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Various Kinds of Automata with Weights

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By extracting the basic properties common to the automata appeared in existing literatures, we develop a general formulation of automata with "weights." We define a pseudoautomaton and derive from it the well-known deterministic automaton, nondeterministic automaton, probabilistic automaton, fuzzy automaton, and so on. Moreover, several interesting automata such as l-semigroup automaton, lattice automaton, dual lattice automaton, mixed boolean automaton, semiring automaton, ring automaton and field automaton which have never appeared in any other paper before are derived.

1. INTRODUCTION

Recently some interesting automata such as fuzzy automata [1–3, 7–11], max-product automata [4], integer-valued generalized automata [16, 18] have been formulated as a generalization of well-known deterministic automata, nondeterministic automata, and probabilistic automata. The common property with these automata is that they have the "weights" of state transitions as well as initial and final distributions. Clearly probabilistic automata can be considered as sorts of these automata with "weights." For example, fuzzy automata are the automata with weights where the values in the interval [0, 1] are adopted as the weights of state transition and the operations max and min are introduced. In addition, max-product automata can be formulated from fuzzy automata by replacing min by ordinary product. And by using $+$ and $\times$ as the operations and the probabilities as the weights, probabilistic automata can be defined. Moreover, integer-valued generalized automata have integers as weights and $+$ and $\times$ as operations.

In this paper we develop a general formulation of automata with weights by extracting the basic properties common to the existing automata and by incorporating the appropriate algebra systems with automata systems and by performing the operations of the algebra systems to the state transition functions and initial and final distribution functions of the pseudoautomata defined later.

Now we shall briefly review the concept of L-fuzzy relations by Goguen [5] as a preparation. The concept of L-fuzzy relations will be found to be an important concept in defining various kinds of automata with weights in Section 3.

2. L-Fuzzy Relations

After Zadeh [6] originated the concept of fuzzy relations as an extension of that of relations in ordinary set theory, Goguen [5] has formulated L-fuzzy relations as a generalization of fuzzy relations. The concept of (L-)fuzzy relations enables us to define fuzzy automata, l-semigroup automata, lattice automata, dual lattice automata, max-product automata, and so on. Moreover, it will be found that (L-)fuzzy relations come to be an important notion in formulating the other kinds of automata with weights such as semiring automata, ring automata, integer-valued generalized automata, and field automata.

Definition 1. An *L-fuzzy relation R* in the product space $X \times X = \{(x_1, x_2) \mid x_1, x_2 \in X\}$ is an L-fuzzy set in $X \times X$ characterized by a membership function μ_R such as

$$\mu_R: X \times X \rightarrow L, \tag{1}$$

where L is called a membership space and is assumed to be a partially ordered set or, more particularly, a lattice.

When L is the unit interval [0, 1], R is a fuzzy relation originated by Zadeh [6]. Moreover, when $L = \{0, 1\}$, R is an ordinary relation and its membership function μ_R reduces to the conventional characteristic function of a nonfuzzy relation.

In this paper the structure of the membership space L is assumed to be a complete lattice ordered semigroup[1] and a complete distributive lattice on account of the concept of composition of L-fuzzy relations denoted below.

Definition 2. Let R_1 and R_2 be two L-fuzzy relations in $X \times X$, then by the *composition* (or *product*) of R_1 and R_2 is meant an L-fuzzy relation in $X \times X$ which is denoted by R_1R_2 and is defined as follows.

(I) If L is a complete lattice ordered semigroup (or l-semigroup) $(L, \vee, *)$, then

$$\mu_{R_1R_2}(x, z) = \bigvee_y [\mu_{R_1}(x, y) * \mu_{R_2}(y, z)], \tag{2}$$

where $\vee$ and $*$ are the operations of lub and semigroup in L, respectively.

[1] A complete lattice which is a semigroup with identity under $*$ and also satisfies the following distributive law is a *complete lattice ordered semigroup* (*l-semigroup* for short) and is denoted as $L = (L, \vee, *)$, where $\vee$ is an operation lub in L. The distributive law is as follows. For each x, y, x_i, y_i in L,

$$x * \left(\bigvee_i y_i\right) = \bigvee_i (x * y_i) \qquad \text{and} \qquad \left(\bigvee_i x_i\right) * y = \bigvee_i (x_i * y).$$

Still more, if semigroup operation $*$ is replaced by $\wedge(=\text{glb})$ in $L = (L, \vee, *)$, then L becomes a *complete distributive lattice*.

(II) If L is a complete distributive lattice $(L, \vee, \wedge)$, then

$$\mu_{R_1R_2}(x, z) = \bigvee_y [\mu_{R_1}(x, y) \wedge \mu_{R_2}(y, z)], \tag{3}$$

where the operations $\vee$ and $\wedge$ are lub and glb in L, respectively.

By the way, as the operations $\vee$ and $\wedge$ are dual in the distributive lattice, the different composition of L-fuzzy relations can be defined by replacing $\vee$ with $\wedge$ as follows.

$$\mu_{R_1R_2}(x, z) = \bigwedge_y [\mu_{R_1}(x, y) \vee \mu_{R_2}(y, z)]. \tag{4}$$

If L-fuzzy relation R is a fuzzy relation by Zadeh, R is characterized by a membership function such as

$$\mu_R: X \times X \to [0, 1].$$

Then two kinds of compositions of fuzzy relations R_1 and R_2 are defined as special case of (3) and (4), i.e.,

$$\mu_{R_1R_2}(x, z) = \sup_y \min[\mu_{R_1}(x, y), \mu_{R_2}(y, z)], \tag{5}$$

$$\mu_{R_1R_2}(x, z) = \inf_y \max[\mu_{R_1}(x, y), \mu_{R_2}(y, z)]. \tag{6}$$

We shall next denote the fundamental properties concerning with L-fuzzy relations. The operation of composition of (L-)fuzzy relations has the associative property, that is, for (L-)fuzzy relations R_1, R_2, and R_3, we have

$$(R_1R_2)R_3 = R_1(R_2R_3). \tag{7}$$

This can be shown by the following. Let μ_{R_1}, μ_{R_2}, and μ_{R_3} be the membership functions which characterize (L-)fuzzy relations R_1, R_2, and R_3, respectively, then, say, in the case of (2),

$$\begin{aligned}
\mu_{(R_1R_2)R_3}(x, w) &= \bigvee_z [\mu_{R_1R_2}(x, z) * \mu_{R_3}(z, w)] \\
&= \bigvee_z \left[\left\{\bigvee_y [\mu_{R_1}(x, y) * \mu_{R_2}(y, z)]\right\} * \mu_{R_3}(z, w)\right] \\
&= \bigvee_{y,z} [\mu_{R_1}(x, y) * \mu_{R_2}(y, z) * \mu_{R_3}(z, w)] \cdots \quad \text{(From distributive law)} \\
&= \bigvee_y [\mu_{R_1}(x, y) * \{[\mu_{R_2}(y, z) * \mu_{R_3}(z, w)]\}] \\
&= \bigvee_y [\mu_{R_1}(x, y) * \mu_{R_2R_3}(y, w)] = \mu_{R_1(R_2R_3)}(x, w).
\end{aligned}$$

It should be noted that the distributive law plays an important role in proving the associative property of (L-)fuzzy relations. Therefore, in general, let $R_1, R_2, \ldots, R_n$ be (L-)fuzzy relations, then the composition $R_1 R_2 \cdots R_n$, say, of the case of (2) is defined by the membership function $\mu_{R_1 R_2 \cdots R_3}$ as

$$\mu_{R_1 R_2 \cdots R_3}(x_1, x_{n+1}) = \bigvee_{x_2, \ldots, x_n} [\mu_{R_1}(x_1, x_2) * \mu_{R_2}(x_2, x_3) * \cdots * \mu_{R_n}(x_n, x_{n+1})]. \quad (8)$$

The same holds for (3)–(6).

If there are two elements 0 and I in l-semigroup $L = (L, \vee, *)$ such that, for all x in L,

$$\begin{aligned} x \vee 0 &= x, & x * 0 &= 0 * x = 0, \\ I \vee x &= I, & x * I &= I * x = x, \end{aligned} \quad (9)$$

then they are called a zero and an identity of L, respectively. For example, let L be ([0, 1], max, $\cdot$) and the operation $\cdot$ be ordinary product, then L is an l-semigroup with zero 0 and identity 1. Moreover, let the cartesian product of [0, 1] be represented as $[0, 1]^2$ and the operations max and $\cdot$ be defined as

$$\max\{(a, b), (c, d)\} = (\max\{a, c\}, \max\{b, d\}),$$
$$(a, b) \cdot (c, d) = (a \cdot c, b \cdot d),$$

for each (a, b), (c, d) in $[0, 1]^2$. Then $L = ([0, 1]^2, \max, \cdot)$ is an l-semigroup with zero (0, 0) and identity (1, 1).

For the l-semigroup L with zero 0 and identity I, let the identity relation E be defined by

$$\mu_E(x, y) = \begin{cases} I \cdots & x = y, \\ 0 \cdots & x \neq y, \end{cases} \quad (10)$$

then we have

$$ER = RE = R, \quad (11)$$

for each L-fuzzy relation R.

Note. As is easily shown, every L-fuzzy relation R over X is representable by matrix if X is a finite set. Let $X = \{x_1, x_2, \ldots, x_n\}$, then R is represented by $n \times n$ matrix as follows.

$$R = [\mu_R(x_i, x_j)], \qquad i, j = 1, 2, \ldots, n.$$

3. Various Kinds of Automata with Weights

In this section a pseudoautomaton is defined and from it we derive various kinds of automata with weights which have or have not appeared in existing papers.

Definition 3. A *pseudoautomaton* is a system[2] such as

$$A = (S, \Sigma, W, \mu, \pi, \eta), \tag{12}$$

where

(1) S is a finite set of *states*;

(2) Σ is a finite set of *input symbols*;

(3) W is a *weighting space*;

(4) μ is a weighting function such as

$$\mu: S \times \Sigma \times S \to W, \tag{13}$$

and is called a *state transition function*. The value $\mu(s, a, s')$ in W represents the *weight* of transition from state s to state s' when the input symbol is a;

(5) π is an *initial distribution function* and is defined as

$$\pi: S \to W; \tag{14}$$

(6) η is a *final distribution function* and is defined as

$$\eta: S \to W. \tag{15}$$

The set of all finite strings over Σ is denoted by Σ^*. The null string is denoted by ϵ and is included in Σ^*, with property $x\epsilon = x = \epsilon x$ for every string x in Σ^*.

Definition 4. An *automaton* A^* is a system

$$A^* = (S, \Sigma, W, \mu^*, \pi, \eta), \tag{16}$$

[2] In this paper the terms of times and outputs are omitted for simplicity. If we adopt the term of times, the state transition function μ, the initial distribution function π, and the final distribution function η are given as

$$\mu: S \times \Sigma \times S \times T \to W,$$
$$\pi: S \times T \to W,$$
$$\eta: S \times T \to W,$$

where T is a subset of real line.

Let Δ be a set of output symbols, then output function δ is defined as

$$\delta: S \times \Sigma \times \Delta \times T \to W.$$

where S, Σ, W, π, and η are the same as that given above. μ^* is the same as μ given above with Σ^* replacing Σ, i.e.,

$$\mu^*: S \times \Sigma^* \times S \to W. \tag{17}$$

Next we shall derive various kinds of automata with weights by introducing the algebra systems to the weighting space W of the pseudoautomaton and by giving the extension rules for obtaining μ^* from μ.

3.1. *L-Semigroup Automaton*

(3.1a). As a weighting space W, complete lattice ordered semigroup (l-semigroup for short) $L = (L, \vee, *)$ with identity I and zero 0 is adopted, where the operations $\vee$ and $*$ are lub and semigroup operation in L, respectively. Then the state transition function μ, the initial distribution function π, and the final distribution function η are given by replacing W in (13), (14), and (15) by L as follows.

$$\mu: S \times \Sigma \times S \to L, \tag{18}$$

$$\pi: S \to L, \tag{19}$$

$$\eta: S \to L. \tag{20}$$

(3.1b). Using the concept of composition of L-fuzzy relations (2), the state transition function μ^* for input strings in Σ^* is obtained recursively as follows.

For ϵ, $x \in \Sigma^*$, and $a \in \Sigma$,

$$\mu^*(s, \epsilon, s') = \begin{cases} I \cdots & \text{if} \quad s = s', \\ 0 \cdots & \text{if} \quad s \neq s', \end{cases} \tag{21}$$

$$\mu^*(s, xa, s') = \bigvee_{s'' \in S} [\mu^*(s, x, s'') * \mu(s'', a, s')], \tag{22}$$

where s, $s' \in S$, and I and 0 are identity and zero of L, respectively.

Remark 1. Suppose that the automaton starts from a certain initial state, say, s_0, the initial distribution function π is concentrated at s_0, i.e.,

$$\pi(s) = \begin{cases} I \cdots & \text{if} \quad s = s_0, \\ 0 \cdots & \text{if} \quad s \neq s_0. \end{cases} \tag{23}$$

Remark 2. Let $F(\subseteq S)$ be a set of final states, then the final distribution function η is defined as

$$\eta(s) = \begin{cases} I \cdots & \text{if} \quad s \in F, \\ 0 \cdots & \text{if} \quad s \notin F. \end{cases} \tag{24}$$

Hence, the definition of η given by (20) is a generalization of that of η by (24).

Remark 3. Given the expression (22) and the initial distribution π and the final distribution η, the *weight*, written by $w(x)$, of input string x by the automaton is defined by

$$w(x) = \bigvee_{s,s' \in S} [\pi(s) * \mu^*(s, x, s') * \eta(s')], \tag{25}$$

where $x \in \Sigma^*$.

Remark 4. As there exists an order relation $\geqslant$ in l-semigroup $L = (L, \vee, *)$, the language $L(A, \lambda)$ accepted by l-semigroup automaton A with parameter λ can be defined by

$$L(A, \lambda) = \{x \in \Sigma^* \mid w(x) \geqslant \lambda\}, \tag{26}$$

where λ is called a threshold (or cut point) and is included in the weighting space L.

3.2. *Max-Product Automaton* [3, 4]

(3.2a). Let the weighting space W be $L' = ([0, 1], \max, \cdot)$ in l-semigroup automaton of 3.1, where the operation $\cdot$ represents ordinary product. Then, obviously, L' is an l-semigroup with identity 1 and zero 0. μ, π, and η are obtained by replacing L in (18)–(20) by [0, 1], i.e.,

$$\mu: S \times \Sigma \times S \rightarrow [0, 1], \tag{27}$$

$$\pi: S \rightarrow [0, 1], \tag{28}$$

$$\eta: S \rightarrow [0, 1]. \tag{29}$$

(3.2b). μ^* and w are obtained by replacing $\vee$ by max and $*$ by $\cdot$ in (21), (22), and (25).

$$\mu^*(s, \epsilon, s') = \begin{cases} 1 \cdots & s = s', \\ 0 \cdots & s \neq s', \end{cases} \tag{30}$$

$$\mu^*(s, xa, s') = \max_{s'' \in S} [\mu^*(s, x, s'') \cdot \mu(s'', a, s')], \tag{31}$$

$$w(x) = \max_{s, s' \in S} [\pi(s) \cdot \mu^*(s, x, s') \cdot \eta(s')]. \tag{32}$$

Remark. Clearly max-product automaton is considered as special case of l-semigroup automaton of 3.1 and is also shown to be a special case of semiring automaton of 3.14 defined soon.

3.3. *Lattice Automaton*

(3.3a). The complete distributive lattice $L = (L, \vee, \wedge)$ is adopted as the weighting space, where $\vee$ and $\wedge$ are the operations lub and glb in L, respectively. μ, π, and η are given from (13)–(15) as

$$\mu: S \times \Sigma \times S \rightarrow L, \tag{33}$$

$$\pi: S \rightarrow L, \tag{34}$$

$$\eta: S \rightarrow L. \tag{35}$$

(3.3b). μ^* and w are obtained as follows by using the concept of composition of L-fuzzy relations (3).

$$\mu^*(s, \epsilon, s') = \begin{cases} I \cdots & s = s', \\ 0 \cdots & s \neq s', \end{cases} \tag{36}$$

$$\mu^*(s, xa, s') = \bigvee_{s'' \in S} [\mu^*(s, x, s'') \wedge \mu(s'', a, s')], \tag{37}$$

$$w(x) = \bigvee_{s, s' \in S} [\pi(s) \wedge \mu^*(s, x, s') \wedge \eta(s')], \tag{38}$$

where I and 0 are maximal and minimal elements of the complete distributive lattice L, respectively. It is noted that the expressions (37) and (38) are obtained by replacing $*$ by $\wedge$ from (22) and (25).

Remark. As the complete distributive lattice is a special case of the complete lattice ordered semigroup, lattice automaton is considered as a special case of l-semigroup automaton of 3.1.

By the way, the operations $\vee$ and $\wedge$ are dual in a complete distributive lattice $L = (L, \vee, \wedge)$, so the dual automaton of lattice automaton can be formulated by the following.

3.4. *Dual Lattice Automaton*

(3.4a). This is the same as (3.3a).

(3.4b). Using the concept of composition of L-fuzzy relations (4), μ^* and w are given as follows.

$$\mu^*(s, \epsilon, s') = \begin{cases} 0 \cdots & s = s', \\ I \cdots & s \neq s', \end{cases} \tag{39}$$

$$\mu^*(s, xa, s') = \bigwedge_{s'' \in S} [\mu^*(s, x, s'') \vee \mu(s'', a, s')], \tag{40}$$

$$w(x) = \bigwedge_{s, s' \in S} [\pi(s) \vee \mu^*(s, x, s') \vee \eta(s')]. \tag{41}$$

Remark. It is noted that, given a certain initial state s_0 and a final state set F, π and η of lattice automaton of 3.3 are given as follows in the same manner as (23) and (24).

$$\pi(s) = \begin{cases} I \cdots & s = s_0, \\ 0 \cdots & s \neq s_0, \end{cases} \tag{42}$$

$$\eta(s) = \begin{cases} I \cdots & s \in F, \\ 0 \cdots & s \notin F. \end{cases} \tag{43}$$

However, π and η of dual lattice automaton of 3.4 are

$$\pi(s) = \begin{cases} 0 \cdots & s = s_0\,, \\ I \cdots & s \neq s_0\,, \end{cases} \tag{44}$$

$$\eta(s) = \begin{cases} 0 \cdots & s \in F, \\ I \cdots & s \notin F. \end{cases} \tag{45}$$

3.5. *(Pessimistic) Fuzzy Automaton* [1–3, 7–11]

(3.5a). As the weighting space, $J = ([0, 1], \max, \min)$ is adopted. Needless to say, J is a complete distributive lattice under the operations max and min. μ, π, and η are as follows.

$$\mu\colon S \times \Sigma \times S \to [0, 1], \tag{46}$$

$$\pi\colon S \to [0, 1], \tag{47}$$

$$\eta\colon S \to [0, 1]. \tag{48}$$

(3.5b). μ^* and w are given by using the concept of composition of fuzzy relations (5), i.e.,

$$\mu^*(s, \epsilon, s') = \begin{cases} 1 \cdots & s = s', \\ 0 \cdots & s \neq s', \end{cases} \tag{49}$$

$$\mu^*(s, xa, s') = \max_{s'' \in S} \min[\mu^*(s, x, s''), \mu(s'', a, s')], \tag{50}$$

$$w(x) = \max_{s, s' \in S} \min[\pi(s), \mu^*(s, x, s'), \eta(s')]. \tag{51}$$

Remark. As $J = ([0, 1], \max, \min)$ is a complete distributive lattice, fuzzy automaton can be considered as special case of lattice automaton of 3.3. Therefore μ^* and w of (49)–(51) are obtained from (36)–(38) by replacing $\vee$ by max, $\wedge$ by min, I by 1, and 0 by 0.

3.6. *Optimistic Fuzzy Automaton* [1–3, 9]

(3.6a). This is the same as (3.5a).

(3.6b). μ^* and w are given by using the composition of fuzzy relations (6), i.e.,

$$\mu^*(s, \epsilon, s') = \begin{cases} 0 \cdots & s = s', \\ 1 \cdots & s \neq s', \end{cases} \tag{52}$$

$$\mu^*(s, xa, s') = \min_{s'' \in S} \max[\mu^*(s, x, s''), \mu(s'', a, s')], \tag{53}$$

$$w(x) = \min_{s, s' \in S} \max[\pi(s), \mu^*(s, x, s'), \eta(s')]. \tag{54}$$

Remark 1. Optimistic fuzzy automaton is the special case of dual lattice automaton of 3.4.

Remark 2. Given an initial state s_0 and a final state set F, π and η of fuzzy automaton of 3.5 are obtained from (42) and (43) by replacing I by 1 and 0 by 0, but π and η of optimistic fuzzy automaton are obtained from (44) and (45) [2].

3.7. *Mixed Fuzzy Automaton* [3]

(3.7a). This is the same as (3.5a).

(3.7b). μ^* and w are defined by using the concept of convex combination of fuzzy sets [6], i.e.,

$$\mu^*(s, x, s') = a\mu^*_{PF}(s, x, s') + b\mu^*_{OF}(s, x, s'), \tag{55}$$

$$w(x) = aw_{PF}(x) + bw_{OF}(x), \tag{56}$$

where μ^*_{PF} and μ^*_{OF} are the state transition functions defined by (pessimistic) fuzzy automaton of 3.5 and optimistic fuzzy automaton of 3.6, respectively. This is the same for w_{PF} and w_{OF}. And a, b ($\geqslant 0$) are real numbers such that $a + b = 1$.

3.8. *Composite Fuzzy Automata* [1, 8]

(3.8a). This is the same as (3.5).

(3.8b). μ^* is obtained by operating between μ^*_{PF} and μ^*_{OF} with probability p. This is the same for w.

3.9. *Nondeterministic Automaton* [12]

(3.9a). $J' = (\{0, 1\}, \max, \min)$ is adopted as the weighting space. Clearly J' forms a distributive lattice (more precisely, a boolean lattice). μ, π, and η are given as follows.

$$\mu: S \times \Sigma \times S \to \{0, 1\}, \tag{57}$$

$$\pi: S \to \{0, 1\}, \tag{58}$$

$$\eta: S \to \{0, 1\}. \tag{59}$$

(3.9b). This is the same as (3.5b).

Remark. Nondeterministic automaton is the special case of fuzzy automaton of 3.5 (or l-semigroup automaton of 3.1).

3.10. *Deterministic Automaton* [12]

(3.10a). This is the same as (3.9a) plus the additional constraints that there exists only one $s' \in S$ such that $\mu(s, a, s') = 1$ for each $s \in S$ and $a \in \Sigma$ and $\mu(s, a, s'') = 0$

for other s'' ($\neq s'$), and that there exists only one $s' \in S$ (that is, s' is an initial state) such that $\pi(s') = 1$ and $\pi(s'') = 0$ for other s'' ($\neq s'$). As for η, let F be a set of final states, then

$$\eta(s) = \begin{cases} 1 \cdots & s \in F, \\ 0 \cdots & s \notin F. \end{cases}$$

(3.10b). This is the same as (3.9b).

Remark. Clearly, deterministic automaton is the special case of nondeterministic automaton of 3.9 and also of probabilistic automaton of 3.18 defined later.

3.11. *Boolean Automaton*

(3.11a). As the weighting space, a complete boolean lattice $B = (B, \vee, \wedge)$ is adopted, where the operations $\vee$ and $\wedge$ are lub and glb in B. Clearly, the boolean lattice is a special case of the distributive lattice. Then μ, π, and η are as follows.

$$\mu: S \times \Sigma \times S \to B, \tag{60}$$

$$\pi: S \to B, \tag{61}$$

$$\eta: S \to B. \tag{62}$$

(3.11b). This is the same as (3.3b).

Remark. Boolean automaton and dual boolean automaton defined next are the special cases of lattice automaton of 3.3 and dual lattice automaton of 3.4, respectively.

3.12. *Dual Boolean Automaton*

(3.12a). This is the same as (3.11a).

(3.12b). This is the same as (3.4b).

3.13. *Mixed Boolean Automaton*

(3.13a). This is the same as (3.11a).

(3.13b). Using the concept of convex combination of B-fuzzy sets [13], μ^* and w are defined as follows.

$$\mu^* = (\alpha \wedge \mu_B^*) \vee (\bar{\alpha} \wedge \mu_{DB}^*), \tag{63}$$

$$w = (\alpha \wedge w_B) \vee (\bar{\alpha} \wedge w_{DB}), \tag{64}$$

where μ_B^* and μ_{DB}^* are the state transition functions which are defined in boolean automaton of 3.11 and dual boolean automaton of 3.12, respectively. α, $\bar{\alpha} \in B$ and $\bar{\alpha}$ is the complement of α.

3.14. *Semiring Automaton*

(3.14a). The weighting space is a semiring $R = (R, +, \times)$ with unity 1 and zero 0.[3] μ, π, and η are given as

$$\mu: S \times \Sigma \times S \to R, \tag{65}$$

$$\pi: S \to R, \tag{66}$$

$$\eta: S \to R. \tag{67}$$

(3.14b). μ^* and w are given as follows.

$$\mu^*(s, \epsilon, s') = \begin{cases} 1 \cdots & s = s', \\ 0 \cdots & s \neq s', \end{cases} \tag{68}$$

$$\mu^*(s, xa, s') = \sum_{s'' \in S} [\mu^*(s, x, s'') \times \mu(s'', a, s')], \tag{69}$$

$$w(x) = \sum_{s, s' \in S} [\pi(s) \times \mu^*(s, x, s') \times \eta(s')]. \tag{70}$$

Remark. As the special case of semiring automaton, there exist l-semigroup automaton of 3.1, max-product automaton of 3.2, lattice automaton of 3.3, fuzzy automaton of 3.5, nondeterministic automaton of 3.9, boolean automaton of 3.12, and so on.

3.15. *Weighted Automaton* [14, 19]

(3.15a). The weighting space is $R = ([0, \infty), +, \cdot)$, where $[0, \infty)$ is a set of nonnegative numbers and the operations $+$ and $\cdot$ are ordinary addition and product, respectively. Obviously, $R = ([0, \infty), +, \cdot)$ is the semiring with unity and zero. Then μ, π, and η are defined from (65)–(67) by replacing R by $[0, \infty)$ as follows.

$$\mu: S \times \Sigma \times S \to [0, \infty), \tag{71}$$

$$\pi: S \to [0, \infty), \tag{72}$$

$$\eta: S \to [0, \infty). \tag{73}$$

[3] The set R with the operations of addition $+$ and multiplication $\times$ is called a *semiring* if the following three conditions are satisfied. (1) $+$ is associative and commutative; (2) $\times$ is associative; (3) $\times$ distributes over $+$, i.e.,

$$a \times (b + c) = a \times b + a \times c, \qquad (b + c) \times a = b \times a + c \times a,$$

for all a, b, c in R. The semiring R is called a *semiring with unity* 1 *and zero* 0 if 1 is identity under $\times$ and 0 is identity under $+$ in R. For example, let R be $([0, \infty), +, \cdot)$ with ordinary addition $+$ and ordinary product $\cdot$, then $[0, \infty)$ (= set of nonnegative numbers) is a semiring with unity 1 and zero 0. Similarly, the set of natural numbers containing 0 is also a semiring with unity and zero under $+$ and $\cdot$. And $R = ([0, 1], \max, \cdot)$ is a semiring with unity and zero. Note that this R is also an l-semigroup. In general, it is found that l-semigroup and complete distributive lattice are the special cases of the semiring with unity and zero.

(3.15b). μ^* and w are defined by letting $+$ be ordinary addition and replacing $\times$ by $\cdot$ (= ordinary product) in (68)–(70).

$$\mu^*(s, \epsilon, s') = \begin{cases} 1 \cdots & s = s', \\ 0 \cdots & s \neq s', \end{cases} \tag{74}$$

$$\mu^*(s, xa, s') = \sum_{s'' \in S} \mu^*(s, x, s'') \cdot \mu(s'', a, s'), \tag{75}$$

$$w(x) = \sum_{s, s' \in S} \pi(s) \cdot \mu^*(s, x, s') \cdot \eta(s'). \tag{76}$$

Remark. Weighted automaton is found to be a special case of semiring automaton of 3.14.

3.16. *Max-Weighted Automaton* [14]

(3.16a). Let the weighting space be $R = ([0, \infty), \max, \cdot)$ with ordinary product $\cdot$. Clearly R is a semiring with unity and zero. μ, π, and η are given in the same way as (71)–(73).

(3.16b). μ^* and w are obtained as follows.

$$\mu^*(s, \epsilon, s') = \begin{cases} 1 \cdots & s = s', \\ 0 \cdots & s \neq s', \end{cases} \tag{77}$$

$$\mu^*(s, xa, s') = \max_{s'' \in S}[\mu^*(s, x, s'') \cdot \mu(s'', a, s')], \tag{78}$$

$$w(x) = \max_{s, s' \in S}[\pi(s) \cdot \mu^*(s, x, s') \cdot \eta(s')]. \tag{79}$$

Remark 1. Max-weighted automaton is the special case of semiring automaton of 3.14.

Remark 2. Max-product automaton of 3.2 can be reduced from max-weighted automaton by replacing $[0, \infty)$ by $[0, 1]$.

3.17. *Natural Numbered Automaton*

(3.17a). The weighting space is $N = (N, +, \cdot)$, where N is a set of natural numbers which contains 0.

$$\mu: S \times \Sigma \times S \to N, \tag{80}$$

$$\pi: S \to N, \tag{81}$$

$$\eta: S \to N. \tag{82}$$

(3.17b). This is the same as (3.15b).

Remark 1. Natural numbered automaton is the special case of weighted automaton of 3.15.

Remark 2. Max-natural numbered automaton can be easily defined in a same manner as max-weighted automaton of 3.16.

3.18. *Probabilistic Automaton* [12, 15]

(3.18a). Let the weighting space be $([0, 1], +, \cdot)$, then μ, π, and η are

$$\mu: S \times \Sigma \times S \to [0, 1], \tag{83}$$

$$\pi: S \to [0, 1], \tag{84}$$

$$\eta: S \to [0, 1], \tag{85}$$

and, in addition, the following constraint of μ, π, and η are assumed. For each $s \in S$ and $a \in \Sigma$,

$$\sum_{s' \in S} \mu(s, a, s') = 1, \qquad \sum_{s' \in S} \pi(s') = 1. \tag{86}$$

As for η, let F be a final state set, then

$$\eta(s) = \begin{cases} 1 \cdots & s \in F, \\ 0 \cdots & s \notin F. \end{cases} \tag{87}$$

(3.18b). This is the same as (3.15b).

Remark. There exists another definition of η different from (87) [16]. That is, in the same way as μ and π of (86), we have

$$\sum_{s \in S} \eta(s) = 1. \tag{88}$$

3.19. *Generalized Probabilistic Automaton* [16, 17]

(3.19a). This is the same as (3.18a) without the assumption that the range of η is not the unit interval [0, 1] but a set of real numbers $(-\infty, \infty)$, i.e.,

$$\eta: S \to (-\infty, \infty). \tag{89}$$

(3.19b). This is the same as (3.18b).

Remark. The language accepted by generalized probabilistic automaton is defined by

$$L(A, \lambda) = \{x \in \Sigma^* \mid w(x) \geqslant \lambda\}, \tag{90}$$

where λ is in $(-\infty, \infty)$. As for probabilistic automaton of 3.18, λ is in [0, 1].

3.20. *Rational Probabilistic Automaton* [18]

(3.20a). This is the same as (3.18a) plus the assumption that the values $\mu(s, a, s')$ and $\pi(s)$ are rational numbers in [0, 1].

(3.20b). This is the same as (3.18b).

3.21. *Ring Automaton*

(3.21a). The weighting space is a ring with identity[4] $R = (R, +, \times)$. μ, π, and η are

$$\mu: S \times \Sigma \times S \to R, \tag{91}$$

$$\pi: S \to R, \tag{92}$$

$$\eta: S \to R. \tag{93}$$

(3.21b). This is the same as (3.14b).

Remark. Ring automaton is the special case of semiring automaton of 3.14. It is noted that weighted automaton of 3.15 and max-weighted automaton of 3.16 which are the special cases of semiring automaton of 3.14 are *not* the special case of ring automaton.

3.22. *Integer-Valued Generalized Automaton* [16, 18]

(3.22a). The weighting space is $Z = (Z, +, \cdot)$, where Z is a set of integers and the operations $+$ and $\cdot$ are ordinary addition and product, respectively. Clearly Z is a ring with identity. μ, π, and η are

$$\mu: S \times \Sigma \times S \to Z, \tag{94}$$

$$\pi: S \to Z, \tag{95}$$

$$\eta: S \to Z. \tag{96}$$

(3.22b). This is the same as (3.15b).

Remark. Integer-valued generalized automaton is a special case of ring automaton of 3.21.

[4] The set R with the operations of addition $+$ and multiplication $\times$ is called a *ring* if (1) R is an Abelian group under $+$, (2) $\times$ is associative, (3) $\times$ distributive over $+$, i.e.,

$$a \times (b + c) = a \times b + a \times c, \qquad (b + c) \times a = b \times a + c \times a$$

for all a, b, c in R. In addition, if R has an identity 1 under $\times$, then R is called a *ring with identity*.

3.23. *Field Automaton*

(3.23a). Let the weighting space be a field $F = (F, +, \times)$,[5] then μ, π, and η are

$$\mu: S \times \Sigma \times S \to F, \tag{97}$$

$$\pi: S \to F, \tag{98}$$

$$\eta: S \to F. \tag{99}$$

(3.23b). This is the same as (3.21b).

Remark. Clearly, field automaton is a special case of ring automaton of 3.21. Integer-valued generalized automaton of 3.22 which is a special case of ring automaton is *not* a special case of field automaton.

3.24. (*Real-Valued*) *Generalized Automaton* [16, 18, 19]

(3.24a). The weighting space is $F = ((-\infty, \infty), +, \cdot)$, where $(-\infty, \infty)$ is a set of real numbers, and $+$, $\cdot$ are ordinary addition and product. μ, π, and η are

$$\mu: S \times \Sigma \times S \to (-\infty, \infty), \tag{100}$$

$$\pi: S \to (-\infty, \infty), \tag{101}$$

$$\eta: S \to (-\infty, \infty). \tag{102}$$

(3.24b). This is the same as (3.15b).

Remark. Real-valued generalized automaton is a special case of field automaton of 3.23.

3.25. *Rational Automaton*

(3.25a). The weighting space is $Q = (Q, +, \cdot)$, where Q is a set of rational numbers, and $+$, $\cdot$ are ordinary addition and product. μ, π, and η are

$$\mu: S \times \Sigma \times S \to Q, \tag{103}$$

$$\pi: S \to Q, \tag{104}$$

$$\eta: S \to Q. \tag{105}$$

(3.25b). This is the same as (3.15b).

Remark. Rational automaton is the special case of real-valued generalized automaton of 3.24 and also of field automaton of 3.23.

[5] The set F with addition $+$ and multiplication $\times$ is called a *field* if (1) F is a ring, and (2) $F - \{0\}$ is a group under x, where 0 is a zero element of F.

4. Conclusion

We have derived various kinds of automata with weights. Some of these automata are scanty of physical images. But, for example, from the fact that the classes of languages defined by rational probabilistic automata of 3.20 and integer-valued generalized automata of 3.22 are equal, various problems concerning with rational probabilistic automata can be solved by investigating the properties of integer-valued generalized automata [16, 18]. Therefore, the automata with weights will play an important role in investigating the properties of well-known automata such as deterministic automata and probabilistic automata. Moreover, they will come to be useful models of, say, learning systems, gamings, and pattern recognitions as in the case of fuzzy automata [1, 7, 8].

By the way, as the special case of field automaton of 3.23, we can define a complex numbered automaton, since the set of complex numbers forms a field. We cannot, however, define a language accepted by this automaton in the same way as (26) because of the fact that there does not exist an order relation $\geqslant$ in the set of complex numbers. But, using the concept of mapping of the set of complex numbers into a certain algebra system with ordering, say, by transforming the complex number z to the absolute value $|z|$, we can define a language by complex numbered automaton A as $L(A, \lambda) = \{x \in \Sigma^* \mid |w(x)| \geqslant |\lambda|\}$. If not permitted to use the concept of mappings, we would have to adopt a ring and a field with orderings [20, 21] as the weighting space.

Recently, probabilistic pushdown automata, probabilistic Turing machines, fuzzy pushdown automata, and fuzzy Turing machines have been reported in many papers as extensions of probabilistic automata and fuzzy automata. So we hope that pushdown automata, linear bounded automata, and Turing machines with "weights" will be formulated as extensions of automata with weights.

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