L-Fuzzy Grammars

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ABSTRACT

An L-fuzzy grammar is defined by assigning the element of lattice to the rewriting rules of a formal grammar. According to the kind of lattice, say, distributive lattice, lattice-ordered group, and lattice-ordered monoid, two type of L-fuzzy grammars are defined. It is shown that some context-sensitive languages can be generated by type 3 $*$-L-fuzzy grammars with cut points. It is also shown that for type 2 L-fuzzy grammars, Chomsky and Greibach normal form can be constructed as an extension of corresponding notion in the theory of formal grammars.

1. INTRODUCTION

Recently, in order to reduce the gap between natural languages and formal languages, stochastic grammars and fuzzy grammars have been formulated by introducing the concept of randomness and fuzziness into the structure of the formal grammars [3-5, 7, 8]. In addition, several interesting grammars with weights are also defined by corresponding the element of the appropriate algebraic system to each rewriting rule of grammars [6].

Now, incorrectness and ambiguity, which natural languages such as English have syntactically and semantically, may be reduced to impose consistency among means of words in the sentence, valuation of means of the word, and so forth on formal languages. Therefore, if the suitable characteristic parameters which describe the property of grammars or languages may be given, then the grammar with such characteristic parameters may be considered as a model for natural languages.

In this paper, we define L-fuzzy grammars by introducing the concept of L-fuzzyy sets [1] into the structure of formal grammars, that is, L-fuzzy grammars are defined by assigning the element of lattice to the rewriting rules of formal grammars. Two types of L-fuzzy grammars will be considered. One is the $\wedge$-L-fuzzy grammar ($\wedge$-*lfg* for short) whose membership space consists of a distributive lattice or a Boolean lattice. A fuzzy grammar is a special case of the $\wedge$-*lfg*. The other is the $*$-L-fuzzy grammar ($*$-*lfg* for short), whose membership space consists of a lattice-ordered group or a lattice-ordered monoid.

Some basic results concerning the families of languages generated by the $\wedge$-*lfg* and the $*$-*lfg* are given in Sec. 3. In Sec. 4, it is shown that the so called Chomsky and Greibach normal form for a type 2 $\wedge$-*lfg* and a type 2 $*$-*lfg* can be constructed as the extension of the corresponding notion in the theory of formal grammars.

2. BASIC DEFINITIONS

We shall review L-fuzzy sets [1] for the purpose of defining L-fuzzy grammars later.

L-FUZZY SETS

An *L-fuzzy set A* in a space $X = \{x\}$ is a function such that

$$A : X \longrightarrow L, \tag{1}$$

where L is called a membership space which is a partially ordered set or, more precisely, lattice-ordered semigroup or lattice [11], and the value $A(x)$ in L represents the "grade of membership" of X in A. When L is the unit interval [0,1], A is the fuzzy set originated by Zadeh [2].

PRODUCT OF L-FUZZY RELATIONS

An *L-fuzzy relation R* is a function from product space $X \times Y = \{(x,y) \mid x \in X, y \in Y\}$ to L, i.e.,

$$R : X \times Y \longrightarrow L. \tag{2}$$

If R_1 and R_2 are two L-fuzzy relations in $X \times X$, then the product of R_1 and R_2 is defined by an L-fuzzy relation denoted as R_1R_2 and is written as follows:

$$R_1R_2(x,z) = \vee\, [R_1(x,y) \circ R_2(y,z)], \tag{3}$$

where $\vee$ is an operation of supremum, $\circ$ is an operation of infimum when L is a distributive lattice of a Boolean lattice, or an operation of group or monoid when L is a lattice-ordered group or a lattice-ordered monoid, respectively. In general, if L is distributive, the product of $R, \cdots, R$ of L-fuzzy relation R in $X \times X$ is defined as follows:

$$R \cdots R(x,y) = R^n(x,y) = \bigvee_{x_1,\cdots,x_{n-1}} [R(x,x_1) \circ R(x_1,x_2) \cdot \circ \cdots \circ R(x_{n-1},y)]. \tag{4}$$

L-FUZZY GRAMMARS

An L-fuzzy grammar (*lfg* for short) is a system such that

$$G = (V_N, V_T, P, S, J, \mu, L), \tag{5}$$

where V_N is the set of nonterminal, V_T is a set of terminal, S is an initial symbol, J is a set of labels of rules, and P is a finite set of productions such as

$$(r)\, u \longrightarrow v\, \mu(r), \tag{6}$$

in which $r \in J$, $u \to v$ is an ordinary rewriting rule, and μ is an L-fuzzy set from J to L, i.e., $\mu : J \to L$. L consists of a distributive lattice, a Boolean lattice, a lattice-ordered group of a lattice-ordered monoid. The value $\mu(r)$ in L is called the grade of the application of the rule r. Concerning with the L-fuzzy relation R in $(V_N \cup V_T)^* \times (V_N \cup V_T)^*$, the value of R is defined as follows:

$$R(\alpha u \beta, \alpha v \beta) = \begin{cases} \mu(r) \text{ if } (r)\, u \longrightarrow v\, \mu(r) \in P, \\ -\infty \text{ if } (r)\, u \longrightarrow v\, \mu(r) \notin P, \end{cases} \tag{7}$$

where $\alpha u \beta$ and $\alpha v \beta$ are in $(V_N \cup V_T)^*$. If L is a complete lattice, $-\infty$ may be assumed as an infinum of L, i.e., $\inf L = -\infty$. In parallel with the standard classification of formal grammars [9], four principal types of *lfg* may be distinguished as type 0, 1, 2, and 3 *lfg*. Furthermore, according to differences among lattices, two types of *lfg* may be considered. That is, $\wedge$-*lfg*, whose membership space consists of a distributive lattice or a Boolean lattice, and $*$-*lfg*, whose membership space consists of a lattice-ordered group or a lattice-ordered monoid, may be considered.

L-FUZZY LANGUAGES

An *L-fuzzy languages*, $L(G)$, is a set of ordered pairs,

$$L(G) = \{(x, \mu_G(x))\}, x \in V_T^*, \tag{8}$$

where $\mu_G(x)$ is the grade of membership of x generated by *lfg* G and is given by

$$\mu_G(x) = R^+(S, x), \text{ where } R^+ = \cup R^i, i \geqq 1. \tag{9}$$

When the derivation chains from S to x are expressed as

$$S \underset{r_{i1}}{\overset{\mu(r_{i1})}{\Longrightarrow}} \alpha_{i1} \underset{r_{i2}}{\overset{\mu(r_{i2})}{\Longrightarrow}} \alpha_{i2} \Longrightarrow \cdots \underset{r_{im}}{\overset{\mu(r_{im})}{\Longrightarrow}} \alpha_{im} = x, \tag{10}$$

the value of $\mu_G(x)$ is also given by

$$\mu_G(x) = \bigvee_i \, [\mu(r_{i1}) \circ \mu(r_{i2}) \circ \cdots \circ \mu(r_{im})], \tag{11}$$

where the supremum $\vee$ is taken over all derivation chains from S to x, $\circ$ is an operation of infinum when an L-fuzzy grammar is a $\wedge$-*lfg*, or an operation of group or monoid when an L-fuzzy grammar is a $*$-*lfg*.

The *language*, $L(G, \lambda)$, which is generated by an *lfg* G with cut point $\lambda(\in L)$ is defined as follows:

$$L(G,\lambda) = \{x \in V_T^* \mid \mu_G(x) \geqq \lambda\}. \tag{12}$$

3. CLASS OF L-FUZZY LANGUAGES

3.1. THE CLASS OF LANGUAGES GENERATED BY $\wedge$-lfg

In this section, to discuss the class of L-fuzzy languages, let us consider the language, $L(G, \lambda)$, by an $\wedge$-*lfg* G with cut point $\lambda(\in L)$.

In general, we may assume that an L-fuzzy relation R assigns a sublattice of membership space L to an L-fuzzy grammar. In Lemma 1, we investigate the property of sublattices. Using this property, it is shown that the class of the languages generated by the type 3 $\wedge$-*lfg* is a regular set.

Theorem 2 shows that the class of languages generated by type 2 $\wedge$-*lfg* with cut point properly contains the class of context-free languages.

In Theorem 3, it is shown that type 1 $\wedge$-*lfg* is recursive.

LEMMA 1. *In the distributive lattice whose number of elements is infinite, the number of elements of the sublattice which is generated from arbitrary finite elements of lattice is finite.*

Proof. In lattice, an arbitrary finite subset of the elements has its supremum and infinum. Also the arbitrary lattice polynominal of distributive lattice can be expressed such that

$$P\{x_1, x_2, \cdots, x_m\} = \vee \{x_{i1} \wedge x_{i2} \wedge \cdots \wedge x_{ik}\}, x_{ij} \in \{x_1, x_2, \cdots, x_m\},$$

or

$$P\{x_1, x_2, \cdots, x_m\} = \wedge \{x_{i1} \vee x_{i2} \vee \cdots \vee x_{ik}\}, x_{ij} \in \{x_1, x_2, \cdots, x_m\}. \tag{13}$$

Therefore, the number of elements of the sublattice which is generated from arbitrary finite elements of L is equal to the number of lattice polynominals. From the above fact, it can be seen that the number of elements of the sublattice is finite. And the number of elements of the sublattice Y is obtained as follows:

$$|Y| \leqq 2 \sum_{k=1}^{n-1} \sum_{j=1}^{\binom{n}{k}} \binom{\binom{n}{k}}{j}, \tag{14}$$

where Y is the sublattice generated from the set of arbitrary elements $X = \{x_1, x_2, \cdots, x_n\}$, and an operation $(\ )$ is an operation of combination.

THEOREM 2. *The class of languages generated by type* 3 $\wedge$-*lfg with cut point* λ *is the class of regular sets.*

Proof. Let $G = (V_N, V_T, P, S, J, \mu, L)$ be the type 3 $\wedge$-*lfg* with $V_N = \{S, A_1, A_2, \cdots, A_m\}$, $|J| = n (n \geqq m + 1)$. Then the equivalence relation $\equiv$ of $x, y (\in V_T^*)$ is defined as follows: For any $A_1 \in V_N \cup \{\epsilon\}$ (ϵ is a null symbol), define

$$x \equiv y \Longleftrightarrow R^+(S, xA_i) = R^+(S, yA_i). \tag{15}$$

Then

$$\begin{aligned} R^+(S, xzA) &= \bigvee_i [R^+(S, xA_i) R^+(xA_i, xzA)], \\ &= \bigvee_i [R^+(S, xA_i) R^+(A_i, zA)], \\ &= \bigvee_i [R^+(S, yA_i) R^+(A_i, zA)], \\ &= \bigvee_i [R^+(S, yA_i) R^+(yA_i, yzA)], \\ &= R^+(S, yzA). \end{aligned} \tag{16}$$

Therefore the equivalence relation $\equiv$ is a right invariant equivalence relation. Here let us define the $m + 2$ array vector consisted of the elements of the membership space L as follows:

$$(R^+(S, xS), R^+(S, xA_1), R^+(S, xA_2), \cdots, R^+(S, xA_m), R^+(S, x)) = (\nu_0, \nu_1, \nu_2, \cdots, \nu_m, \nu_{m+1}), \tag{17}$$

where $x \in V_T^*$, and $S, A_1, A_2, \cdots, A_m \in V_N$, then the number of equivalent classes by the relation $\equiv$ is less than the number of the $m + 2$ array vectors $(\nu_0, \nu_1, \nu_2, \cdots, \nu_m, \nu_{m+1})$. And from Lemma 1, it is evident that the number of vectors is finite. Consequently, the language generated by the type 3 $\wedge$-*lfg* G is the union of the equivalent class of a right invariant equivalence relation $\equiv$ of finite index. We conclude that the class of the languages generated by the type 3 $\wedge$-*lfg*'s is the class of regular sets.

THEOREM 3. *The class of languages generated by the type* 2 $\wedge$-*lfg's with cut point properly contains the class of context-free languages.*

Proof. In the first place, let us show that the context-sensitive language is generated by the type 2 $\wedge$-*lfg* with cut point. Consider the type 2 $\wedge$-*lfg* with the productions

(1) $S \longrightarrow AB\ \nu_1$,	(4) $B \longrightarrow cB\ \nu_4$,
(2) $A \longrightarrow aAb\ \nu_2$,	(5) $B \longrightarrow c\ \nu_3$,
(3) $A \longrightarrow ab\ \nu_3$,	(6) $S \longrightarrow CD\ \nu_6$,

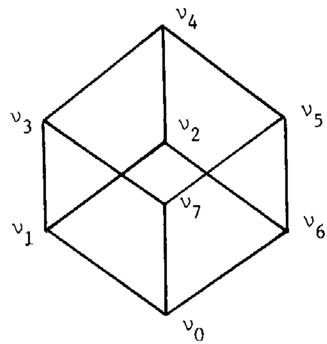

Fig. 1. Membership space of L.

(7) $C \longrightarrow aC\ \nu_5$, (9) $D \longrightarrow bDc\ \nu_2$,

(8) $C \longrightarrow a\ \nu_4$, (10) $D \longrightarrow bc\ \nu_4$.

where the membership space L is as in Fig 1.

Now, let us generate $a^2b^2c^2$ by Λ-*lfg* in practice. The derivation chains of $a^2b^2c^2$ is obtained as follows:

$$(1)\ S \underset{1}{\overset{\nu_1}{\Longrightarrow}} AB \underset{2}{\overset{\nu_2}{\Longrightarrow}} aAbB \underset{3}{\overset{\nu_3}{\Longrightarrow}} aabbB \underset{4}{\overset{\nu_4}{\Longrightarrow}} aabbcB \underset{3}{\overset{\nu_3}{\Longrightarrow}} aabbcc,$$

$$(2)\ S \underset{6}{\overset{\nu_6}{\Longrightarrow}} CD \underset{5}{\overset{\nu_5}{\Longrightarrow}} aCD \underset{4}{\overset{\nu_4}{\Longrightarrow}} aaD \underset{2}{\overset{\nu_2}{\Longrightarrow}} aabDc \underset{4}{\overset{\nu_4}{\Longrightarrow}} aabbcc,$$

The derivation chains of $a^2b^2c^2$ may be obtained from another source than the derivation chains of (1) and (2). However, the same rules of the derivation chains as (1) and (2) are used in those derivation chains. Thus the membership grade of $a^2b^2c^2$ is obtained as follows:

$$\mu_G(a^2b^2c^2) = (\nu_1 \wedge \nu_2 \wedge \nu_3 \wedge \nu_4 \wedge \nu_3) \vee (\nu_6 \wedge \nu_5 \wedge \nu_4 \wedge \nu_2 \wedge \nu_4)$$
$$= \nu_1 \vee \nu_6 = \nu_2$$

In general, the membership grade of $a^nb^nc^n$, $a^mb^mc^n$, and $a^mb^nc^n$ is obtained as $\mu_G(a^nb^nc^n) = \nu_2$, $\mu_G(a^mb^mc^n) = \nu_1$, and $\mu_G(a^mb^nc^n) = \nu_6$, respectively. Therefore, let the cut point $\lambda = \nu_2$, then the type 1 language is obtained, that is,

$$L(G, \nu_2) = \{a^nb^nc^n \mid n \geqq 1\}.$$

In the next place, we show that any context-free languages can be generated by type 2 Λ-*lfg*. It is enough to consider the case that the class of type 2 Λ-*lfg* whose membership space, L, is equal to $\{0, I\}$.

Suppose that the membership function μ of type 2 Λ-*lfg* G assigns the element I for each rule $r \in J$, then type 2 Λ-*lfg* G may be regarded as the context-free grammar in the sense of formal grammar. Therefore the class of the languages $L(G,I)$ is the class of context-free languages.

THEOREM 4. *Type 1 $\wedge$-lfg is recursive.*

Proof. From the definition of type 1 $\wedge$-*lfg*, it is easily determined whether (ϵ,ν) is in $L(G)$ or not by inspecting the productions of G. Then we can assume that P does not contain a production $(r)S\to\epsilon\,\mu(r)$.

Suppose that there is a derivation $S \overset{*}{\underset{G}{\Rightarrow}} x\,(x\in V_T^+)$ whose derivation chain C is given as follows:

$$C\,;S \xrightarrow[r_1]{\mu(r_1)} \alpha_1 \xrightarrow[r_2]{\mu(r_2)} \alpha_2 \Longrightarrow \cdots \xrightarrow[r_m]{\mu(r_m)} \alpha_m = x,$$

where r_i is the label of rule and $\mu(r_i)$ is the grade of rule r_i. In addition, suppose that in the derivation chain C, α_i is the same as α_j, $i<j$. Now, consider the shorter derivation chain C', which is obtained by replacing the subchain

$$\alpha_i \xrightarrow[r_{i+1}]{\mu(r_{i+1})} \cdots \Longrightarrow \alpha_j \xrightarrow[r_{j+1}]{\mu(r_{j+1})} \alpha_{j+1}$$

in C by

$$\alpha_i \xrightarrow[r_{j+1}]{\mu(r_{j+1})} \alpha_{j+1}.$$

Clearly C' is a derivation chains from S to x. And following equation holds.

$$\mu(r_1)\wedge\cdots\wedge\mu(r_i)\wedge\cdots\wedge\mu(r_j)\wedge\mu(r_{j+1})\wedge\cdots\wedge\mu(r_m)$$
$$\leqq \mu(r_1)\wedge\cdots\wedge\mu(r_i)\wedge\mu(r_{j+1})\wedge\cdots\wedge\mu(r_m). \quad (18)$$

Thus the derivation chain C will not take part in the value $\mu_G(x)$. Therefore the value of $\mu_G(x)$ can be given by taking the supremum over all loop-free derivation chains from S to x. That is to say, if G is a type 1 $\wedge$-*lfg*, $|x|=n$, and $|V_N\cup V_T|=k$, then the value of $\mu_G(x)$ can be given by taking the supremum over finite set of derivation chains from S to x of length $l\leqq 1+k+\cdots+k^n$.

Next, in order to show the algorithm whether $(x,\nu)\in L(G)$ or not, we have to show a way for generating all finite derivation chains from S to x of length $l\leqq 1+k+\cdots+k^n=l_0$. As already mentioned, it can be determined whether $(\epsilon,\nu)\in L(G)$ or not by inspecting the productions of G. Therefore, we also assume that the set of productions P does not contain $(r)\,S\to\epsilon\,\mu(r)$. Now, let us define the set Q_m as the set of ordered pairs $(\alpha,R^m(\alpha))$ in which α is the string of length at most $n(=|x|)$ in $(V_N\cup V_T)^*$ and $R^m(\alpha)$ is the membership grade of α in G such that $S\overset{*}{\underset{G}{\Rightarrow}}\alpha$ by the derivations of at most m steps.

Formally,

$$Q_m=\{(\alpha,R^{m-1}(S,\beta)\wedge R(\beta,\alpha))\mid(\beta,R^{m-1}(S,\beta))\in Q_{m-1},$$
$$R(\beta,\alpha)=\mu(r),\text{ and }|\alpha|\leqq n\}. \quad (19)$$

Clearly, $Q_1 = \{(\alpha, R(S, \alpha)) \mid R(S, \alpha) = \mu(r), |\alpha| \leqq n\}$. We construct consecutively $Q_1, Q_2, Q_3, \cdots, Q_m$ until $m = l_0$ or $Q_m = \phi$, whichever happens first. It is apparent that if $Q_m = \phi$, then $Q_m = Q_{m+1} = \cdots = \phi$ since Q_m depends only on Q_{m-1}. From above, it is evident that the set Q_m is finite set and all of the ordered pairs of $(x, \mu_G(x))$ generated by all derivations of $l(\leqq l_0)$ steps can be obtained. That (x, ν) is the element of $L(G)$ is equal to that (x, ν_i) is in $\cup_m Q_m$ and $\nu = \sup \{\nu_i \mid (x, \nu_i) \in \cup_m Q_m\}$, where $x\ V_T^+$ and $\nu, \nu_i \in L$. Therefore, it can be determined recursively whether (x, ν) is the element of $L(G)$ or not. It can also be determined recursively whether x is the element of $L(G, \lambda)$ or not, since x is the element of $L(G, \lambda)$ equal to (x, ν_i) in $\cup_m Q_m$ and $\lambda \leqq \sup\ \{\nu_i | (x, \nu_i) \in \cup_m Q_m\}$

*3.2 THE CLASS OF LANGUAGES GENERATED BY *-lfg*

In this section, we shall consider the class of languages generated by **-lfg's* with cut points. As mentioned in Sec. 1, **-lfg* is defined as the *lfg* whose membership space is a lattice-ordered group or a lattice-ordered monoid. Here we would like to mention simply the property of a lattice-ordered group and a lattice-ordered monoid. Both a lattice-ordered group (for short L_g) and a lattice-ordered monoid (for short L_m) have the following properties.

(a) Both L_g and L_m are lattice. Let the operation $*$ be an operation of group in L_g or an operation of monoid in L_m.

(b) Both an operation in L_g and an operation in L_m preserve the order, that is, if $p \leqq q$, then $a * p * b \leqq a * q * b$, for all a, b in L_g or L_m.

(c) The distributive law holds such as

$$a * (p \vee q) * b = a * p * b \vee a * q * b,$$

$$a * (p \wedge q) * b = a * p * b \wedge a * q * b.$$

From the properties of L_g or L_m, an interesting property can be seen in the class of languages generated by **-lfg*.

THEOREM 5. *The class of the languages $L(G, \lambda)$ generated by type 3 *-lfg G with cut point properly includes the class of regular sets.*

Proof. At first, we shall show that context-sensitive languages and context-free languages are generated by the type 3 **-lfg* with cut point. Consider the type 3 **-lfg* G whose membership space is L_g such that $L_g = (Q^+ \times Q^+ \times Q^+ \times Q^+ \times Q^+ \times Q^+, \times, \vee, \wedge)$, where Q^+ is positive rational numbers, $\times$ is a multiple operation, $\vee$ is an operation of supremum, and $\wedge$ is an operation of infinum. The productions are

(1) $S \longrightarrow aS \quad (2, 1/2, 1, 1, 2, 1/2)$,

(2) $S \longrightarrow aA \quad (2, 1/2, 1, 1, 2, 1/2)$,

(3) $A \longrightarrow bA$ $(1, 1, 2, 1/2, 1/2, 2)$,
(4) $A \longrightarrow bB$ $(1, 1, 2, 1/2, 1/2, 2)$,
(5) $B \longrightarrow aB$ $(1/2, 2, 1, 1, 1, 1)$,
(6) $B \longrightarrow aC$ $(1/2, 2, 1, 1, 1, 1)$,
(7) $C \longrightarrow bC$ $(1, 1, 1/2, 2, 1, 1)$,
(8) $C \longrightarrow b$ $(1, 1, 1/2, 2, 1, 1)$.

Then the L-fuzzy languages, $L(G)$, generated by this type 3 $*$-*lfg* G are

$$L(G) = \{(a^i b^j a^k b^l, (2^{i-k}, 2^{k-i}, 2^{j-l}, 2^{l-j}, 2^{i-j}, 2^{j-i})) \mid i, j, k, l \geqq 1\}.$$

Therefore context-sensitive languages are obtained by choosing the suitable cut points. For example,

$$L(G, (1, 1, 1, 1, 1, 1)) = \{a^n b^n a^n b^n \mid n \geqq 1\},$$
$$L(G, (1, 1, 1, 1, 1, 0)) = \{a^n b^m a^n b^m \mid n \geqq m \geqq 1\},$$
$$L(G, (1, 1, 1, 0, 1, 1)) = \{a^n b^n a^n b^m \mid n \geqq m \geqq 1\},$$
$$L(G, (0, 1, 1, 1, 1, 1)) = \{a^m b^n a^n b^n \mid n \geqq m \geqq 1\},$$
$$L(G, (1, 0, 1, 0, 1, 1)) = \{a^n b^n a^m b^l \mid n \geqq m \geqq 1, n \geqq l \geqq 1\}.$$

Also, context-free languages are obtained by choosing the suitable cut points. For example,

$$L(G, (0, 0, 0, 0, 1, 1)) = \{a^n b^n a^m b^l \mid n, m, l \geqq 1\},$$
$$L(G, (1, 1, 0, 0, 0, 0)) = \{a^n b^m a^n b^l \mid n, m, l \geqq 1\},$$
$$L(G, (0, 0, 1, 1, 0, 0)) = \{a^m b^n a^l b^n \mid n, m, l \geqq 1\}.$$

When the membership space is not L_g but L_m, context-sensitive languages and context-free languages are obtained in the same manner as in the case of L_g. From the definition of type 3 $*$-*lfg*, it is evident that any regular sets are generated by type 3 $*$-*lfg* with cut point. In the above type 3 $*$-*lfg* G with cut point $\lambda = (0, 0, 0, 0, 0, 0)$, the language $L(G, \lambda)$ is the regular set.

Example. Consider the type 3 $*$-*lfg* G whose membership space is the lattice-ordered group L_g such that $L_g = (Q^+ \times Q^+ \times Q^+ \times Q^+, \times, \vee, \wedge)$. The productions are

(1) $S \longrightarrow aS$ $(2, 1/2, 1, 1)$,

(2) $S \longrightarrow bS$ $(1/2, 2, 2, 1/2)$,

(3) $S \longrightarrow cS$ $(1, 1, 1/2, 2)$,

(4) $S \longrightarrow a$ $(2, 1/2, 1, 1)$,

(5) $S \longrightarrow b \quad (1/2, 2, 2, 1/2)$,

(6) $S \longrightarrow c \quad (1, 1, 1/2, 2)$.

Let us actually generate $x \in \{a, b, c\}^+$ in which the number of $a's$ $|x_a|$, the number of $b's$ $|x_b|$ and the number of $c's$ $|x_c|$ are all two, and denote the set of word as $X_2 = \{x | x \in V_T^*, |x_a| = |x_b| = |x_c| = 2\}$. For the element $abccab \in X_2$, the derivation chain C is obtained as

$$C; S \xRightarrow[1]{(2,\,1/2,\,1,\,1)} aS \xRightarrow[2]{(1/2,\,2,\,2,\,1/2)} abS \xRightarrow[3]{(1,\,1,\,1/2,\,2)} abcS \xRightarrow[3]{(1,\,1,\,1/2,\,2)} abccS$$

$$\xRightarrow[1]{(2,\,1/2,\,1,\,1)} abccaS \xRightarrow[5]{(1/2,\,2,\,2,\,1/2)} abccab.$$

Thus, the membership grade of $abccab$ is obtained as

$$\begin{aligned}\mu_G\,(abccab) &= (2, 1/2, 1, 1) \times (1/2, 2, 2, 1/2) \times (1, 1, 1/2, 2)\\ &\quad \times (1, 1, 1/2, 2) \times (2, 1/2, 1, 1) \times (1/2, 2, 2, 1/2)\\ &= (1, 1, 1, 1).\end{aligned}$$

In general, the membership grade of $x \in \{x | x \in V_T^*, |x_a| = i, |x_b| = j, |x_c| = k$, and $i, j, k \geqq 1\}$ is given as

$$\mu_G(x) = (2^{i-j}, 2^{j-i}, 2^{j-k}, 2^{k-j}).$$

Therefore many languages can be obtained by suitable cut points. For example,

$$L(G, (1, 1, 1, 1)) = \{x | x \in V_T^*, |x_a| = |x_b| = |x_c| \geqq 1\},$$

$$L(G, (1, 0, 1, 0)) = \{x | x \in V_T^*, |x_a| \geqq |x_b| \geqq |x_c| \geqq 1\},$$

$$L(G, (1, 1/4, 1, 1/2)) = \{x | x \in V_T^*, |x_a| \geqq |x_b| \geqq |x_a| - 2, \\ |x_b| \geqq |x_c| \geqq |x_b| - 1\},$$

$$L(G, (1/4, 1/4, 1/4, 1/4)) = \{x | x \in V_T^*, ||x_a| - |x_b|| \leqq 2, \\ ||x_b| - |x_c|| \leqq 2\},$$

$$L(G, (1, 1, 1, 0)) = \{x | x \in V_T^*, |x_a| = |x_b| \geqq |x_c|\}.$$

4. NORMAL FORM OF TYPE 2 *lfg*

In formal grammars, Chomsky and Greibach normal forms can be given for any context-free grammars [9]. In this section, we shall show that Chomsky and Greibach normal forms for type 2 *lfg* can be constructed as the extension of the corresponding notion in the theory of formal grammars.

Chomsky and Greibach normal forms can be constructed for any $\wedge$-*lfg* but not necessarily for $*$-*lfg*. That reason is reduced to the difference of membership space, since we must consider both words and the grade of membership of those words generated by *lfg*.

Definition 6. An L-fuzzy language $L(G)$ and an L-fuzzy language $L(G')$ are equivalent if and only if $\mu_G(x) = \mu_{G'}(x)$, for any $x \in V_T^*$.

LEMMA 7. *For a type 2 $\wedge$-lfg G with ϵ-rule, there exists the type 2 ϵ-free $\wedge$-lfg G' such that*

$$L(G') = L(G) - \{(\epsilon, \mu_G(\epsilon))\}. \tag{20}$$

Proof. Let $W_1 = \{A \mid (r)\, A \to \epsilon\ \mu(r)\}$, $W_{k+1} = W_k \cup \{A \mid (r)\, A \to \xi\ \mu(r), \xi \in W_k^*\}$, where A is a nonterminal of G and $\mu(r)$ is the grade of the application of the rule r of G. It is evident that if $A \in W_i$, then $A \overset{*}{\underset{G}{\Rightarrow}} \epsilon$. So the set of productions of G' is constructed as follows:

When $(r)\, A \to \xi\ \mu(r)$ is a production of G, for the string $\omega (\neq \epsilon)$ given by taking the string away any element of W_n from the string ξ, let the production $(r')\, A \to \omega\ \mu'(r')$ be the production of G'. Suppose that ω is obtained by taking away the element $A_1, A_2, \cdots, A_m$ of W_n, the value $\mu'(r')$ is given by

$$\mu'(r') = \mu(r) \wedge R^+(A_1, \epsilon) \wedge R^+(A_2, \epsilon) \wedge \cdots \wedge R^+(A_m, \epsilon). \tag{21}$$

Suppose that $(r)\, B \to \xi\ \mu(r)$ is a production of G and $B \in W_n$, then the production $(r')\, B \to \xi\ \mu'(r')$ is given as the element of the set of production P', and its value is $\mu'(r') = \mu(r)$.

Here suppose that the grade of membership of $x \in V_T^+$ by G is given as

$$\mu_G(x) = \bigvee_i [R(S, \alpha_{i1}) \wedge \cdots \wedge R(\alpha_{ij}, \alpha_i A \beta_i) \wedge R(\alpha_i A \beta_i, \alpha_i \xi \beta_i) \wedge \cdots$$

$$\wedge R(\alpha_i \psi \beta_i, \alpha_i \omega \beta_i) \wedge \cdots \wedge R(\alpha_{in}, x)],$$

and the ϵ-rule is applied only to the relation between $\alpha_i \xi \beta_i$ and $\alpha_i \omega \beta_i$, then the following equation should hold from the construction of G'.

$$\mu_G(x) = \bigvee_i [R'(S, \alpha_{i1}) \wedge \cdots \wedge R'(\alpha_{ij}, \alpha_i A \beta_i) \wedge R'(\alpha_i A \beta_i, \alpha_i \omega \beta_i) \wedge \cdots$$

$$\wedge R(\alpha_{in}, x)]$$

$$= \mu_{G'}(x).$$

In the same way, $\mu_G(x) = \mu_{G'}(x)$ holds for any string $x \in V_T^+$.

LEMMA 8. *Given type 2 $\wedge$-lfg, we can find an equivalent type 2 $\wedge$-lfg with no productions of the form $(r)\, A \to B\ \mu(r)$, where A and B are nonterminals, r is a label of rule, and $\mu(r)$ is a value of membership function of rule r.*

Proof. It may be assumed that G is an ϵ-free and a loop-free type 2 $\wedge$-*lfg*. We construct the set of production P' of G' from P by first including all productions not of the form $(r)\, A \to B\ \mu(r)$.

Suppose that $A \overset{*}{\underset{G}{\Rightarrow}} B$ and there is a loop-free derivation chain from A to B such as

$$A \underset{r_1}{\overset{\mu(r_1)}{\Longrightarrow}} B_1 \underset{r_2}{\overset{\mu(r_2)}{\Longrightarrow}} B_2 \Longrightarrow \cdots \Longrightarrow B_m \underset{r_m}{\overset{\mu(r_m)}{\Longrightarrow}} B.$$

And suppose that the production $(r_0)\, B \to \xi\ \mu(r_0)$ is in P, where $\xi \in V_N$. Then we add to P' all productions of the form $(r')\, A \to \xi\ \mu'(r')$, where the value of membership function of r', $\mu'(r')$, is given by

$$\mu'(r') = \mu(r_1) \wedge \mu(r_2) \wedge \cdots \wedge \mu(r_m) \wedge \mu(r_0). \tag{22}$$

From the construction of the set of productions P', it is evident that if $S \overset{*}{\underset{G}{\Rightarrow}} x$, then $S \overset{*}{\underset{G'}{\Rightarrow}} x$ and $\mu_G(x) = \mu_{G'}(r')$.

THEOREM 9. *Chomsky normal form equivalent to a given type 2 $\wedge$-lfg can be constructed. Chomsky normal form for lfg has productions of the form $(r_i)\, A \to BC\ \mu(r_i)$ or $(r_j)\, A \to a\ \mu(r_j)$, where A, B, and C are nonterminal symbols, a is a terminal symbol, and $\mu(r_i)$ is the value of the membership function of rule r_i.*

Proof. From Lemma 7 and Lemma 8, it may be assumed that the productions of a given type 2 $\wedge$-*lfg* G are of the form $(r_i)\, A \to B_1 B_2 \cdots B_m\ \mu(r_i)$, $m \geqq 2$, or $(r_j)\, A \to a\ \mu(r_j)$, where A is a nonterminal symbol, and $B_1, B_2, \cdots, B_m$ are in $V_N \cup V_T$. If productions are of the form $(r_j)\, A \to a\ \mu(r_j)$, then those are already in an acceptable form. Now consider the productions of the form $(r_i)\, A \to B_1 B_2 \cdots B_m\ \mu(r_i), m \geqq 2$. Each terminal B_k is replaced by a new symbol C_k, which appears on the right of no other productions. If B_k is a terminal symbol, then a new production $(r_k')\, C_k \to B_k\ I$ is created and the value I is given such that

$$I = \mu(r_1) \wedge \mu(r_2) \wedge \cdots \wedge \mu(r_n), \tag{23}$$

when the set of label is $J = \{r_1, r_2, \cdots, r_n\}$. And the production $(r_i)\, A \to B_1 B_2 \cdots B_m\ \mu(r_i)$ is replaced by $(r_i')\, A \to C_1 C_2 \cdots C_m\ \mu'(r_i')$, where $B_k = C_k$ if B_k is a nonterminal symbol, and $\mu(r_i) = \mu'(r_i')$.

By these replacements, we have now the set of productions P' of G' with productions being either of the form $(r_i')\, A \to C_1 C_2 \cdots C_m\ \mu'(r_i')$ or $(r_j')\, A \to a\ \mu'(r_j')$, where $A, C_1, C_2, \cdots, C_m \in V_N'$, $a \in V_T'$ $(= V_T)$, and $\mu(r_j') = \mu(r_j)$. Next we modify G' by replacing the production of the form $(r_i')\, A \to C_1 C_2 \cdots C_m\ \mu'(r_i')$ by the set of productions $\{(r_i'')\, A \to C_1 D_1\ \mu''(r_i''), (r_{i+1}'')\, D_1 \to C_2 D_2\ I, \cdots, (r_{i+m-1}'')\, D_{m-1} \to C_{m-1} C_m\ I\}$, where D_k is an additional new nonterminal symbol to V_N', and $\mu''(r_i'') = \mu'(r_i')$. Then the new set of productions P'' of G'' is obtained. Clearly P'' is the set of productions either of the form $(r_i'')\ A \to BC\ \mu''(r_i'')$ or $(r_j'')\ A \to a\ \mu''(r_j'')$. So let G'' be $(V_N'', V_T'', P'', S, J'', \mu'')$, where $V_N'' = V_N \cup \{C_k\} \cup \{D_k\}$, and $V_T'' = V_T$, then G'' is the normal form equivalent to a given type 2 $\wedge$-*lfg* G.

COROLLARY 10. *Greibach normal form equivalent to a given type 2 $\wedge$-lfg can be constructed. Greibach normal form for type 2 $\wedge$-lfg has productions of the form $(r_i)\ A \to a\alpha\ \mu(r_i)$, where A is a nonterminal symbol, a is a terminal symbol, and $\alpha \in V_N^*$.*

Proof. Since the proof is much the same as the proof in fuzzy languages [3], we shall omit the proof.

Next we continue the discussion on the normal form for type 2 $*$-*lfg*. Before mentioning the construction of the normal form for type 2 $*$-*lfg*, we would like to mention the property of definability on $*$-*lfg*.

Definition 11. Definability on $*$-*lfg* is that the grade of any word generated by $*$-*lfg* is obtained uniquely as the bounded value.

Generally speaking, derivation chains with loop are admitted in type 0, 1, and 2 grammars. That is, suppose that there is a derivation $S \overset{*}{\underset{G}{\Rightarrow}} x$ whose derivation chain C is given as

$$C; S \Rightarrow \cdots \Rightarrow \alpha_i \Rightarrow \cdots \Rightarrow \alpha_j \Rightarrow \alpha_{j+1} \Rightarrow \cdots \Rightarrow \alpha_m = x,$$

and $\alpha_i = \alpha_j$. Then the following derivation chain C' is also the derivation chain from S to x.

$$C'; S \Rightarrow \cdots \Rightarrow \alpha_i \Rightarrow \cdots \Rightarrow \alpha_i \Rightarrow \cdots \Rightarrow \alpha_j \Rightarrow \alpha_{j+1} \Rightarrow \cdots \Rightarrow \alpha_m = x.$$

Also, the derivation chains with infinite loop subchain $\alpha_i \Rightarrow \cdots \Rightarrow \alpha_j$ are the derivation chains from S to x.

In this case, the value $\mu_G(x)$ is given by

$$\mu_G(x) = \bigvee_{n=1}^{\infty} [\mu(r_1) * \cdots * (\mu(r_i) * \cdots * \mu(r_j))^n * \mu(r_{j+1}) * \cdots * \mu(r_m)].$$

Now let $\mu(r_i) * \cdots * \mu(r_j) = \nu$ and suppose that $\nu < \nu^2 < \cdots < \nu^n < \cdots$, where $\nu^2 = \nu * \nu, \cdots, \nu^n = \nu * \overset{n}{\cdots} * \nu$, then the value $\mu_G(x)$ can not be obtained uniquely as the bounded value. In the above case, $*$-*lfg* is not definable. Evidently, both any $\wedge$-*lfg* and loop-free $*$-*lfg* are definable.

LEMMA 12. *Given a definable ϵ-free type 2 $*$-lfg whose membership space is a complete lattice ordered group or a complete lattice ordered monoid, we can find an equivalent ϵ-free type 2 $*$-lfg with no loops.*

Proof. Let G be a given definable ϵ-free type 2 $*$-*lfg*. As G is ϵ-free type

2 $*$-*lfg*, we may consider only the case that there are loops between nonterminals A and B, that is, there are derivations $A \overset{*}{\underset{G}{\Rightarrow}} B$ and $B \overset{*}{\underset{G}{\Rightarrow}} A$.

It may be assumed that G has no productions of the form $(r)\, A \to B\, \mu(r)$ except that there is a loop between nonterminals A and B, because we can construct the equivalent grammar with no productions of the form $(r)\, A \to B\, \mu(r)$ in the same way as in Lemma 8 when there is no loop between nonterminals A and B. Suppose that there is a loop between nonterminals A and B such that $(r_0)\, A \to B\, \sigma_0$, $(r_i)\, A \to \xi_i, i = 1, 2, \cdots, n$, $(r_0')\, B \to A\, \nu_0$, and $(r_j')\, B \to \zeta_j\, \nu_j, j = 1, 2, \cdots, m$, are the set of all productions having A and B as the premise. It is evident that ζ_j can be derived from A by means of an infinite number of derivations which can be obtained by applying the productions $(r_0)\, A \to B\, \sigma_0$ and $(r_0')\, B \to A\, \nu_0$ an arbitrary number of times before applying the production $(r_j')\, B \to \zeta_j\, \nu_j$. Therefore the value of $R^+(A, \zeta_j)$ is given as follows:

$$\begin{aligned} R^+(A,\zeta_j) &= R(A,B) * R(B,\zeta_j) \vee R(A,B) * R(B,A) * R(A,B) * R(B,\zeta_j) \vee \cdots \\ &\qquad \vee (R(A,B) * R(B,A))^n * R(A,B) * R(B,\zeta_j) \vee \cdots, \\ &= \sigma_0 * \nu_j \vee \sigma_0 * \nu_0 * \sigma_0 * \nu_j \vee \cdots \vee (\sigma_0 * \nu_0)^n * \sigma_0 * \nu_j \vee \cdots, \\ &= \bigvee_{n=0}^{\infty} (\sigma_0 * \nu_0)^n * \sigma_0 * \nu_j. \end{aligned} \tag{24}$$

Since G is a definable $*$-*lfg*, the value $R^+(A, \zeta_j)$ must be a boundary value, i.e., it is required that $(\sigma_0 * \nu_0)^n * \sigma_0 * \nu_j \leqq b$ for any natural number n, where b is an element of the membership space L. When the membership space L is a complete lattice-ordered group, L is Archimedean, i.e., the following equation holds.

$$(\sigma_0 * \nu_0)^n \leqq b * \nu_j' * \sigma_0' \Longrightarrow \sigma_0 * \nu_0 \leqq e, \tag{25}$$

where ν_j' and σ_0' are inverse element of ν_j and σ_0, respectively, and e is the unit element of L. From the property of the operation $*$ in a lattice ordered group, it is clear that

$$e \geqq \sigma_0 * \nu_0 \geqq (\sigma_0 * \nu_0)^2 \geqq \cdots \geqq (\sigma_0 * \nu_0)^n \geqq \cdots. \tag{26}$$

So the value $R^+(A, \zeta_j)$ is given by $R^+(A, \zeta_j) = \sigma_0 * \nu_j$.

When the membership space L is a complete lattice-ordered monoid, L is residuated, i.e., the following equation holds.

$$\bigvee_{n=0}^{\infty} \{(\sigma_0 * \nu_0)^n * \sigma_0 * \nu_j\} \leqq b \Longrightarrow \exists\, u, u = (\sigma_0 * \nu_0)^n = \{v \mid v * \sigma_0 * \nu_j \leqq b\}. \tag{27}$$

Since the above equation requires that $\sigma_0 * \nu_0 \leqq e$, the value $R^+(A, \zeta_j)$ is also given such that $R^+(A, \zeta_j) = \sigma_0 * \nu_j$. Similary, it can be shown that $R^+(B, \xi_i) = \nu_0 * \sigma_i$.

Let G' be the $*$-*lfg* modified by replacing the set of productions

$$\{(r_0)\, A \to B\, \sigma_0, (r_i)\, A \to \xi_i\, \sigma_i, i = 1, 2, \cdots, n, (r_0')\, B \to A\, \nu_0, \quad (r_j')\, B \to \zeta_j\, \nu_j, j = 1, 2, \cdots, m\}$$

by the set of productions

$$\{(r_j')\, A \to \zeta_j\, \sigma_0 * \nu_j, j = 1, 2, \cdots, m, (r_i)\, B \to \xi_i\, \nu_0 * \sigma_i, i = 1, 2, \cdots, n\}.$$

Then it can easily be seen that there are no loops in G' and $\mu_G(x) = \mu_{G'}(x)$ for any string $x \in V_T^*$.

THEOREM 13. *Given a definable type 2 $*$-lfg G whose membership space is a complete lattice-ordered group or a lattice-ordered monoid, there exists an equivalent type 2 $*$-lfg G' with no loops.*

Proof. As in Lemma 12 we discussed the case where ϵ-free type 2 $*$-*lfg* has loops, we shall discuss here on loops depending only on the ϵ-rule.

Suppose that the $*$-*lfg* G has productions such as $(r_0)\, A \to AA\, \nu$, $(r_1)\, A \to \epsilon\, \rho$, $(r_2)\, A \to \xi\, \sigma$, where A is a nonterminal, $\xi \neq \epsilon$, and $\xi \overset{*}{\underset{G}{\Rightarrow}} \epsilon$. Then ξ^i can be derived from A by means of infinite number of derivations which can be obtained by applying the productions $(r_0)\, A \to AA\, \nu$ and $(r_1)\, A \to \epsilon\, \rho$ an arbitrary number of times before replacing A by ξ. Therefore the value $R^+(A, \xi^i)$, $i = 0, 1, 2, \cdots$, are obtained as follows:

$$\begin{aligned} R^+(A, \epsilon) &= R(A, \epsilon) \vee R(A, AA) * R(AA, A) * R(A, \epsilon) \\ &\quad \vee R(A, AA) * R(AA, AAA) * R(AAA, AA) * R(AA, A) \\ &\quad * R(A, \epsilon) \vee \cdots, \\ &= \rho \vee \nu * \rho * \rho \vee (\nu * \rho)^2 * \rho \vee \cdots \vee (\nu * \rho)^n * \rho \vee \cdots. \end{aligned} \tag{28}$$

Similary, we can show that

$$\begin{aligned} R^+(A, \xi) &= R(A, \xi) \vee R(A, AA) * R(AA, A) * R(A, \xi) \vee \cdots, \\ &= \sigma \vee \nu * \rho * \sigma \vee (\nu * \rho)^2 * \sigma \vee \cdots \vee (\nu * \rho)^n * \sigma \vee \cdots, \\ R^+(A, \xi^2) &= \nu * \sigma * \sigma \vee \nu * \nu * \rho * \sigma^2 \vee \nu * (\nu * \rho)^2 * \sigma^2 \\ &\quad \vee \cdots \vee \nu * (\nu * \rho)^n * \sigma^2 \vee \cdots, \\ &\;\;\vdots \end{aligned}$$

$$R^+(A, \xi^m) = \nu^{m-1} * \sigma^m \vee \nu^{m-1} * (\nu * \rho) * \sigma^m$$
$$\vee \cdots \vee \nu^{m-1} * (\nu * \rho)^n * \sigma^m \vee \cdots,$$
$$\vdots \tag{29}$$

Since G is a definable $*$-*lfg* and its membership space is a complete lattice-ordered group or a complete lattice monoid, the value $R^+(A, \xi^i)$, $i = 0, 1, 2, \cdots$, are obtained in the same way as in the proof of Lemma 12 such that $R^+(A, \epsilon) = \rho$, $R^+(A, \xi) = \sigma$, $R^+(A, \xi^2) = \nu\sigma^2, \cdots, R^+(A, \xi^m) = \nu^{m-1}\sigma^m, \cdots$.

Let G' be the $*$-*lfg* modified by replacing the production $(r_0)\ A \to AA\ \nu$ by the production $(r_0')\ A \to A\xi\ \nu\sigma$. Then it can easily be seen that G' has no loops and $\mu_G(x) = \mu_{G'}(x)$ for any string $x \in V_T^*$.

THEOREM 14. *Given a definable type* 2 $*$-*lfg G whose membership space is a complete lattice-ordered group or a complete lattice-ordered monoid, there exists an equivalent Chomsky normal form.*

Proof. It may be assumed that G has neither loops nor productions of the form $(r)\ A \to B\ \mu(r)$ from Lemma 12 and Theorem 13. Therefore Chomsky normal form grammar G' can be constructed in the same way as in Theorem 9, that is, it can be constructed by means of replacing the operation $\wedge$ and value I in Theorem 9 by the operation $*$ and value e, respectively, in which the operation $\wedge$ is the operation of infinum, the operation $*$ is the operation of group or monoid, and value e is the unit element.

Example. Consider the following definable type 2 $*$-*lfg* G in which $V_N = \{A, B, S\}$, $V_T = \{a, b\}$, and membership space is a complete lattice ordered group such that $L = (Q^+ \times Q^+, \times, \vee, \wedge)$. And the productions are given by

(1) $S \longrightarrow aAB\ (2, 1/2)$,	(5) $A \longrightarrow \epsilon\ \ (1/3, 1/4)$,
(2) $A \longrightarrow AA\ \ (2, 3)$,	(6) $B \longrightarrow bB\ (1/2, 2)$,
(3) $A \longrightarrow aA\ \ (2, 1/2)$,	(7) $B \longrightarrow A\ \ (1/3, 3)$,
(4) $A \longrightarrow B\ \ (1, 1/2)$,	(8) $B \longrightarrow a\ \ (1/2, 2)$.

To find the equivalent $*$-*lfg* in Chomsky normal form, we proceed as follows.

Step 1. As loops between nonterminals A and B are derived by applying the productions (4) $A \to B\ (1, 1/2)$ and (7) $B \to A\ (1/3, 3)$ an aribtrary number of times, Lemma 12 is applied to the above productions.

The resulting set of productions is

P'; (1′) $S \longrightarrow aAB\ (2, 1/2)$,	(3′) $A \longrightarrow aA\ \ (2, 1/2)$,
(2′) $A \longrightarrow AA\ \ (2, 3)$,	(4′) $A \longrightarrow bB\ \ (1/2, 1)$,

(5′) $A \longrightarrow b$ (1/2, 1),
(6′) $A \longrightarrow \epsilon$ (1/3, 1/4),
(7′) $B \longrightarrow bB$ (1/2, 2),
(8′) $B \longrightarrow aA$ (2/3, 3/2),
(9′) $B \longrightarrow AA$ (2/3, 9),
(10′) $B \longrightarrow b$ (1/2, 2),
(11′) $B \longrightarrow \epsilon$ (1/9, 3/4).

Step 2. Since there are productions (2′) $A \rightarrow AA$ (2, 3) and (6′) $A \rightarrow \epsilon$ (1/3, 1/4) in P', the following new set of productions P'' is obtained by means of the same modification as in the proof of Theorem 13.

P''; (1″) $S \longrightarrow aAB$ (2, 1/2),
(2″) $A \longrightarrow aAA$ (4, 3/2),
(3″) $A \longrightarrow aA$ (2, 1/2),
(4″) $A \longrightarrow bB$ (1/2, 1),
(5″) $A \longrightarrow b$ (1/2, 1),
(6″) $A \longrightarrow \epsilon$ (1/3, 1/4),
(7″) $B \longrightarrow bB$ (1/2, 2),
(8″) $B \longrightarrow aA$ (2/3, 3/2),
(9″) $B \longrightarrow AA$ (2/3, 9),
(10″) $B \longrightarrow b$ (1/2, 2),
(11″) $B \longrightarrow \epsilon$ (1/9, 3/4).

Step 3. Since the set of productions P'' does not derive loops, Chomsky normal form with the set of productions P''' can be obtained in the similar manner of Theorem 9.

P'''; (1‴) $S \longrightarrow D_1B$ (2, 1/2),
(2‴) $D_1 \longrightarrow C_1A$ (1, 1),
(3‴) $A \longrightarrow D_1A$ (4, 3/2),
(4‴) $A \longrightarrow C_1A$ (2, 1/2),
(5‴) $A \longrightarrow C_2B$ (1/2, 1),
(6‴) $A \longrightarrow b$ (1/2, 1),
(7‴) $A \longrightarrow \epsilon$ (1/3, 1/4),
(8‴) $B \longrightarrow C_1A$ (2/3, 3/2),
(9‴) $B \longrightarrow AA$ (2/3, 9),
(10‴) $B \longrightarrow C_2B$ (1/2, 2),
(11‴) $B \longrightarrow b$ (1/2, 2),
(12‴) $B \longrightarrow \epsilon$ (1/9, 3/4),
(13‴) $C_1 \longrightarrow a$ (1, 1),
(14‴) $C_2 \longrightarrow b$ (1, L).

COROLLARY 15. *Greibach normal form equivalent to a given definable type 2 *-lfg whose membership space is a complete lattice-ordered group or a complete lattice-ordered monoid can be constructed.*

5. CONCLUSIONS

We have defined L-fuzzy grammars by introducing the concept of L-fuzzy sets and mentioned some properties of L-fuzzy grammars and languages generated by L-fuzzy grammars. In this paper, we have mainly discussed L-fuzzy grammars whose membership spaces are commutative. The interesting results for the properties of languages generated by L-fuzzy grammars whose membership spaces are not commutative will be obtained, since the membership grade of a string generated by such L-fuzzy grammars depends not only on the rules but also on the order of applying rules.

The membership grade of a string generated by an L-fuzzy grammar may be reregarded as an element which is described by some arguments of characteristic parameters. Therefore, it may be interesting to obtain the grade of similarity of incorrectness or ambiguity of strings by means of defining the distance between strings generated by an L-fuzzy grammar.

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