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FUZZY-FUZZY AUTOMATA

M. MIZUMOTO and K. TANAKA

Department of Information and Computer Sciences, Faculty of Engineering Science, Osaka University, Toyonaka, Osaka 560, Japan

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Based on the concept of fuzzy sets of type 2 (or fuzzy–fuzzy sets) defined by L. A. Zadeh, fuzzy–fuzzy automata are newly formulated and somé properties of these automata are investigated. It is shown that fuzzy–fuzzy languages characterized by fuzzy–fuzzy automata are closed under the operations of union, intersection, concatenation, and Kleene closure in the sense of fuzzy sets of type 2, but are not closed under complement. The power of fuzzy–fuzzy automata as an acceptor is the same as that of ordinary fuzzy automata and finite automata, though fuzzy–fuzzy automata include fuzzy automata and finite automata as special cases. Finally, fuzzy–fuzzy grammars are illustrated and it is shown that fuzzy–fuzzy grammars with context-free rules can generate context-sensitive languages.

1 INTRODUCTION

Since L. A. Zadeh[1] formulated the concept of fuzzy sets which can deal with ill-defined objects, a number of researchers have been engaged in the studies on fuzzy sets and their applications to automata, languages, pattern recognition, decision making, logic, control, and so on. Fuzzy automata were firstly defined by Wee and Fu[2] as learning automata, and then fuzzy automata have been applied to the areas of learning control, interaction with random environment, gaming, time series pattern recognition, fuzzy program and fuzzy languages. In parallel with these applications, the properties of fuzzy automata have been investigated[3–5] and various types of fuzzy automata, say, max-product automata, mini-max automata, L-fuzzy automata and R-fuzzy automata have been formulated.[6–9]

In this paper, based on the concept of fuzzy sets of type 2 (or fuzzy-fuzzy sets)[10,11] formulated by Zadeh, fuzzy-fuzzy automata are newly defined and some properties of the automata are investigated. It is shown that fuzzy-fuzzy languages characterized by fuzzy-fuzzy automata are closed under the operations of union, intersection, concatenation, and Kleene closure in the sense of fuzzy sets of type 2. The power of fuzzy-fuzzy automata as an acceptor is the same as that of ordinary fuzzy automata and finite automata, though fuzzy-fuzzy automata include fuzzy automata and finite automata as special cases. Finally, fuzzy-fuzzy grammars are illustrated and it is shown that fuzzy-fuzzy grammars with context-free rules can generate context-sensitive languages.

2 FUZZY SETS OF TYPE 2

We shall briefly review some of the basic properties relating to fuzzy sets of type 2 for the purpose of fuzzy-fuzzy automata discussed later.

Fuzzy Sets of Type 2

A *fuzzy set of type 2*, A, in a set X is the fuzzy set which is characterized by a *fuzzy membership function* μ_A as

$$\mu_A : X \to [0, 1]^{[0,1]} \tag{1}$$

with the value $\mu_A(x)$ being called a *fuzzy grade* and being a fuzzy set in [0, 1] (or in the subset J of [0, 1]).

A fuzzy set of type 2, A, is represented as follows.

$$A = \mu_A(x_1)/x_1 + \mu_A(x_2)/x_2 + \ldots + \mu_A(x_n)/x_n = \sum_i \mu_A(x_i)/x_i, \quad x_i \in X \tag{2}$$

where the operation + stands for logical sum (OR).

And a fuzzy grade $\mu_A(x)$ (that is, fuzzy set in $J \subseteq [0, 1]$) is also expressed as

$$\mu_A(x) = f(u_1)/u_1 + f(u_2)/u_2 + \ldots + f(u_m)/u_m = \sum_i f(u_i)/u_i, \quad u_i \in J \tag{3}$$

where the function f is a membership function for the fuzzy grade $\mu_A(x)$ (fuzzy set in J) and the value $f(u_i)$ ($\in [0, 1]$) represents the grade for $u_i \in J$.

Example Suppose that $X = \{$Jack, Bill, Joseph, George, Abe$\}$ is a set of men and that A is a fuzzy set of type 2 of *handsome* men in X. Then we may have

A = *middle*/Jack + *not low*/Bill + *low*/Joseph + *very high*/George + *high*/Abe,

where the fuzzy grades named *middle, low, high* are assumed to be fuzzy sets in $J = \{0, 0.1, \ldots, 0.9, 1\} \subseteq [0, 1]$ and, for example, are expressed as follows.

middle = 0.3/0.3 + 0.7/0.4 + 1/0.5 + 0.7/0.6 + 0.3/0.7

low = 1/0 + 0.9/0.1 + 0.7/0.2 + 0.4/0.3

high = 0.4/0.7 + 0.7/0.8 + 0.9/0.9 + 1/1

Moreover, the fuzzy grades *not low* and *very high* can be defined from the fuzzy grades *low* and *high* by the use of the concept of linguistic hedges.[12] That is to say,

not low = 0.1/0.1 + 0.3/0.2 + 0.6/0.3 + 1/0.4 + 1/0.5 + . . . + 1/1,

very high = 0.16/0.7 + 0.49/0.8 + 0.81/0.9 + 1/1.

The operations of fuzzy sets of type 2 are defined by using the extension principle.[10]

Let $\mu_A(x)$ and $\mu_B(x)$ be two fuzzy grades for fuzzy sets of type 2, A and B, respectively, and be represented as

$$\mu_A(x) = f(u_1)/u_1 + f(u_2)/u_2 + \ldots + f(u_n)/u_n = \sum_i f(u_i)/u_i, \qquad (4)$$

$$\mu_B(x) = g(w_1)/w_1 + g(w_2)/w_2 + \ldots + g(w_m)/w_m = \sum_j g(w_j)/w_j. \qquad (5)$$

Then the operations of fuzzy sets of type 2 (in other words, fuzzy grades) are given by the following:

Union
$$A \cup B \Leftrightarrow \mu_{A \cup B}(x) = \mu_A(x) \sqcup \mu_B(x) = \left(\sum_i f(u_i)/u_i\right) \sqcup \left(\sum_j g(w_j)/w_j\right) = \sum_{i,j} (f(u_i) \wedge g(w_j))/(u_i \vee w_j) \qquad (6)$$

Intersection
$$A \cap B \Leftrightarrow \mu_{A \cap B}(x) = \mu_A(x) \sqcap \mu_B(x) = \sum_{i,j} (f(u_i) \wedge g(w_j))/(u_i \wedge w_j) \qquad (7)$$

Complement
$$\bar{A} \Leftrightarrow \mu_{\bar{A}}(x) = \neg\, \mu_A(x) = \sum_i f(u_i)/(1 - u_i) \qquad (8)$$

where $\vee$ and $\wedge$ represent max and min, respectively. We shall call the operations for fuzzy grades, that is, $\sqcup$ as join, $\sqcap$ as meet, and $\neg$ as negation hereafter.

Example Let $J = \{0, 0.1, 0.2, \ldots, 0.9, 1\}$ and let fuzzy grades $\mu_A(x)$ and $\mu_B(x)$ be given as

$$\mu_A(x) = 0.5/0 + 0.7/0.1 + 0.3/0.2,$$

$$\mu_B(x) = 0.9/0 + 0.6/0.1 + 0.2/0.2.$$

Then we have

$$\mu_A(x) \sqcup \mu_B(x)$$
$$= (0.5/0 + 0.7/0.1 + 0.3/0.2) \sqcup (0.9/0 + 0.6/0.1 + 0.2/0.2)$$
$$= \frac{0.5 \wedge 0.9}{0 \vee 0} + \frac{0.5 \wedge 0.6}{0 \vee 0.1} + \frac{0.5 \wedge 0.2}{0 \vee 0.2} + \frac{0.7 \wedge 0.9}{0.1 \vee 0} + \frac{0.7 \wedge 0.6}{0.1 \vee 0.1} + \frac{0.7 \wedge 0.2}{0.1 \vee 0.2} + \frac{0.3 \wedge 0.9}{0.2 \vee 0} + \frac{0.3 \wedge 0.6}{0.2 \vee 0.1} + \frac{0.3 \wedge 0.2}{0.2 \vee 0.2}$$
$$= 0.5/0 + 0.5/0.1 + 0.2/0.2 + 0.7/0.1 + 0.6/0.1 + 0.2/0.2 + 0.3/0.2 + 0.3/0.2 + 0.2/0.2$$
$$= 0.5/0 + (0.5 \vee 0.7 \vee 0.6)/0.1 + (0.2 \vee 0.2 \vee 0.3 \vee 0.3 \vee 0.2)/0.2$$
$$= 0.5/0 + 0.7/0.1 + 0.3/0.2.$$

Similarly, we have

$$\mu_A(x) \sqcap \mu_B(x) = 0.7/0 + 0.6/0.1 + 0.2/0.2$$

$$\neg\mu_A(x) = 0.5/1 + 0.7/0.9 + 0.3/0.8$$

A fuzzy grade $\mu_A(x) = \sum_i f(u_i)/u_i$ in $J \subseteq [0, 1]$ is said to be *normal* if

$$\max_i f(u_i) = 1. \qquad (9)$$

Otherwise it is *subnormal*. Let $J = \{u_1, u_2, \ldots, u_n\}$ be a subset of $[0, 1]$ which satisfies $u_1 < u_2 < \ldots < u_n$, then a fuzzy grade $\mu_A(x)$ in J is said to be *convex* if for any integers i, k with $i \leqq k$, the following is satisfied, i.e.,

$$f(u_j) \geqq \min\{f(u_i), f(u_k)\} \qquad (10)$$

where j is any integer satisfying $i \leqq j \leqq k$. Furthermore, a fuzzy grade which is convex and normal is referred to be as a *normal convex* fuzzy grade.

Example Various types of fuzzy grades in $J = \{0.1, 0.2, 0.3, 0.4\}$ are listed below:

$\mu_A(x) = 0.8/0.1 + 0.3/0.2 + 0.5/0.3 + 0.9/0.4$
(subnormal, non-convex)

$\mu_A(x) = 0.3/0.1 + 0.6/0.2 + 0.8/0.3 + 0.5/0.4$
(subnormal, convex)

$\mu_A(x) = 0.7/0.1 + 0.2/0.2 + 1/0.3 + 0.3/0.4$
(normal, non-convex)

$\mu_A(x) = 0.5/0.1 + 0.8/0.2 + 1/0.3 + 0.7/0.4$
(normal, convex)

It was shown by Mizumoto and Tanaka[11] that convex fuzzy grades in J under $\sqcup$, $\sqcap$, and $\urcorner$ form a commutative semiring and normal convex fuzzy grades form a distributive lattice.

3 FUZZY-FUZZY AUTOMATA

We shall define a fuzzy-fuzzy automaton using the concept of fuzzy sets of type 2 and show that fuzzy-fuzzy languages characterized by fuzzy-fuzzy automata are closed under the operations of union, intersection, concatenation, Kleene closure in the sense of fuzzy sets of type 2, but are not closed under complement. Furthermore, λ-fuzzy-fuzzy languages with the fuzzy threshold λ are regular languages, though fuzzy-fuzzy automata include ordinary fuzzy automata and finite automata as special cases.

Fuzzy-Fuzzy Automata

A *fuzzy-fuzzy automaton* (FFA for short) over the alphabet Σ is a system

$$A = (S, S_0, \mu_A, F) \tag{11}$$

where

i S is a finite set of internal states.

ii S_0 is called a *fuzzy initial state* of A and is a fuzzy set of type 2 in S characterized by a fuzzy membership function as

$$\mu_{S_0}: S \to [0, 1]^{[0,1]} \tag{12}$$

iii μ_A is a *fuzzy transition function* which is a conditional fuzzy membership function† such that for s_i, $s_{i+1} \in S$ and $a_i \in \Sigma$,

$$\mu_A(s_{i+1}/s_i, a_i) \in [0, 1]^{[0,1]} \tag{13}$$

This represents the grade of transition from state s_i to state s_{i+1} when the input is a_i.

iv $F \subseteq S$ is a set of final states.

Remark If $\mu_{S_0}(s_i)$ and $\mu_A(s_{i+1}/s_i, a_i)$ are assumed to be convex fuzzy grades, then FFA is reduced to a semiring automaton.[7] If μ_{S_0} and μ_A are normal convex fuzzy grades, A becomes a lattice automaton.[7] If μ_{S_0} and μ_A are in $[0, 1]$, then A is an ordinary fuzzy automaton.[2~5] Moreover, if μ_{S_0} and μ_A are in $\{0, 1\}$, then A is a (non-)deterministic automaton.

Now we shall first obtain the state equation of FFA A.‡

Given the initial fuzzy state S_0 and the input a_0, the next fuzzy state S_1 is characterized by the following fuzzy membership function μ_{S_1} which is obtained from the expression (I) in the footnote.

$$\mu_{S_1}(s_1) = \bigsqcup_{s_0 \in S} [\mu_{S_0}(s_0) \sqcap \mu_A(s_1/s_0, a_0)] \tag{14}$$

In general, for input string $x = a_0 a_1 \ldots a_n$, we can obtain the fuzzy state S_{n+1} by the following:

$$\mu_{S_{n+1}}(s_{n+1}) = \bigsqcup_{s_0, s_1, \ldots, s_n \in S} [\mu_{S_0}(s_0) \sqcap \mu_A(s_1/s_0, a_0) \sqcap \cdots \sqcap \mu_A(s_{n+1}/s_n, a_n)]. \tag{15}$$

Let $\mu_A(s_{n+1}/s_0, a_0 a_1 \ldots a_n)$ be

$$\mu_A(s_{n+1}/s_0, a_0 a_1 \ldots a_n) = \bigsqcup_{s_0, s_1, \ldots, s_n \in S} [\mu_A(s_1/s_0, a_0) \sqcap \mu_A(s_2/s_1, a_1) \sqcap \cdots \sqcap \mu_A(s_{n+1}/s_n, a_n)]$$

† A fuzzy set of type 2, $B(x)$, in Y is said to be conditioned on x if its fuzzy membership function (that is, *conditional fuzzy membership function*) depends on x as a parameter. We shall denote the fuzzy membership function of $B(x)$ as $\mu_B(y/x)$. Suppose that the parameter x ranges over a set X. Then the function $\mu_B(y/x)$ defines a mapping from X to the family of fuzzy sets of type 2 defined on Y. Through this mapping, a fuzzy set of type 2, A, in X induces a fuzzy set of type 2, B, in Y which is defined by

$$\mu_B(y) = \bigsqcup_{x \in X} [\mu_A(x) \sqcap \mu_B(y/x)] \tag{I}$$

‡ In this paper it is assumed that $\mu_{S_0}(s_i)$ and $\mu_A(s_{i+1}/s_i, a_i)$ are normal convex fuzzy grades.

then (15) will be expressed by

$$\mu_{S_{n+1}}(s_{n+1}) = \bigsqcup_{s_0 \in S} [\mu_{S_0}(s_0) \sqcap \mu_A(s_{n+1}/s_0, a_0 a_1 \ldots a_n)]. \quad (16)$$

Therefore, from the fact that the fuzzy state S_{n+1} is derived from the initial fuzzy state S_0 and the input string $x = a_0 a_1 \ldots a_n$, (16) can be replaced as follows.

$$rp_A(s/S_0, x) = \bigsqcup_{s_0 \in S} [\mu_{S_0}(s_0) \sqcap \mu_A(s/s_0, x)] \quad (17)$$

where rp_A represents a response function.

Hence, the fuzzy grade of input string $x \in \Sigma^*$ accepted by FFA A is defined by the following.

$$f_A(x) = \bigsqcup_{s \in F} rp_A(s/S_0, x), \quad x \in \Sigma^* \quad (18)$$

Fuzzy-Fuzzy Languages

A *fuzzy-fuzzy language* $L(A)$ (FFL for short) is a fuzzy set of type 2 in Σ^* which is characterized by a fuzzy membership function f_A defined in (18).

Now we shall investigate the closure properties of FFL characterized by FFA.

Theorem 1 Let A_1 and A_2 be two FFA, then there exists an FFA A such that, in the sense of fuzzy sets of type 2,

$$L(A) = L(A_1) \cup L(A_2)$$

where the operation $\cup$ for FFL's is defined in the same way as in (6).

Proof For FFA A_1 and A_2, that is,

$$A_1 = (S^1, S_0^1, \mu_{A_1}, F^1), \quad (19)$$

$$A_2 = (S^2, S_0^2, \mu_{A_2}, F^2), \quad (20)$$

define an FFA A as

$$A = (S, S_0, \mu_A, F) \quad (21)$$

where $S = S^1 \cup S^2 (S^1 \cap S^2 = \phi)$, $F = F^1 \cup F^2$, and μ_{S_0} and μ_A are given as follows.

$$\mu_{S_0}(s) = \begin{cases} \mu_{S_0^1}(s) \ldots s \in S^1, \\ \mu_{S_0^2}(s) \ldots s \in S^2 \end{cases}$$

$$\mu_A(s'/s, a) = \begin{cases} \mu_{A_1}(s'/s, a) \ldots s', s \in S^1 \\ \mu_{A_2}(s'/s, a) \ldots s', s \in S^2 \\ 1/0 \ldots \text{otherwise.} \end{cases}$$

Theorem 2 For two FFA A_1 and A_2, there exists an FFA A such that

$$L(A) = L(A_1) \cap L(A_2)$$

Proof Suppose that FFA A_1 and A_2 are as in (19), (20), respectively. Then an FFA $A = (S, S_0, \mu_A, F)$ is obtained as follows. $S = S^1 \times S^2$ (direct product of S^1 and S^2), $F = F^1 \times F^2$, and

$$\mu_{S_0}(s_1, s_2) = \mu_{S_0^1}(s_1) \sqcap \mu_{S_0^2}(s_2) \quad \ldots s_1 \in S^1, s_2 \in S^2$$

$$\mu_A((s_1', s_2')/(s_1, s_2), a) = \mu_{A_1}(s_1'/s_1, a) \sqcap \mu_{A_2}(s_2'/s_2, a).$$

Theorem 3 $L(A) = L(A_1) * L(A_2)$ (Concatenation)†

Proof $S = S^1 \cup S^2 (S^1 \cap S^2 = \phi)$, $F = F^2$, and for $\mu_i = \bigsqcup_{s \in S} \mu_{S_0^i}(s)$ $(i = 1, 2)$, define

$$\mu_{S_0}(s) = \begin{cases} \mu_{S_0^1}(s) \sqcap \mu_2 \ldots s \in S^1 \\ \mu_{S_0^2}(s) \sqcap \mu_1 \ldots s \in S^2. \end{cases}$$

$$\mu_A(s'/s, a) = \begin{cases} \mu_{A_1}(s'/s, a) \ldots s', s \in S^1 \\ \mu_{A_2}(s'/s, a) \ldots s', s \in S^2 \\ \mu_{S_0^2}(s') \ldots s \in S^1, s' \in S^2 \\ 1/0 \ldots \text{otherwise} \end{cases}$$

Theorem 4 $L(A) = L(A_1)^*$ (Kleene closure)

Proof $S = S^1$, $F = F^1$, $S_0 = S_0^1$, and

$$\mu_A(s'/s, a) = \mu_{A_1}(s'/s, a) \sqcup \mu_{S_0^1}(s').$$

λ-Fuzzy-Fuzzy Languages

The set of all input strings accepted by an FFA A with *fuzzy threshold* λ (λ is assumed to be a normal convex fuzzy grade) is defined as†

$$L(A, \lambda) = \{x \mid f_A(x) \sqsupseteq \lambda, x \in \Sigma^*\}$$

and is called a *λ-fuzzy-fuzzy language* (λ-FFL for short).

Theorem 5 FFL's are not closed under the complement (8) in the sense of fuzzy sets of type 2.

Proof In general we have $L(A, \lambda_1) \supseteq L(A, \lambda_2)$ if $\lambda_1 \sqsubseteq \lambda_2$. That is, $L(A, \lambda)$ is non-increasing for λ. In

† Let L_1 and L_2 be two FFL's, then

Concatenation $L_1 * L_2 \Leftrightarrow \mu_{L_1 * L_2}(x) = \bigsqcup_u [\mu_{L_1}(u) \sqcap \mu_{L_2}(v)]$, $x = uv$.

Kleene Closure $L_1^* = L_1 \cup L_1 * L_1 \cup L_1 * L_1 * L_1 \cup \ldots$

† In general, the order relation $\sqsubseteq$ over normal convex fuzzy grades can be defined by the following:[11]

$$\mu_A \sqsubseteq \mu_B \Leftrightarrow \mu_A \sqcap \mu_B = \mu_A \Leftrightarrow \mu_A \sqcup \mu_B = \mu_B$$

where μ_A and μ_B are normal convex fuzzy grades.

contradiction to this, the complement $\overline{L(A, \lambda)}$ is non-decreasing, that is, $\overline{L(A, \lambda_1)} \subseteq \overline{L(A, \lambda_2)}$ if $\lambda_1 \sqsubseteq \lambda_2$.

Theorem 6 λ-FFL $L(A, \lambda)$ is a regular language.

Proof Define the relation R on Σ^* by the definition. For any $x, y \in \Sigma^*$ and any $s \in S$, let

$$x \, R \, y \Leftrightarrow rp_A(s/S_0, x) = rp_A(s/S_0, y).$$

Then R is an equivalence relation on Σ^*. Furthermore, for any $z \in \Sigma^*$,

$$rp_A(s/S_0, xz) = \bigsqcup_{t \in S} [rp_A(t/S_0, x) \sqcap \mu_A(s/t, z)]$$

$$= \bigsqcup_{t \in S} [rp_A(t/S_0, y) \sqcap \mu_A(s/t, z)] = rp_A(s/S_0, yz)$$

Thus we have $xz \, R \, yz$ which says that R is a congruence relation. Moreover, we can easily show that R has finite rank and that $L(A, \lambda)$ is the union of some of the equivalence classes. Hence $L(A, \lambda)$ is a regular language.

Remark From the above Theorem 6 we find that the capacity of FFA as acceptors is equal to that of ordinary fuzzy automata[4] in spite of the generalization of fuzzy automata. Therefore, even if the concept of fuzzy sets of type 2 is applied to formal grammars with type 3 rules, the generative power of regular fuzzy-fuzzy grammars is shown to be equal to that of ordinary regular grammars from the fact that fuzzy-fuzzy automata can be easily transformed into regular fuzzy-fuzzy grammars and vice versa. However, the generative power of context-free fuzzy-fuzzy grammars can be enhanced as is shown in the following example.

Let the fuzzy productions P of the context-free fuzzy-fuzzy grammars $G = (V_N, V_T, S, P)$ be as follows.

$$S \xrightarrow{\alpha} AB, \quad S \xrightarrow{\beta} CD, \quad A \xrightarrow{\gamma} aAb, \quad A \xrightarrow{\gamma} ab$$

$$B \xrightarrow{\gamma} cB, \quad B \xrightarrow{\gamma} c, \quad C \xrightarrow{\gamma} aC, \quad C \xrightarrow{\gamma} a,$$

$$D \xrightarrow{\gamma} bDc, \quad D \xrightarrow{\gamma} bc,$$

where $V_N = \{S, A, B, C, D\}$, $V_T = \{a, b, c\}$, S is an initial symbol, and α, β, and γ are fuzzy grades such that

$$\alpha = 0.5/0 + 1/0.5 + 0.5/1$$

$$\beta = 1/0.5$$

$$\gamma = 1/0.5 + 0.5/1$$

Thus, a string, say, $a^2b^2c^2$ is obtained by the following fuzzy derivation chain.

$$S \xrightarrow{\alpha} AB \xrightarrow{\gamma} aAbB \xrightarrow{\gamma} aAbcB \xrightarrow{\gamma} a^2b^2cB \xrightarrow{\gamma} a^2b^2c^2$$

The fuzzy grade of the generation of $a^2b^2c^2$ by this derivation is obtained as

$$\alpha \sqcap \gamma \sqcap \gamma \sqcap \gamma \sqcap \gamma = \alpha$$

Similarly, for the same string $a^2b^2c^2$, the following derivation is also possible.

$$S \xrightarrow{\beta} CD \xrightarrow{\gamma} aCD \xrightarrow{\gamma} aCbDc \xrightarrow{\gamma} a^2bDc \xrightarrow{\gamma} a^2b^2c^2.$$

In this case we have β. Furthermore we can also have the different fuzzy derivation chains of $a^2b^2c^2$, all the fuzzy grade of which are easily shown to be α or β. Hence the fuzzy grade of the generation of $a^2b^2c^2$ by this context-free fuzzy-fuzzy grammar G is given as follows.

$$f_G(a^2b^2c^2) = \alpha \sqcup \beta = \gamma.$$

In general, fuzzy-fuzzy language $L(G)$ characterized by G is

$$L(G) = \sum_{i \geqq 1} \gamma/a^ib^ic^i + \sum_{\substack{i,j \geqq 1 \\ i \neq j}} \alpha/a^ib^ic^j + \sum_{\substack{i,j \geqq 1 \\ i \neq j}} \beta/a^ib^jc^j.$$

Therefore, γ-fuzzy-fuzzy language $L(G, \gamma)$ with fuzzy threshold γ is given as

$$L(G, \gamma) = \{x \mid f_G(x) \sqsupseteq \gamma\} = \{a^ib^ic^i \mid i \geqq 1\}$$

and it is a context-sensitive language, which is not a context-free language.

It should be noted that ordinary context-free fuzzy grammars cannot generate context-sensitive languages[13].

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