ from (40) as follows.

$$\begin{aligned} p(a) &= f(a) \cdot \sum_z g(z) + \sum_{u \geqslant a} f(u) \cdot \sum_{w \leqslant a} f(w) \cdot g(a) \cdots \text{from (40)} \\ &= f(a) \cdot \sum_z g(z) + f(a) \cdot g(a) \cdots \text{from the convexity of } f \\ &= f(a) \cdot \sum_z g(z) = f(a) \cdot 1 \cdots \text{from the normality of } g \\ &= f(a). \end{aligned} \tag{65}$$

Therefore we obtain $\mu_A \sqcap (\mu_A \sqcup \mu_B) = \mu_A$. The same holds for the absorption law $\mu_A \sqcup (\mu_A \sqcap \mu_B) = \mu_A$. ∎

Note. In the proof of the absorption law, we did not use the normality of f (or μ_A) and the convexity of g (or μ_B) in spite of the use of the convexity of f and the normality of g. Thus, it is found that in general the absorption laws hold as long as μ_A is convex and μ_B is normal. More precisely, we find that it is not necessary to assume the normality of g (or μ_B). That is to say, if μ_A is convex, then the absorption laws can be shown to be satisfied so long as the maximal grade of f is less than or equal to the maximal grade of g, i.e., $\sum_{u \in J} f(u) \leqslant \sum_{z \in J} g(z)$ from the fact that in (65) $f(a) \cdot \sum_z g(z) = f(a)$ if $f(a) \leqslant \sum_z g(z)$ for any $a \in J$.

THEOREM 10. *Normal convex fuzzy grades in J form a distributive lattice under $\sqcup$ and $\sqcap$, where the greatest element is* $1/1$ *and the least element is* $1/0$.

Proof. It is obvious from Theorems 1, 4, 7, 8, and 9. It is noted that for normal convex fuzzy grades, the order relation $\sqsubseteq$ defined in (45) is equal to $\dot{\subseteq}$ in (46). ∎

From Theorem 10 the following theorem is derived.

THEOREM 11. *Fuzzy sets of type* 2 *in a set X form a distributive lattice under the operations $\cup$ and $\cap$ defined in* (30) *and* (31), *respectively, where the grades characterizing these fuzzy sets of type* 2 *are normal convex fuzzy grades in $J \subseteq [0, 1]$.*

Thus, it is shown that fuzzy sets of type 2 whose grades are normal convex fuzzy grades in J are the special cases of L-fuzzy sets by Goguen (1967).

EXAMPLE 5. Let fuzzy grades in $J = \{0, 0.5, 1\}$ be

$$\mu_i = a_1/0 + a_2/0.5 + a_3/1 \tag{66}$$

with a_1, a_2, a_3 being in $\{0, 0.5, 1\}$. Then all fuzzy grades μ_i $(i = 1, 2, \ldots, 27)$ are as in Table I, in which the fuzzy grades denoted by (μ_i) stands for normal

TABLE I

Fuzzy Grades $\mu_i = a_1/0 + a_2/0.5 + a_3/1$

((μ_i): Normal Convex Fuzzy Grades)

Fuzzy grades	a_1	a_2	a_3	Fuzzy grades	a_1	a_2	a_3
μ_1	0	0	0	(μ_{15})	1	0.5	0.5
μ_2	0.5	0	0	(μ_{16})	0	1	0.5
(μ_3)	1	0	0	(μ_{17})	0.5	1	0.5
μ_4	0	0.5	0	(μ_{18})	1	1	0.5
μ_5	0.5	0.5	0	(μ_{19})	0	0	1
(μ_6)	1	0.5	0	μ_{20}	0.5	0	1
(μ_7)	0	1	0	μ_{21}	1	0	1
(μ_8)	0.5	1	0	(μ_{22})	0	0.5	1
(μ_9)	1	1	0	(μ_{23})	0.5	0.5	1
μ_{10}	0	0	0.5	μ_{24}	1	0.5	1
μ_{11}	0.5	0	0.5	(μ_{25})	0	1	1
μ_{12}	1	0	0.5	(μ_{26})	0.5	1	1
μ_{13}	0	0.5	0.5	(μ_{27})	1	1	1
μ_{14}	0.5	0.5	0.5				

convex fuzzy grades. These normal convex fuzzy grades constitute a distributive lattice in Fig. 1.

The satisfaction or failure of each law for various kinds of fuzzy grades under $\sqcup$, $\sqcap$, and $\neg$ is summarized in Table II, where $\bigcirc$ stands for the satisfaction of law, $\times$ stands for the failure, and $\triangle$ represents that a part of identity laws is not satisfied, that is, $\mu_A \sqcup 1 \neq 1$ and $\mu_A \sqcap 0 \neq 0$. The properties of grades for ordinary fuzzy sets and binary grades (values of characteristic functions) for ordinary sets are also listed in Table II. This table also contains the properties of fuzzy grades under $\sqcup$ and $\sqcap$, which will be defined in Section 4.

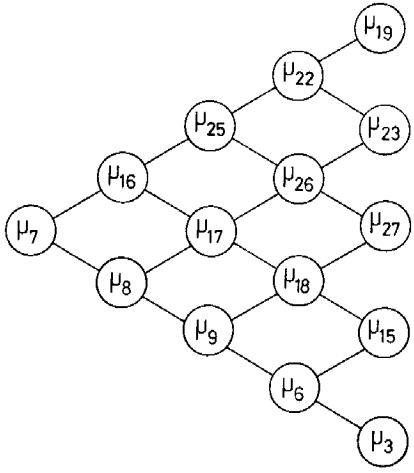

FIG. 1. Distributive lattice by normal convex grades (μ_i) under $\sqcup$ and $\sqcap$.

4. SOME PROPERTIES OF FUZZY GRADES UNDER $\sqcap$ AND $\sqcup$

In this section we shall briefly investigate some algebraic properties of fuzzy grades under the operations of join $\sqcup$ and meet $\sqcap$ which are slightly different from the operations of join $\sqcup$ and meet $\sqcap$ in (30), (31). The operations of $\sqcup$ and $\sqcap$ are obtained by replacing min($\wedge$) by the algebraic product, that is, by letting $f(u_i)\, g(w_j)$ instead of $f(u_i) \wedge g(w_j)$ in (30) and (31).

Therefore, the operations of join $\sqcup$ and meet $\sqcup$ for fuzzy grades are defined by the following:

TABLE II

Satisfaction or Failure of Laws for Various Kinds of Fuzzy Grades under $\sqcup$, $\sqcap$, and $\neg$

	Idempotent laws	Commutative laws	Associative laws	Absorption laws
Fuzzy grades	○	○	○	×
Normal fuzzy grades	○	○	○	×
Convex fuzzy grades	○	○	○	×
Normal convex fuzzy grades	○	○	○	○
Grades (for ordinary fuzzy sets)	○	○	○	○
Binary grades (for ordinary sets)	○	○	○	○
(Convex) fuzzy grades under $\sqcup$, $\sqcap$, and $\neg$	×	○	○	×
Normal (convex) fuzzy grades under $\sqcup$, $\sqcap$, and $\neg$	×	○	○	×

	Distributive laws	Involution laws	DeMorgan's laws	Identity laws	Complement laws
Fuzzy grades	×	○	○	△	×
Normal fuzzy grades	×	○	○	○	×
Convex fuzzy grades	○	○	○	△	×
Normal convex fuzzy grades	○	○	○	○	×
Grades (for ordinary fuzzy sets)	○	○	○	○	×
Binary grades (for ordinary sets)	○	○	○	○	○
(Convex) fuzzy grades under $\sqcup$, $\sqcap$, and $\neg$	×	○	○	△	×
Normal (convex) fuzzy grades under $\sqcup$, $\sqcap$, and $\neg$	×	○	○	○	×

Let fuzzy grades μ_A and μ_B be represented as $\mu_A = \sum_{u \in J} f(u)/u$ and $\mu_B = \sum_{w \in J} g(w)/w$, then

$$\mu_A \sqcup \mu_B = \left(\sum_{u \in J} f(u)/u\right) \sqcup \left(\sum_{w \in J} g(w)/w\right)$$

$$= \sum_{u,w \in J} f(u)\, g(w)/u \vee w, \tag{67}$$

$$\mu_A \sqcap \mu_B = \sum_{u,w \in J} f(u)\, g(w)/u \wedge w, \tag{68}$$

where $f(u)\, g(w)$ stands for the algebraic product of $f(u)$ and $g(w)$.

THEOREM 12. *Under the operations* $\sqcup$, $\sqcap$, *and* $\neg$ *in* (67), (68), (32), *arbitrary fuzzy grades satisfy such laws as commutative laws, associative laws, involution laws, and De Morgan's laws. But normal convex fuzzy grades* (*needless to say, any fuzzy grades, normal fuzzy grades, convex fuzzy grades*) *do not satisfy idempotent laws, absorption laws, distributive laws, and complement laws. A part of identity laws, i.e.,*

$$\mu_A \sqcup 1 = \mu_A, \qquad \mu_A \sqcap 0 = \mu_A$$

are satisfied for any fuzzy grades. Another part of identity laws, i.e.,

$$\mu_A \sqcup 1 = 1, \qquad \mu_A \sqcap 0 = 0$$

can be satisfied for normal fuzzy grades and normal convex fuzzy grades only.

Proof. The proofs of satisfaction of laws can be executed similarly to the proofs of Theorems 1 and 7. On the other hand, the failure of laws can be illustrated by the following examples. ▮

EXAMPLE 6. It will be sufficient to show the examples of normal convex fuzzy grades which do not satisfy the idempotent laws, absorption laws, distributive laws, and complement laws.

Failure of idempotent laws. Let $J = \{0.1, 0.2, 0.3, 0.4\}$ and let μ_A be a normal convex fuzzy grade such that

$$\mu_A = 0.3/0.1 + 1/0.2 + 0.8/0.3 + 0.4/0.4,$$

then from (67) and (68) we have

$$\mu_A \sqcup \mu_A = 0.09/0.1 + 1/0.2 + 0.8/0.3 + 0.4/0.4 \neq \mu_A,$$
$$\mu_A \sqcap \mu_A = 0.3/0.1 + 1/0.2 + 0.64/0.3 + 0.16/0.4 \neq \mu_A.$$

Failure of absorption laws. Let μ_A and μ_B be

$$\mu_A = 0.4/0.1 + 0.8/0.2 + 1/0.3 + 0.6/0.4,$$
$$\mu_B = 0.3/0.1 + 1/0.2 + 0.7/0.3 + 0.5/0.4,$$

then

$$\mu_A \sqcap (\mu_A \sqcup \mu_B) = 0.4/0.1 + 0.8/0.2 + 1/0.3 + 0.36/0.4 \neq \mu_A,$$
$$\mu_A \sqcap (\mu_A \sqcup \mu_B) = 0.16/0.1 + 0.8/0.2 + 1/0.3 + 0.6/0.4 \neq \mu_A.$$

It is noted that $\mu_A \sqcap (\mu_A \sqcup \mu_B) \neq \mu_A \sqcup (\mu_A \sqcap \mu_B)$ in general.

Failure of distributive laws. If μ_A, μ_B, and μ_C are

$$\mu_A = 0.7/0.1 + 1/0.2 + 0.9/0.3,$$
$$\mu_B = 0.4/0.1 + 1/0.2 + 0.3/0.3,$$
$$\mu_C = 0.8/0.1 + 1/0.2 + 0.1/0.3,$$

then

$$\mu_A \sqcap (\mu_B \sqcup \mu_C) = 0.56/0.1 + 1/0.2 + 0.9/0.3,$$
$$(\mu_A \sqcap \mu_B) \sqcup (\mu_A \sqcap \mu_C) = 0.56/0.1 + 1/0.2 + 0.81/0.3.$$

Similarly,

$$\mu_A \sqcup (\mu_B \sqcap \mu_C) = 0.7/0.1 + 1/0.2 + 0.27/0.3$$
$$(\mu_A \sqcup \mu_B) \sqcap (\mu_A \sqcup \mu_C) = 0.56/0.1 + 1/0.2 + 0.27/0.3.$$

Failure of complement laws. If μ_A is

$$\mu_A = 0.7/0.1 + 1/0.2 + 0.4/0.3,$$

then

$$\neg\mu_A = 0.7/0.9 + 1/0.8 + 0.4/0.7.$$

Thus,

$$\mu_A \sqcup \neg\mu_A = 0.7/0.9 + 1/0.8 + 0.4/0.7 \neq 1/1 \ (=1),$$
$$\mu_A \sqcap \neg\mu_A = 0.7/0.1 + 1/0.2 + 0.4/0.3 \neq 1/0 \ (=0).$$

Furthermore, we can easily show that if μ_A and μ_B are normal, then $\mu_A \sqcup \mu_B$, $\mu_A \sqcap \mu_B$, $\neg\mu_A$ are normal. It is not known, however, whether $\mu_A \sqcup \mu_B$ and $\mu_A \sqcap \mu_B$ are convex or not under the assumption that μ_A and μ_B are convex. The authors have shown that the convexity of $\mu_A \sqcup \mu_B$ and $\mu_A \sqcap \mu_B$ holds if J consists of three elements. They have not been able to show the convexity in general cases. But we hope that the positive answer

will be obtained in general cases. Thus we leave this problem as an open problem.

Open problem. Are $\mu_A \sqcup \mu_B$ and $\mu_A \sqcap \mu_B$ convex fuzzy grades when μ_A and μ_B are convex fuzzy grades?

In the sequel, we find that normal convex fuzzy grades (needless to say, arbitrary fuzzy grades, convex fuzzy grades, normal fuzzy grades) under the operations of $\sqcup$ and $\sqcap$ do not, in general, constitute algebraic structures such as a commutative semiring or a distributive lattice. In Table II the algebraic properties of fuzzy grades under $\sqcup$ and $\sqcap$ are listed.

5. Conclusion

In the foregoing discussion, we have concerned ourselves with elementary operations for fuzzy grades. Various kinds of operations, however, will be defined as an extension of the operations such as algebraic product and algebraic sum for ordinary fuzzy sets. The algebraic properties of fuzzy grades under such operations will be presented in subsequent papers.

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References

Brown, J. G. (1971), A note on fuzzy sets, *Inform. Control* **18**, 32.

DeLuca, A., and Termini, S. (1972), Algebraic properties of fuzzy sets, *J. Math. Anal. Appl.* **40**, 373.

Goguen, J. A. (1967), *L*-fuzzy sets, *J. Math. Anal. Appl.* **18**, 145.

Mizumoto, M., and Tanaka, K. (1974), Fuzzy-fuzzy automata, Seminar Reports on "Studies on Time Series Pattern Recognition Systems," Res. Inst. for Math. Sci., Kyoto Univ., Nov. 1974 (in Japanese); also in *KYBERNETES*, to appear.

Zadeh, L. A. (1965), Fuzzy sets, *Inform. Control* **8**, 338.

Zadeh, L. A. (1972), A fuzzy-set-theoretic interpretation of linguistic hedges, *J. Cybernetics* **2**, 4.

Zadeh, L. A. (1973), The concept of a linguistic variable and its application to approximate reasoning, Memo. No. ERL-M411, Univ. of California.

Zadeh, L. A. (1974), Fuzzy logic and its application to approximate reasoning, *Information Processing* **74**, 591.

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