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B-Fuzzy Grammars

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Fuzzy grammars on Boolean lattices (B-fuzzy grammars) are newly defined and their basic properties are investigated. B-fuzzy grammars are defined as the extension of fuzzy grammars by Lee and Zadeh, where the grades of the application of rewriting rules of B-fuzzy grammars are the elements of Boolean lattice rather than the elements of unit interval [0, 1].

It is shown that type 2 B-fuzzy grammars can generate type 1 languages though type 2 fuzzy grammars cannot generate type 2 languages. And the closure properties of B-fuzzy grammars are also studied.

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1. INTRODUCTION

In [8] we have derived various kinds of formal grammars with weights by incorporating the algebra systems with the formal grammars systems. There we evaluate the weights of sentences by corresponding the element of the appropriate algebra (say, lattice ordered semigroup and distributive lattice) to each rewriting rule of a formal grammar and by performing the operations of the algebras in the case of the application of the rewriting rules in a derivation. Among these sorts of formal grammars with weights are the well-known probabilistic grammars [2, 5, 12], fuzzy grammars [2, 6, 7, 10, 11, 15], and weighted grammars [12]. For example, probabilistic grammars are the formal grammars with weights such that the values in the range [0, 1] are adopted as the weights (or probabilities) where, needless to say, the condition of probability is assumed to be satisfied, and + (addition) and × (multiplication) are used as the operations. In addition, weighted grammars can be formulated by extending the range [0, 1] in probabilistic grammars to the set of non-negative real numbers [0, ∞). Fuzzy grammars are defined by the values in [0, 1] to the weights and by using max and min as the operations.

In this paper B-fuzzy grammars are newly defined as the extension of fuzzy grammars. B-fuzzy grammar is a fuzzy grammar in which the weight (or grade) of the application of the rewriting rule is an element of a Boolean lattice and the operations $\vee$ (= lub) and $\wedge$ (= glb) are used in the case of the application of the rewriting rules in a derivation.

We show that type 2 (context-free) B-fuzzy grammars can generate type 1 (context-sensitive) languages although type 2 fuzzy grammars cannot generate type 1 languages [10]. But the generative power of type 3 (regular) B-fuzzy grammars is shown to be equal to that of the ordinary type 3 grammars by using the newly defined B-fuzzy automata. Moreover, the closure properties of B-fuzzy languages characterized by type 2 and 3 B-fuzzy grammars are studied. For example, type 3 B-fuzzy languages are not closed under the complement in the sense of B-fuzzy sets. However, the complement of type 3 B-fuzzy language can be shown to be characterized by another kind of B-fuzzy grammar, namely, $\wedge\vee$B-fuzzy grammar.

2. B-FUZZY SETS AND B-FUZZY LANGUAGES

In this chapter we shall briefly review B-fuzzy sets and B-fuzzy relations by J. G. Brown [1] and J. A. Goguen [3] for the purpose of B-fuzzy grammars defined later. Moreover, using the concept of B-fuzzy sets, we shall define B-fuzzy languages and their operations such as union, intersection, complement, concatenation, and Kleene closure.

L-Fuzzy Sets

After L. A. Zadeh [14] introduced the concept of fuzzy sets, J. A. Goguen [3] defined L-fuzzy sets as an extension of fuzzy sets.

An *L-fuzzy set* A in a space $X = \{x\}$ is characterized by a membership function μ_A as follows.

$$\mu_A : X \to L \tag{1}$$

where L is called a *membership space* and the value $\mu_A(x)$ in L represents the *grade of membership* of x in A.

A membership space L may be assumed to be a partially ordered set or, more particularly, a lattice [3].

When L is the unit interval [0, 1], an L-fuzzy set A is a fuzzy set originated by L. A. Zadeh [14]. If L contains only two points 0 and 1, then A is a non-fuzzy set and its membership function reduces to the conventional characteristic function of a non-fuzzy set.

B-Fuzzy Sets

Let L be a Boolean lattice B, then an L-fuzzy set becomes a B-fuzzy set introduced by J. G. Brown [1]. Therefore, B-fuzzy set A is an L-fuzzy set which is characterized by a membership function such as

$$\mu_A : X \to B, \tag{2}$$

where X is a non-nullset and B is a Boolean lattice.

B-fuzzy set A can be represented as a set of ordered pairs of x and its grade $\mu_A(x)$ as follows.

$$A = \{(x, \mu_A(x))\}, x \in X. \tag{3}$$

The notions of containment, equality, union, intersection, and complement of B-fuzzy sets are easily derived as extensions of the notions in the ordinary set theory.

Let A and C be two B-fuzzy sets in X, and let μ_A, μ_C be membership functions of A and C, respectively, then for all x in X

Containment $$A \subseteq C \Leftrightarrow \mu_A(x) \leqq \mu_C(x) \tag{4}$$

Equality $$A = C \Leftrightarrow \mu_A(x) = \mu_C(x) \tag{5}$$

Union $$A \cup C \Leftrightarrow \mu_{A \cup C}(x) = \mu_A(x) \vee \mu_C(x) \tag{6}$$

Intersection $$A \cap C \Leftrightarrow \mu_{A \cap C}(x) = \mu_A(x) \wedge \mu_C(x) \tag{7}$$

Complement $$\bar{A} \Leftrightarrow \mu_{\bar{A}}(x) = \overline{\mu_A(x)} \tag{8}$$

where the operations $\leqq$, $\vee$, $\wedge$, and $^-$ represent an order relation, lub, glb, and complement in a Boolean lattice B, respectively.

Moreover, let I and 0 be greatest element and least element of B, respectively, then universal B-fuzzy set and empty B-fuzzy set are defined by the following: For all x in X,

Universal B-Fuzzy Set

$$U \Leftrightarrow \mu_U(x) = I. \tag{9}$$

Empty B-Fuzzy Set

$$\phi \Leftrightarrow \mu_\phi(x) = 0. \tag{10}$$

J. G. Brown [1] introduced an interesting definition concerning with convex combination of B-fuzzy sets.

Let A, C, and Λ be B-fuzzy sets. The convex combination of A, C, and Λ is a B-fuzzy set $(A, C; \Lambda)$ and is defined by the relation:

Convex Combination

$$(A, C; \Lambda) = (\Lambda \cap A) \cup (\bar{\Lambda} \cap C). \tag{11}$$

A basic property of the convex combination of A, C, and Λ is expressed by

$$A \cap C \subseteq (A, C; \Lambda) \subseteq A \cup C, \text{ for all } \Lambda. \tag{12}$$

The operations $\subseteq$, $\cup$, $\cap$, and $^-$ on B-fuzzy sets have a number of algebraic properties. Some of these are as follows:

If A, C, and D are B-fuzzy sets and U and ϕ are universe and empty B-fuzzy sets, respectively, then we have

i) $A \subseteq A$ (reflexive law) (13)

ii) $A \subseteq C,\ C \subseteq A \Rightarrow A = C$ (anti-symmetric law) (14)

iii) $A \subseteq C,\ C \subseteq D \Rightarrow A \subseteq D$ (transitive law) (15)

iv) $A \cup A = A,\ A \cap A = A$ (idempotent law) (16)

v) $A \cup C = C \cup A,\ A \cap C = C \cap A$ (commutative law) (17)

vi) $\left.\begin{array}{l}(A \cup C) \cup D = A \cup (C \cup D)\\ (A \cap C) \cap D = A \cap (C \cap D)\end{array}\right\}$ (associative law) (18)

vii) $\left.\begin{array}{l}A \cup (A \cap C) = A\\ A \cap (A \cup C) = A\end{array}\right\}$ (absorption law) (19)

viii) $\left.\begin{array}{l}A \cup (C \cap D) = (A \cup C) \cap (A \cup D)\\ A \cap (C \cup D) = (A \cap C) \cup (A \cap D)\end{array}\right\}$ (distributive law) (20)

ix) $\bar{\bar{A}} = A$ (involution law) (21)

x) $\left.\begin{array}{l}\overline{(A \cup C)} = \bar{A} \cap \bar{C}\\ \overline{(A \cap C)} = \bar{A} \cup \bar{C}\end{array}\right\}$ (De Morgan's law) (22)

xi) $\left.\begin{array}{ll}A \cup U = U, & A \cap U = A\\ A \cup \phi = A, & A \cap \phi = \phi\end{array}\right\}$ (identity law) (23)

xii) $A \cup \bar{A} = U,\quad A \cap \bar{A} = \phi$ (complement law) (24)

From the properties concerning with B-fuzzy sets, we see that B-fuzzy sets form a Boolean lattice. It should be noted that fuzzy sets by Zadeh do not form a Boolean lattice but a distributive lattice.

B-Fuzzy Relations

A *B-fuzzy relation* R in the product space $X \times X = \{(x, y) \mid x, y \in X\}$ is a B-fuzzy set in $X \times X$ characterized by a membership function μ_R as

$$\mu_R : X \times X \to B, \tag{25}$$

where B is a Boolean lattice. More generally, an *n-ary B-fuzzy relation* in a product space $X = X_1 \times X_2 \times \ldots \times X_n$ is a B-fuzzy set in X characterized by an n-variate membership function $\mu_R(x_1, x_2, \ldots, x_n)$, $x_i \in X_i$, $i = 1, 2, \ldots, n$.

Composition of B-Fuzzy Relations

Let R_1 and R_2 be two B-fuzzy relations in $X \times X$, then by the *composition* (or *product*) of R_1 and R_2 is meant a B-fuzzy relation in $X \times X$ which is denoted by $R_1 \circ R_2$ and is defined by

$$\mu_{R_1 \circ R_2}(x, z) = \bigvee_y \, [\mu_{R_1}(x, y) \wedge \mu_{R_2}(y, z)], \tag{26}$$

where $\vee$ and $\wedge$ are the operations of lub and glb in B, respectively.

It is noted that we can define another kind of composition of B-fuzzy relations (denoted by $R_1 \cdot R_2$) by changing $\vee$ and $\wedge$ in (26), i.e.,

$$\mu_{R_1 \cdot R_2}(x, z) = \bigwedge_y \, [\mu_{R_1}(x, y) \vee \mu_{R_2}(y, z)]. \tag{27}$$

In what follows, in order to avoid a confusing multiplicity of the composition, we shall be using (26) for the most part as our definition of the composition of B-fuzzy relations.

Note that the operation of composition of B-fuzzy relations has the associative property, i.e.,

$$R_1 \circ (R_2 \circ R_3) = (R_1 \circ R_2) \circ R_3. \tag{28}$$

The same holds for the composition in (27).

Therefore let $R_1, R_2, \ldots, R_n$ be B-fuzzy relations on X, then the composition $R_1 \circ R_2 \ldots \circ R_n$ of $R_1, R_2, \ldots, R_n$ can be defined as

$$\mu_{R_1 \circ R_2 \circ \ldots \circ R_n}(x_1, x_{n+1}) = \bigvee_{x_2, \ldots, x_n} [\mu_{R_1}(x_1, x_2) \wedge \mu_{R_2}(x_2, x_3) \wedge \ldots \wedge \mu_{R_n}(x_n, x_{n+1})]. \tag{29}$$

B-Fuzzy Languages

Using the concept of B-fuzzy sets, we shall define B-fuzzy languages. Let Σ be a finite non-empty alphabet. The set of all finite strings over Σ is denoted by Σ^*. The null string is denoted by ε and is included in Σ^*.

A *B-fuzzy language L* is a B-fuzzy set in Σ^* characterized by a membership function μ_L such as

$$\mu_L : \Sigma^* \to B. \tag{30}$$

The B-fuzzy language L may be represented by a set of ordered pairs of string x in Σ^* and its grade of membership $\mu_L(x)$ in B, i.e.,

$$L = \{(x, \mu_L(x))\}, \; x \in \Sigma^*. \tag{31}$$

The operations of containment, equality, union, intersection, and complement for B-fuzzy languages are defined in the same way as those of B-fuzzy sets mentioned before (see (4) $\sim$(8)). Moreover the notions of concatenation and Kleene closure of B-fuzzy languages can be defined as an extension of that of ordinary formal languages [4] by the following;

Let L_1 and L_2 be two B-fuzzy languages in Σ^*, and μ_{L_1} and μ_{L_2} be membership functions of L_1 and L_2, respectively.

Concatenation

The *concatenation* of L_1 and L_2 is a B-fuzzy language denoted by $L_1 \circ L_2$ and is defined as follows. Let a string x in Σ^* be expressed as a concatenation of a prefix string u and a suffix string v, that is, $x = uv$. Then

$$\mu_{L_1 \circ L_2}(x) = \bigvee_u [\mu_{L_1}(u) \wedge \mu_{L_2}(v)]. \tag{32}$$

Furthermore, we can define the following concatenation of B-fuzzy languages by changing $\vee$ and $\wedge$ in (32), that is,

$$\mu_{L_1 \cdot L_2}(x) = \bigwedge_u [\mu_{L_1}(u) \vee \mu_{L_2}(v)], \tag{33}$$

where $\vee$ in (32) and $\wedge$ in (33) are taken over all prefix u of x.

Kleene Closure

By using the concept of concatenation $L_1 \circ L_2$ or $L_1 \cdot L_2$, *Kleene closure* of a B-fuzzy language L (denoted as L^* or $\hat{L}$) is defined by

$$L^* = \{\varepsilon\} \cup L \cup L \circ L \cup L \circ L \circ L \cup \ldots \tag{34}$$

$$\hat{L} = \{\varepsilon\} \cap L \cap L \cdot L \cap L \cdot L \cdot L \cap \ldots \tag{35}$$

3. B-FUZZY GRAMMARS

In this chapter B-fuzzy grammars are defined as an extension of fuzzy grammars by Lee and Zadeh [2, 6, 8, 10, 11, 15] by using the concept of B-fuzzy sets and B-fuzzy relations.

It is shown that type 2 (or context-free) B-fuzzy grammars can generate type 1 (or context-sensitive) languages by setting a threshold. But type 3 (or regular) B-fuzzy grammars can generate type 3 (or regular) languages only.

3.1. B-Fuzzy Grammars

DEFINITION 3.1 A *B-fuzzy grammar* (*BG* for short) is a system such that

$$BG = (V_N, V_T, P, S, J, \mu, B), \tag{36}$$

where

1) V_N is a nonterminal alphabet.

2) V_T is a terminal alphabet.

3) S is an initial symbol in V_N.

4) P is a finite set of productions such as

$$(r) \quad u \to v \quad \mu(r), \tag{37}$$

where $r \in J$, and $u \to v$ is an ordinary rewriting rule with $u \in V_N^* - \{\varepsilon\}$ and $v \in (V_N \cup V_T)^*$. $\mu(r)$ is the grade of the application of the production r, which is denoted in (6).

5) J is a set of (production) labels as shown in (4), i.e., $J = \{r\}$.

6) μ is a membership function such as

$$\mu : J \to B. \tag{38}$$

μ may be called a *B-fuzzy function.*

7) B is a Boolean lattice. It is assumed that B is finite.

Next we shall explain a derivation chain with fuzzy grades (that is, B-fuzzy derivation chain).

If $(r)\ u \to v\ \mu(r)$ is in P, and α and β are any strings in $(V_N \cup V_T)^*$, then

$$\alpha u \beta \xrightarrow[r]{\mu(r)} \alpha v \beta, \tag{39}$$

and $\alpha v \beta$ is said to be directly derivable from $\alpha u \beta$ with the grade $\mu(r)$ by the production r. If $\alpha_1, \alpha_2, \ldots, \alpha_m$ are strings in $(V_N \cup V_T)^*$ and

$$\alpha_0 \xrightarrow[r_1]{\mu(r_1)} \alpha_1, \alpha_1 \xrightarrow[r_2]{\mu(r_2)} \alpha_2, \ldots, \alpha_{m-1} \xrightarrow[r_m]{\mu(r_m)} \alpha_m$$

then α_m is said to be derivable from α_0 by the productions $r_1, r_2, \ldots, r_m$. The expression

$$\alpha_0 \xrightarrow[r_1]{\mu(r_1)} \alpha_1 \xrightarrow[r_2]{\mu(r_2)} \alpha_2 \longrightarrow \ldots \xrightarrow[r_m]{\mu(r_m)} \alpha_m \tag{40}$$

will be referred to as a *B-fuzzy derivation chain* of length m from α_0 to α_m by the productions $r_1, r_2, \ldots, r_m$.

When $\alpha_0 = S$, $\alpha_m = x (\in V_T^*)$ in (40), that is,

$$S \xrightarrow[r_1]{\mu(r_1)} \alpha_1 \xrightarrow[r_2]{\mu(r_2)} \alpha_2 \longrightarrow \ldots \xrightarrow[r_m]{\mu(r_m)} x, \tag{41}$$

S is said to generate a terminal string x by the productions $r_1, r_2, \ldots, r_m$. In general, there are more than one B-fuzzy derivation chain from S to x.

We shall next explain B-fuzzy languages characterized by B-fuzzy grammars.

DEFINITION 3.2 The grade of the generation of terminal string x ($\in V_T^*$) by a B-fuzzy grammar BG, which is denoted as $\mu_{BG}(x)$, is given as follows by using the concept of the composition of B-fuzzy relations of (26) and by the B-fuzzy derivation chain from S to x of (41). Clearly $\mu_{BG}(x)$ is in B.

$$\mu_{BG}(x) = \vee\ [\mu(r_1) \wedge \mu(r_2) \wedge \ldots \wedge \mu(r_m)], \tag{42}$$

where lub $\vee$ is taken over all the B-fuzzy derivation chains from S to x.

It is assumed that the grade of the terminal string x is 0, i.e., $\mu_{BG}(x) = 0$, if there do not exist any derivation chains for x from S, where 0 is a least element in a finite Boolean lattice B.

We can define another kind of B-fuzzy grammar denoted by $\wedge\vee BG$ by using the composition of B-fuzzy relations in (27). The grade of the generation of terminal string x by $\wedge\vee BG$ is given by

$$\mu_{\wedge\vee BG}(x) = \begin{cases} \wedge\,[\mu(r_1) \vee \mu(r_2) \vee \ldots \vee \mu(r_m)] \ldots \text{ if there exist derivation chains for } x. \\ I \ldots \text{if there exist no derivation chains for } x. \end{cases} \tag{43}$$

where $\wedge$ is taken over all the derivation chains from S to x.

Therefore we may call the B-fuzzy grammar defined in Definition 3.1 a $\vee\wedge BG$ due to the duality of $\wedge\vee BG$. In this paper, unless stated especially, by a "B-fuzzy grammar" we shall mean a $\vee\wedge BG$†.

DEFINITION 3.3 A *B-fuzzy language* characterized by B-fuzzy grammar BG is a B-fuzzy set in V_T^* defined by the membership function $\mu_{BG}(x)$ as denoted in (42) and is shown to $L(BG)$.

DEFINITION 3.4 Let BG be a B-fuzzy grammar and λ an element of B, then a *λ-B-fuzzy language* by BG with a threshold λ is a subset of V_T^* and is defined by

$$L(BG, \lambda) = \{x \,\varepsilon\, V_T^* \mid \mu_{BG}(x) > \lambda\}. \tag{44}$$

†Using the convex combination of B-fuzzy sets, we can also define a *convex B-fuzzy grammar* (CBG) of $\vee\wedge BG$ and $\wedge\vee BG$ with the property that the grade of the generation of x is defined as

$$\mu_{CBG}(x) = [\alpha \wedge \mu_{\vee\wedge BG}(x)] \vee [\bar{\alpha} \wedge \mu_{\wedge\vee BG}(x)],$$

where α, $\bar{\alpha}$ is in B and $\bar{\alpha}$ is the complement of α.

Obviously, if $\alpha = I$, then CBG becomes $\vee\wedge BG$, and if $\alpha = O$, then CBG becomes $\wedge\vee BG$.

Moreover, we can also define another language by the following.

DEFINITION 3.5 For a B-fuzzy grammar BG and the threshold λ in B, a language $L(BG, =, \lambda)$ is defined as

$$L(BG, =, \lambda) = \{x \in V_T^* \mid \mu_{BG}(x) = \lambda\}. \tag{45}$$

Now we shall give an example of the B-fuzzy grammar with CF rules and show that this type 2 (or context-free) B-fuzzy grammar can generate type 1 (or context-sensitive) language by setting a threshold. Therefore, from this example it is founded that the class of type 2 λ-B-fuzzy languages properly contains the class of context-free languages.

Example 3.1 Let BG be the type 2 B-fuzzy grammar $(V_N, V_T, P, S, J, \mu, B)$, where $V_N = \{S, A, B, C, D\}$, $V_T = \{a, b, c\}$, B is the Boolean lattice in Figure 1, and P is as follows.

1) $S \to AB\, x_1$,
2) $S \to CD\, x_2$,
3) $A \to aAb\, I$,
4) $A \to ab\, I$,
5) $B \to cB\, I$,
6) $B \to c\, I$
7) $C \to aC\, I$
8) $C \to a\, I$
9) $D \to bDc\, I$
10) $D \to bc\, I$.

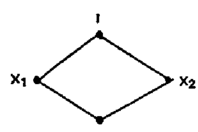

FIGURE 1 Structure of B.

A string, say, $a^2b^2c^2$ is obtained by the following fuzzy derivation chain.

$$\underline{S} \xrightarrow[1]{x_1} \underline{A}B \xrightarrow[3]{I} aAb\underline{B} \xrightarrow[5]{I} a\underline{A}bcB \xrightarrow[4]{I} a^2b^2c\underline{B} \xrightarrow[6]{I} a^2b^2c^2,$$

where the underbar in the intermediate strings represents the location where the next rule was applied.

The grade of the generation of $a^2b^2c^2$ by this derivation is given as follows.

$$x_1 \wedge I \wedge I \wedge I \wedge I = x_1.$$

Similarly, for the same string $a^2b^2c^2$, the following derivation is also possible.

$$\underline{S} \xrightarrow[2]{x_2} \underline{C}D \xrightarrow[7]{I} aC\underline{D} \xrightarrow[9]{I} a\underline{C}bDc \xrightarrow[8]{I} a^2b\underline{D}c \xrightarrow[10]{I} a^2b^2c^2.$$

In this case we have x_2. Furthermore we can also give the different fuzzy derivation chains of $a^2b^2c^2$, all the grades of which are easily shown to be x_1 or x_2.

Hence the grade of the generation of $a^2b^2c^2$ by this B-fuzzy grammar BG is given from (42) as follows.

$$\mu_{BG}(a^2b^2c^2) = x_1 \vee x_2 = I.$$

Continuing in this manner we can obtain the grade $\mu_{BG}(x)$ for each terminal string $x \in V_T^*$. Hence the B-fuzzy language generated by this type 2 B-fuzzy grammar is from Definition 3.3 as follows.

$$L(BG) = \{(a^ib^ic^i, I) \mid i \geqq 1\} \cup \{(a^ib^ic^j, x_1) \mid i, j \geqq 1, i \neq j\} \cup \{(a^ib^jc^j, x_2) \mid i, j \geqq 1, i \neq j\}.$$

Therefore we have from (44) and (45)

$$L(BG, x_1) = L(BG, x_2) = L(BG, =, I) = \{a^ib^ic^i \mid i \geqq 1\}.$$

It should be noted that a B-fuzzy grammar reduces to a conventional phrase structure grammar when the grades of productions are all equal to I (greatest element of Boolean lattice).

Therefore, from above example we have the following theorem.

THEOREM 3.1 *The class of λ-B-fuzzy languages generated by type* 2 (*or context-free*) *B-fuzzy grammars properly contains the class of type* 2 *languages.*

We have shown that the generative power of B-fuzzy grammars with type 2 rules is enhanced by introducing a Boolean lattice as the grades of productions though fuzzy grammars by Lee and Zadeh cannot generate type 1 languages [10].

Next we shall show that type 3 (or regular) B-fuzzy grammars cannot generate type 2 languages but type 3 languages only by using B-fuzzy automata in spite of the generative power of type 2 B-fuzzy grammars.

3.2. Type 3 B-Fuzzy Grammars

At first we shall define B-fuzzy automata as an extension of fuzzy automata by Wee and Fu [9, 11, 13] and show that the languages accepted by B-fuzzy automata are type 3 languages and that B-fuzzy automata are equivalent to type 3 B-fuzzy grammars.

DEFINITION 3.6 A *B-fuzzy automaton A* over the alphabet Σ is a system

$$A = (S, s_1, \{F(a) \mid a \in \Sigma\}, G, B), \tag{45}$$

where

1) $S = \{s_1, s_2, \ldots, s_n\}$ is a non-empty finite set of internal states.
2) s_1 is an initial state in S.
3) G is a subset of S (the set of final states).
4) $F(a)$ is a *fuzzy transition matrix* of order n such that

$$F(a) = [f_A(s_i, a, s_j)], \tag{46}$$

where $s_i, s_j \in S$, $a \in \Sigma$, and $n = \#(S)$. And f_A is a membership function of a B-fuzzy set in $S \times \Sigma \times S$; i.e.,

$$f_A : S \times \Sigma \times S \to B, \tag{47}$$

where B is a finite Boolean lattice. f_A may be called a *B-fuzzy transition function* and the value $f_A(s_i, a, s_j)$ represents the grade (or weight) of transition from state s_i to state s_j when the input is a.

Remark If Boolean lattice B consists of two element $\{0, I\}$, B-fuzzy automaton is reduced to a conventional (non-)deterministic finite automaton.

The grade of transition for the input string is defined as follows by using the concept of the composition of B-fuzzy relations in (26).

DEFINITION 3.7 For $\varepsilon, x, y \in \Sigma^*$ and $s, t \in S$

$$f_A(s, \varepsilon, t) = \begin{cases} I & \text{if } s = t, \\ 0 & \text{if } s \neq t. \end{cases} \tag{48}$$

$$f_A(s, xy, t) = \bigvee_{q \in S} [f_A(s, x, q) \wedge f_A(q, y, t)]. \tag{49}$$

where I and 0 are the greatest and least elements of the Boolean lattice B. The operations $\vee$ and $\wedge$ are lub and glb of B, respectively.

Using the above definition, the domain of transition matrix $F(a)$, $a \in \Sigma$, of a B-fuzzy automaton A can be extended from Σ to Σ^* as follows.

DEFINITION 3.8 For a string $x = a_1 a_2 \ldots a_m \in \Sigma^*$, $a_i \in \Sigma \cup \{\varepsilon\}$, define $n \times n$ fuzzy transition matrix $F(x)$ by the following.

$$\left.\begin{array}{l} \text{i) } F(\varepsilon) = E, \\ \text{ii) } F(x) = F(a_1) \circ F(a_2) \circ \ldots \circ F(a_m). \end{array}\right\} \tag{50}$$

where E is an $n \times n$ identity matrix such as

$$E = [e_{ij}] \Leftrightarrow e_{ij} = \begin{cases} I & i = j, \\ 0 & i \neq j. \end{cases}$$

The operation $\circ$ of B-fuzzy matrices are defined by

$$C = A \circ B \Leftrightarrow c_{ij} = \vee_k [a_{ik} \wedge b_{kj}],$$

where c_{ij}, a_{ik}, b_{kj} are in the Boolean lattice B. Note that this operation of matrices corresponds to the composition of B-fuzzy relations in (26).

Obviously, the (i, j)th element of fuzzy transition matrix $F(x)$ for an input string x is $f_A(s_i, x, s_j)$ defined before in (49).

DEFINITION 3.9 The B-fuzzy set in Σ^* defined by a B-fuzzy automaton A is characterized by the following membership function μ_A and is denoted by $L(A)$.

$$\mu_A(x) = \bigvee_{s_f \in G} f_A(s_1, x, s_f), \tag{51}$$

where s_1 is the initial state of A and $s_f \in G$ is the final state.

$\mu_A(x)$ is designated as the grade of transition of A, when started with the initial state to enter into the state in G after scanning the input string x. Then an input string x is said to be accepted by A with grade (or weight) $\mu_A(x)$.

Example 3.2 Let $\Sigma = \{a\}$ and let a B-fuzzy automaton be

$$A = (\{s_1, s_2, s_3\}, s_1, \{F(a)\}, \{s_3\}, B),$$

where $F(a)$ is given as

$$F(a) = \begin{array}{c} \\ s_1 \\ s_2 \\ s_3 \end{array} \begin{array}{c} \begin{array}{ccc} s_1 & s_2 & s_3 \end{array} \\ \begin{bmatrix} I & x_2 & x_2 \\ x_1 & x_2 & I \\ x_1 & 0 & x_2 \end{bmatrix} \end{array},$$

and B is a Boolean lattice in Figure 1.

Then, for example

$$\begin{aligned} \mu_A(a) &= f_A(s_1, a, s_3) = x_2 \\ \mu_A(aa) &= f_A(s_1, aa, s_3) \\ &= [f_A(s_1, a, s_1) \wedge f_A(s_1, a, s_3)] \\ &\quad \vee [f_A(s_1, a, s_2) \wedge f_A(s_2, a, s_3)] \\ &\quad \vee [f_A(s_1, a, s_3) \wedge f_A(s_3, a, s_3)] \\ &= (I \wedge x_2) \vee (x_2 \wedge I) \vee (x_2 \wedge x_2) \\ &= x_2 . \end{aligned}$$

DEFINITION 3.10 Let $A = (S, s_1, \{F(a) \mid a \in \Sigma\}, G, B)$ be a B-fuzzy automaton and λ the element of B. The set of all input strings accepted by A with parameter λ is defined as

$$L(A, \lambda) = \{x \mid \mu_A(x) > \lambda\},\ x \in \Sigma^*. \tag{52}$$

λ is called a *threshold* of A and $L(A, \lambda)$ a *λ-B-fuzzy language.*

THEOREM 3.2 *λ-B-fuzzy language $L(A, \lambda)$ is a regular language.*

Proof Let a B-fuzzy automaton A be $A = (S, s_1, \{F(a) \mid a \in \Sigma\}, G, B)$. Define the relation R on Σ^* by the definition. For any $x, y \in \Sigma^*$ and any $t \in S$, let

$$xRy \Leftrightarrow f_A(s_1, x, t) = f_A(s_1, y, t),$$

where s_1 is an initial state of A.

Then R is clearly an equivalence relation on Σ^*. Furthermore, for any $z \in \Sigma^*$,

$$\begin{aligned} f_A(s_1, xz, t) &= \bigvee_{q \in S} [f_A(s_1, x, q) \wedge f_A(q, z, t)] \\ &= \bigvee_{q \in S} [f_A(s_1, y, q) \wedge f_A(q, z, t)] \\ &= f_A(s_1, yz, t). \end{aligned}$$

Therefore we have $xz\ R\ yz$.

Hence, R is a right congruence relation on Σ^*. Let the number of elements of the Boolean lattice B be m and the number of states of A be n, then the number of equivalence classes by R is at most m^n. Anyhow R has finite rank. Moreover, it is easily verified that $L(A, \lambda)$ is the union of some of the equivalence classes. Hence $L(A, \lambda)$ is a regular language.

THEOREM 3.3 *Given type* 3 *B-fuzzy grammar, there exists B-fuzzy automaton A such that*

$$L(A) = L(3\text{-}BG)$$

and vice versa.

Proof ($\Leftarrow$) Let type 3 B-fuzzy grammar be $3\text{-}BG = (V_N, V_T, P, \sigma, J, \mu, B)$, then B-fuzzy automaton A is $A = (S, s_1, \{F(a) \mid a \in V_T\}, G, B)$ and is defined as follows. It is assumed that Boolean lattice B of $3\text{-}FG$ is the same as that of A. The set of states of A is $S = \{\phi\} \cup \{\langle A\rangle \mid A \in V_N\}$. The initial state s_1 is $\langle\sigma\rangle$, where σ is the initial symbol of $3\text{-}BG$. The set of final states is $\{\langle\phi\rangle\}$. The fuzzy transition matrix $F(a)$, $a \in V_T$, of order n $(= \#(S))$ is obtained by the following.

i) For each non-terminal production $(r)\,X \to aY\ \mu(r)$ in P with $a \in V_T$ and X, $Y \in V_N$, define

$$f_A(\langle X\rangle, a, \langle Y\rangle) = \mu(r).$$

ii) For each terminal production $(r)\ X \to a\ \mu(r)$ in P,

$$f_A(\langle X\rangle, a, \langle\phi\rangle) = \mu(r),$$

where $\langle\phi\rangle$ is final state of A.

iii) All the other f_A are defined to be equal to 0 (the least element of the Boolean lattice B).

($\Rightarrow$) For a B-fuzzy automaton $A = (S, s_1, \{F(a_k) \mid a_k \in \Sigma\}, G, B)$, let type 3 B-fuzzy grammar be $3\text{-}BG = (V_N, \Sigma, P, \sigma, J, \mu, B)$, where $V_N = \{\langle s_i\rangle \mid s_i \in S\}$, $\sigma = \langle s_1\rangle$, and the productions of 3-BG are obtained by the following. To the element $f_A(s_i, a_k, s_j)$ of fuzzy transition matrix $F(a_k)$ correspond the rewriting rule such as $\langle s_i\rangle \to a_k\langle s_j\rangle$, where $1 \leqq i, j \leqq n$ $1 \leqq k \leqq h$, $n = \#(S)$, and $h = \#(V_T)$. Then the number of the corresponding rules, that is, the number of their labels is n^2h. On the other hand, the terminal rules are given as follows. For each rules $\langle s_i\rangle \to a_k\langle s_j\rangle$ obtained above, if s_j is in G, that is, s_j is a final state, then we construct a terminal rules such as $\langle s_i\rangle \to a_k$. Thus the number of labels of the terminal rules is hnq, where $q = \#(G)$.

Hence, the total number of labels, i.e., $\#(J)$ is $hn^2 + hnq\ (= t)$. We can appropriately attach the labels to the rules obtained above without overlapping. Here it is assumed that the label r of the non-terminal rule $(r)\ \langle s_i\rangle \to a_k\langle s_j\rangle$ is in $\{1, 2, \ldots, hn^2\}$, and the label r of the terminal rule $(r)\ \langle s_i\rangle \to a_k$ is in $\{hn^2+1, \ldots, t\}$.

Next, we shall obtain the grade $\mu(r)$ of the rule r.

i) For each non-terminal rule $(r)\ \langle s_i\rangle \to a_k\langle s_j\rangle$, the grade $\mu(r)$ is given as

$$\mu(r) = f_A(s_i, a_k, s_j).$$

ii) The grade $\mu(r)$ for the terminal rule $(r)\ \langle s_i\rangle \to a_k$ is given as follows if this terminal rule is obtained from the rule $(p)\ \langle s_i\rangle \to a_k\langle s_f\rangle$ with $s_f \in G$.

$$\mu(r) = f_A(s_i, a_k, s_f),$$

where s_f is in G, $hn^2+1 \leqq r \leqq t$, and $1 \leqq p \leqq hn^2$.

From the above formulations of 3-BG and A we can easily show that $\mu_A(x) = \mu_{3-BG}(x)$ for each x in V_T^*. Therefore the theorem holds.

Example 3.3 Let type 3 B-fuzzy grammar be $3\text{-}BG = (V_N, V_T, P, \sigma, J, \mu, B)$, where $V_N = \{\sigma, A, B\}$, $V_T = \{a, b\}$, and the fuzzy productions are

1) $\sigma \to aA$	x_1,	5) $A \to bB$	x_1
2) $\sigma \to bB$	x_2,	6) $B \to bA$	y_2
3) $A \to aA$	x_3,	7) $A \to b$	y_3
4) $B \to a\sigma$	y_1,	8) $B \to a$	I

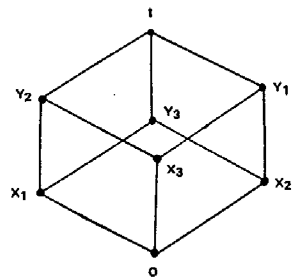

FIGURE 2 Structure of B.

And B is a Boolean lattice as in Figure 2. Then we can construct a B-fuzzy automaton A by the following. Let A be $A = (S, \langle\sigma\rangle, \{F(a), F(b)\}, \langle\phi\rangle, B)$, where $\Sigma = V_T$ and B is the same as in Figure 2. $S = \{\langle\sigma\rangle, \langle A\rangle, \langle B\rangle, \langle\phi\rangle\}$, and the initial state is $\langle\sigma\rangle$. Fuzzy transition matrices $F(a)$ and $F(b)$ are as follows.

$$F(a) = \begin{array}{c} \\ \langle\sigma\rangle \\ \langle A\rangle \\ \langle B\rangle \\ \langle\phi\rangle \end{array} \begin{array}{c} \begin{array}{cccc} \langle\sigma\rangle & \langle A\rangle & \langle B\rangle & \langle\phi\rangle \end{array} \\ \begin{bmatrix} 0 & x_1 & 0 & 0 \\ 0 & x_3 & 0 & 0 \\ y_1 & 0 & 0 & I \\ 0 & 0 & 0 & 0 \end{bmatrix} \end{array},$$

$$F(b) = \begin{bmatrix} 0 & 0 & x_2 & 0 \\ 0 & 0 & x_1 & y_3 \\ 0 & y_2 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Example 3.4 Let $A = (S, s_1, \{F(a) \mid a \in \Sigma\}, G, B)$ be a B-fuzzy automaton, where $S = \{s_1, s_2\}, \Sigma = \{a, b\}, G = \{s_2\}$, B is in Figure 2, and fuzzy transition matrices $F(a)$ and $F(b)$ are

$$F(a) = \begin{array}{c} \\ s_1 \\ s_2 \end{array} \begin{array}{c} \begin{array}{cc} s_1 & s_2 \end{array} \\ \begin{bmatrix} x_1 & x_3 \\ y_3 & x_2 \end{bmatrix} \end{array}, \qquad F(b) = \begin{array}{c} \\ s_1 \\ s_2 \end{array} \begin{array}{c} \begin{array}{cc} s_1 & s_2 \end{array} \\ \begin{bmatrix} y_2 & y_1 \\ x_1 & x_3 \end{bmatrix} \end{array}.$$

Then type 3 B-fuzzy grammar $3\text{-}BG = (V_N, V_T, \sigma, P, J, \mu, B)$ is defined as follows. $V_N = \{\langle s_1\rangle, \langle s_2\rangle\}$, $V_T = \Sigma = \{a, b\}$, $\sigma = \langle s_1\rangle$, and the productions are

1)	$\langle s_1\rangle \to a\langle s_1\rangle$	x_1,	7)	$\langle s_2\rangle \to b\langle s_1\rangle$	x_1
2)	$\langle s_1\rangle \to a\langle s_2\rangle$	x_3,	8)	$\langle s_2\rangle \to b\langle s_2\rangle$	x_3
3)	$\langle s_1\rangle \to b\langle s_1\rangle$	y_2,	9)	$\langle s_1\rangle \to a$	x_3
4)	$\langle s_1\rangle \to b\langle s_2\rangle$	y_1,	10)	$\langle s_1\rangle \to b$	y_1
5)	$\langle s_2\rangle \to a\langle s_1\rangle$	y_3,	11)	$\langle s_2\rangle \to a$	x_2
6)	$\langle s_2\rangle \to a\langle s_2\rangle$	x_2,	12)	$\langle s_2\rangle \to b$	x_3.

THEOREM 3.4 *Type 3 λ-B-fuzzy language $L(3\text{-}BG, \lambda)$ with the threshold λ generated by type 3 B-fuzzy grammar 3-BG is a regular language.*

Proof It is obvious from Theorem 3.2 and 3.3.

4. CLOSURE PROPERTIES OF B-FUZZY LANGUAGES

In this chapter we shall show that a family of *B*-fuzzy languages by type 3 *B*-fuzzy grammars is closed under the operations of union, intersection, concatenation and Kleene closure in the fuzzy sense, but is not closed under the complement. It is, however, shown that the complement of *B*-fuzzy language by type 3 *B*-fuzzy grammar can be characterized by type 3 $\wedge\vee BG$.

Moreover, a family of *B*-fuzzy languages by type 2 *B*-fuzzy grammars is closed under the union, concatenation, Kleene closure in the fuzzy sense. But this family is not closed under the intersection and complement.

4.1. Closure properties of type 3 B-fuzzy languages

At first we shall consider type 3 *B*-fuzzy languages.

THEOREM 4.1 *For two type 3 B-fuzzy grammars $3\text{-}BG^{(1)}$ and $3\text{-}BG^{(2)}$, let $L(3\text{-}BG^{(1)})$ and $L(3\text{-}BG^{(2)})$ be the B-fuzzy languages by $3\text{-}BG^{(1)}$ and $3\text{-}BG^{(2)}$, respectively. Then, in the fuzzy sense, there exists a 3-BG such that*

$$L(3\text{-}BG) = L(3\text{-}BG^{(1)}) \cup L(3\text{-}BG^{(2)}). \tag{53}$$

Proof Let $3\text{-}BG^{(1)}$ and $3\text{-}BG^{(2)}$ be as follows.

$$3\text{-}BG^{(1)} = (V_N^{(1)}, V_T^{(1)}, P^{(1)}, S_1, J^{(1)}, \mu^{(1)}, B),$$

$$3\text{-}BG^{(2)}, = (V_N^{(2)}, V_T^{(2)}, P^{(2)}, S_2, J^{(2)}, \mu^{(2)}, B),$$

where it is assumed that $V_N^{(1)} \cap V_N^{(2)} = \phi$ and $J^{(1)} \cap J^{(2)} = \phi$. Now consider a 3-*BG*, that is,

$$3\text{-}BG = (V_N, V_T, P, S, J, \mu, B),$$

where V_N, V_T, P, J, and μ are given as follows.

$$V_N = V_N^{(1)} \cup V_N^{(2)} \cup \{S\},$$
$$V_T = V_T^{(1)} \cup V_T^{(2)},$$
$$P = P^{(1)} \cup P^{(2)} \cup P_I \cup P_{II},$$
$$J = J^{(1)} \cup J^{(2)} \cup J_I \cup J_{II},$$

where $J^{(1)} \cap J^{(2)} \cap J_I \cap J_{II} = \phi$ is assumed.

P_I, J_I, P_{II}, and J_{II} are defined as follows.

[*I*]: For each initial rule $(r)\ S_1 \to w$ in $P^{(1)}$, we construct a new initial rule of P such as $(r')\ S \to w$, where r' is a different new label, and S is a new initial symbol in 3-*BG*. Let P_I and J_I be the set of all new initial rules obtained above and the set of labels corresponding to these new rules, respectively. Formally,

$$P_I = \{(r')\ S \to w \mid (r)\ S_1 \to w \in P^{(1)}\},$$
$$J_I = \{r' \mid (r')\ S \to w \in P_I\}.$$

[*II*]: We can get P_{II} and J_{II} for the initial rules in $P^{(2)}$ in a similar way as [*I*]. That is,

$$P_{II} = \{(r'')\ S \to w \mid (r)\ S_2 \to w \in P^{(2)}\},$$
$$J_{II} = \{r'' \mid (r'')\ S \to w \in P_{II}\}.$$

Finally, *B*-fuzzy function μ is defined by the following.

$$\mu(r') = \mu^{(1)}(r) \ldots r' \in J_I,$$
$$\mu(r'') = \mu^{(2)}(r) \ldots r'' \in J_{II},$$
$$\mu(r) = \mu^{(1)}(r) \ldots r \in J^{(1)},$$
$$\mu(r) = \mu^{(2)}(r) \ldots r \in J^{(2)}.$$

Example 4.1 Let $V_N^{(1)}$, $V_T^{(1)}$, and $P^{(1)}$ of 3-$BG^{(1)}$ be as follows. $V^{(1)} = \{S_1, A\}$, $V_T^{(1)} = \{a, b\}$, and

$$(r_1) \quad S_1 \to aA \quad I,$$
$$(r_2) \quad A \to bS_1 \quad x_1,$$
$$(r_3) \quad A \to b \quad y_2.$$

The Boolean lattice B is in Figure 2. Then B-fuzzy language by 3-$BG^{(1)}$ is

$$L(3\text{-}BG^{(1)}) = \{(ab, y_2)\} \cup \{((ab)^n, x_1) \mid n \geqq 2\}.$$

On the other hand, $V_N^{(2)}$, $V_T^{(2)}$, and $P^{(2)}$ of 3-$BG^{(2)}$ are as follows. $V_N^{(2)} = \{S_2, B\}$, $V_T^{(2)} = \{a, b, c\}$, and

$$P^{(2)} = \begin{Bmatrix} (p_1)\ S_2 \to aB & I \\ (p_2)\ B \to bS_2 & x_2 \\ (p_3)\ B \to b & y_1 \\ (p_4)\ S_2 \to c & y_3 \end{Bmatrix}$$

Then $L(3\text{-}BG^{(2)})$ is

$$L(3\text{-}BG^{(2)}) = \{(ab, y_1)\} \cup \{((ab)^n, x_2) \mid n \geqq 2\} \cup \{((ab)^n c, x_2) \mid n \geqq 1\} \cup \{(c, y_3)\}.$$

Therefore

$$L(3\text{-}BG^{(1)}) \cup L(3\text{-}BG^{(2)}) = \{(ab, I)\} \cup \{((ab)^n, y_3) \mid n \geqq 2\} \cup \{((ab)^n c, x_2) \mid n \geqq 1\} \cup \{(c, y_3)\}.$$

By the way, 3-BG constructed from 3-$BG^{(1)}$ and 3-$BG^{(2)}$ by Theorem 4.1 is as follows. $V_N = \{S, S_1, S_2, A, B\}$, $V_T = \{a, b, c\}$, and P is given as

$$\begin{array}{lll} (r_1') & S \to aA & I \\ (r_1) & S_1 \to aA & I \\ (r_2) & A \to bS_1 & x_1 \\ (r_3) & A \to b & y_2 \\ (p_1'') & S \to aB & I \\ (p_4'') & S \to c & y_3 \\ (p_1) & S_2 \to aB & I \\ (p_2) & B \to bS_2 & x_2 \\ (p_3) & B \to b & y_1 \\ (p_4) & S_2 \to c & y_3. \end{array}$$

For the string ab, two fuzzy derivation chains are possible, that is,

$$S \xrightarrow{I} aA \xrightarrow{y_2} ab,$$

$$S \xrightarrow{I} aB \xrightarrow{y_1} ab.$$

Therefore, the grade of generation of the string ab is given as

$$\mu_{3\text{-}BG}(ab) = (I \wedge y_2) \vee (I \wedge y_1) = y_2 \vee y_1 = I.$$

Similarly, for the string $(ab)^n$, $n \geqq 2$, we have

$$S \xrightarrow{I} aA \xrightarrow{x_1} abS_1 \xrightarrow{I} abaA \xrightarrow{x_1} \dots \xrightarrow{y_2} (ab)^n,$$

$$S \xrightarrow{I} aB \xrightarrow{x_2} abS_2 \xrightarrow{I} abaB \xrightarrow{x_2} \dots \xrightarrow{y_1} (ab)^n.$$

Therefore, we have

$$\mu_{3\text{-}BG}((ab)^n) = x_1 \vee x_2 = y_3.$$

For the string $(ab)^n c$, $n \geqq 1$,

$$S \xrightarrow{I} aB \xrightarrow{x_2} abS_2 \xrightarrow{I} abaB \xrightarrow{x_2} \dots \longrightarrow (ab)^n S_2 \xrightarrow{y_3} (ab)^n c$$

Therefore, $\mu_{3\text{-}BG}((ab)^n c) = I \wedge x_2 \wedge y_3 = x_2$.

Moreover, we have $\mu_{3-BG}(c) = y_3$. Therefore, $L(3\text{-}BG)$ is

$$\begin{aligned} L(3\text{-}BG) &= \{(ab, I)\} \cup \{((ab)^n, y_3) \mid n \geqq 2\} \\ &\quad \cup \{((ab)^n c, x_2) \mid n \geqq 1\} \cup \{(c, y_3)\} \\ &= L(3\text{-}BG^{(1)}) \cup L(3\text{-}BG^{(2)}). \end{aligned}$$

THEOREM 4.2 *For two B-fuzzy languages* $L(3\text{-}BG^{(1)})$ *and* $L(3\text{-}BG^{(2)})$ *by* $3\text{-}BG^{(1)}$ *and* $3\text{-}BG^{(2)}$, *respectively, there exists* $3\text{-}BG$ *such that*

$$L(3\text{-}BG) = L(3\text{-}BG^{(1)}) \cap L(3\text{-}BG^{(2)}). \tag{54}$$

Proof For two $3\text{-}BG^{(1)}$ and $3\text{-}BG^{(2)}$, that is,

$$3\text{-}BG^{(1)} = (V_N^{(1)}, V_T, P^{(1)}, S_1, J^{(1)}, \mu^{(1)}, B),$$

$$3\text{-}BG^{(2)} = (V_N^{(2)}, V_T, P^{(2)}, S_2, J^{(2)}, \mu^{(2)}, B),$$

let us define $3\text{-}BG$ as follows.

$$3\text{-}BG = (V_N, V_T, P, S, J, \mu, B).$$

At first, the rules in P are given by the following.
[I]: For two non-terminal rules in $P^{(1)}$ and $P^{(2)}$ such that the terminal symbols of the right hand side of these two rules are equal, say, $a \in V_T$. That is, for

$$(r)\ A_1 \to aA_2 \in P^{(1)} \quad \text{and} \quad (p)\ B_1 \to aB_2 \in P^{(2)},$$

let a new non-terminal rule in P be defined as follows.

$$(r, p) \quad \langle A_1, B_1 \rangle \longrightarrow a\langle A_2, B_2 \rangle.$$

[*II*]: For the two terminal rules in $P^{(1)}$ and $P^{(2)}$ such as

$(r)\ A_1 \to a \in P^{(1)}$ and $(p)\ B_1 \to a \in P^{(2)}$,

with the same terminal symbol, define a new terminal rule in P as follows

$$(r, p) \ \langle A_1, B_1 \rangle \to a.$$

Let P_a be the set of new rules obtained in [*I*] and [*II*] and J_a be the set of labels corresponding to these new rules. Then P and J in 3-*BG* are given as

$$P = \bigcup_{a \in V_T} P_a, \quad \text{and} \quad J = \bigcup_{a \in V_T} J_a.$$

Moreover, V_N is the set of pairs $\langle A_1, B_1 \rangle$ obtained in [*I*] and [*II*], and $S = \langle S_1, S_2 \rangle$. Clearly we have

$$V_N \subseteq V_N^{(1)} \times V_N^{(2)} \quad \text{and} \quad J \subseteq J^{(1)} \times J^{(2)}.$$

Finally, *B*-fuzzy function μ is defined by

$$\mu(r, p) = \mu^{(1)}(r) \wedge \mu^{(2)}(p)$$

for each (r, p) in J.

THEOREM 4.3 *For the two fuzzy languages* $L(3\text{-}BG^{(1)})$ *and* $L(3\text{-}BG^{(2)})$ *by* $3\text{-}BG^{(1)}$ *and* $3\text{-}BG^{(2)}$, *respectively, there exists* 3-*BG which realizes the concatenation of* $L(3\text{-}BG^{(1)})$ *and* $L(3\text{-}BG^{(2)})$ (*see* (32)).

$$L(3\text{-}BG) = L(3\text{-}BG^{(1)}) \circ L(3\text{-}BG^{(2)}). \tag{55}$$

Proof Let

$$3\text{-}BG^{(1)} = (V_N^{(1)}, V_T^{(1)}, P^{(1)}, S_1, J^{(1)}, \mu^{(1)}, B),$$
$$3\text{-}BG^{(2)} = (V_N^{(2)}, V_T^{(2)}, P^{(2)}, S_2, J^{(2)}, \mu^{(2)}, B),$$

where $V_N^{(1)} \cap V_N^{(2)} = \phi$ and $J^{(1)} \cap J^{(2)} = \phi$.

We construct a new 3-*BG*,

$$3\text{-}BG = (V_N, V_T, P, S, J, \mu, B),$$

where $S = S_1$, $V_N = V_N^{(1)} \cup V_N^{(2)}$, $V_T = V_T^{(1)} \cup V_T^{(2)}$, and $J = J^{(1)} \cup J^{(2)}$.

We introduce the following notations in order to get P. Let $P_{nt}^{(i)}$ and $P_{tr}^{(i)}$ be the sets of non-terminal rules and terminal rules, respectively, in $P^{(i)}$, $i = 1, 2$. And let $J_{nt}^{(i)}$ and $J_{tr}^{(i)}$ be the sets of the labels corresponding to the rules in

$P_{nr}^{(i)}$ and $P_{tr}^{(i)}$, respectively. Then, clearly, we have $P^{(i)} = P_{nr}^{(i)} \cup P_{tr}^{(i)}$ and $J^{(i)} = J_{nr}^{(i)} \cup J_{tr}^{(i)}$ for each $i \in \{1, 2\}$.

Now we shall obtain the rules of 3-*BG*.

For each terminal rule $(r)\ A \to a$ in $P_{tr}^{(1)}$ and the initial symbol S_2 of 3-$BG^{(2)}$, we construct a new rule $(r)\ A \to aS_2$, where the label is not changed. Let P' be the set of such rules, then P of 3-*BG* is given as

$$P = P_{nr}^{(1)} \cup P' \cup P^{(2)}.$$

The *B*-fuzzy function μ of 3-*BG* is as follows.

$$\mu(r) = \begin{cases} \mu^{(1)}(r) \ldots r \in J_S (= J_{S_1}^{(1)}) \\ \mu^{(1)}(r) \ldots r \in J_{nr}^{(1)} \\ \mu^{(1)}(r) \ldots r \in J' \\ \mu^{(2)}(r) \ldots r \in J^{(2)} \end{cases}$$

where J_S is the set of labels of initial rules $S \to w$ of P.

THEOREM 4.4 *For a fuzzy language* L(3-*BG*) *by* 3-*BG*, *there exists* 3-*BG*′ *which realizes Kleene closure* (*see* (34)).

$$L(3\text{-}BG') = L(3\text{-}BG)^*. \tag{56}$$

Proof For the 3-$BG = (V_N, V_T, P, S, J, \mu, B)$, let 3-$BG'$ be 3-$BG' = (V'_N, V_T, P', S', J', \mu', B)$, where $V'_N = \{S'\} \cup V_N$. P' is obtained from [*I*], [*II*] and [*III*] denoted later. It is assumed that the mappings t_1, t_2, and t_3 in [*I*], [*II*], and [*III*], respectively, are all one to one mappings from labels to new labels, and the obtained new labels are all different from each other.

[*I*]: For each initial rule $(r)\, S \to w$ in P, we construct a new initial rule $(t_1(r))S' \to w$ in P'. Let P_I be the set of such new initial rules and J_I be the set of the labels corresponding to these initial rules. Formally,

$$P_I = \{(t_1(r))S' \to w \mid (r)\, S \to w \in P,\, r \in J_S\},$$
$$J_I = \{t_1(r)\}.$$

[*II*]: For each terminal initial rule $(r)\, S \to a$ in P, define a new rule $(t_2(r))$ $S' \to aS$. Then, let

$$P_{II} = \{(t_2(r))\, S' \to aS \mid (r)\, S \to a\, P,\, r \in J_S\},$$

$$J_{II} = \{t_2(r)\}.$$

[*III*]: For each terminal rule $(r)\ A \to a$ in P, construct a new rule $(t_3(r))\ A \to aS$ in P' and let

$$P_{III} = \{(t_3(r))\ A \to aS \mid (r)\ A \to a \in P\},$$
$$J_{III} = \{t_3(r)\}.$$

Then P' and J' in 3-*BG* are given from [*I*], [*II*], and [*III*] as follows.

$$P' = P \cup P_I \cup P_{II} \cup P_{III} \cup \{(p_0)\ S' \to \varepsilon\},$$
$$J' = J \cup J_I \cup J_{II} \cup J_{III} \cup \{p_0\},$$

where all labels in J' are different from each other.

It is noted that in any derivation we start with an initial rule in P_I or P_{II} (not in P, P_{III}), and then rules in P or P_{III} are used throughout in the derivation.

Finally we shall obtain *B*-fuzzy function.

$$\mu'(p) = \begin{cases} \mu(r) \ldots p = t_1(r) \in J_I \\ \mu(r) \ldots p = t_2(r) \in J_{II} \\ \mu(r) \ldots p = t_3(r) \in J_{III} \\ \mu(p) \ldots p \in J \\ \quad I \ldots p = p_0. \end{cases}$$

THEOREM 4.5 *Type* 3 *B-fuzzy languages are not closed under the complement.*

Proof In general we have that $L(BG, \lambda_1) \supseteq L(BG, \lambda_2)$ if and only if $\lambda_1 \leqq \lambda_2$ for any *B*-fuzzy grammar *BG*. That is, $L(BG, \cdot)$ is non-increasing. In contradication to this, the complement of $\overline{L(BG, \cdot)}$ is non-decreasing, that is, $\overline{L(BG, \lambda_1)} \supseteq \overline{L(BG, \lambda_2)}$ iff $\lambda_1 \geqq \lambda_2$.

Now we shall show that the complement of type 3 *B*-fuzzy language can be defined by type 3-$\wedge\vee$*BG* (43).

THEOREM 4.6 *For a type* 3 *B-fuzzy language L*(3-*BG*) *by* 3-*BG there exists a* $\wedge\vee$*BG which realizes the complement of L*(3-*BG*), *that is,*

$$L(3\text{-}\wedge\vee BG) = \overline{L(3\text{-}BG)}. \tag{57}$$

Proof For a 3-$BG = (V_N, V_T, P, S, J, \mu, B)$, let 3-$\wedge\vee BG$ be $(V_N, V_T, P, S, J, \mu', B)$. It should be noted that the grade $\mu_{\wedge\vee BG}(x)$ of generation of a string x in V_T^* is defined from (43) by changing $\wedge$ with $\vee$ in (42). Moreover, V_N, V_T, P, S, J, and B of 3-$\wedge\vee BG$ are assumed to be the same as that of

3-BG. B-fuzzy function $\mu'(r)$ of 3-$\wedge\vee BG$ is obtained as follows. For each r in J,

$$\mu'(r) = \overline{\mu(r)}. \tag{58}$$

Then we can easily prove the following property using De Morgan's law, that is,

$$\wedge [\bar{\mu}_1 \vee \bar{\mu}_2 \vee \dots \vee \bar{\mu}_m] = \overline{\vee\ [\mu_1 \wedge \mu_2 \wedge \dots \wedge \mu_m]}$$

where μ_i and $\bar{\mu}_i$ are in Boolean lattice B and $\bar{\mu}_i$ is the complement of μ_i.

Therefore we have $\mu_{\wedge\vee BG}(x) = \overline{\mu_{BG}(x)}$, $x \in V_T^*$.

4.2. Closure properties of type 2 B-fuzzy languages

In this section we shall consider the closure properties of type 2 (or context-free) B-fuzzy languages. It is shown that the family of type 2 B-fuzzy languages by type 2 B-fuzzy grammars is closed under the union, concatenation, and Kleene closure in the fuzzy sense. But this family is not closed under the intersection and complement.

THEOREM 4.7 *For two type 2 B-fuzzy grammars* 2-$BG^{(1)}$ *and* 2-$BG^{(2)}$, *let* L(2-$BG^{(1)}$) *and* L(2-$BG^{(2)}$) *be the type 2 B-fuzzy languages defined by* 2-$BG^{(1)}$ *and* 2-$BG^{(2)}$, *respectively. Then, in the fuzzy sense, there exists a* 2-BG *such as*

$$L(2\text{-}BG) = L(2\text{-}BG^{(1)}) \cup L(2\text{-}BG^{(2)}). \tag{59}$$

Proof Let 2-$BG^{(1)}$ and 2-$BG^{(2)}$ be as follows.

$$2\text{-}BG^{(1)} = (V_N^{(1)}, V_T^{(1)}, P^{(1)}, S_1, J^{(1)}, \mu^{(1)}, B)$$

$$2\text{-}BG^{(2)} = (V_N^{(2)}, V_T^{(2)}, P^{(2)}, S_2, J^{(2)}, \mu^{(2)}, B)$$

where it is assumed that $V_N^{(1)} \cap V_N^{(2)} = \phi$ and $J^{(1)} \cap J^{(2)} = \phi$. 2-$BG$ is given as

$$2\text{-}BG = (V_N, V_T, P, S, J, \mu, B).$$

Then $V_N = V_N^{(1)} \cup V_N^{(2)} \cup \{S\}$, where S is an initial symbol of 2-BG and is not in $V_N^{(1)} \cup V_N^{(2)}$. And $V_T = V_T^{(1)} \cup V_T^{(2)}$. The set of productions P of 2-BG is obtained as

$$P = P^{(1)} \cup P^{(2)} \cup \{(k)\ S \to S_1, (m)\ S \to S_2\}.$$

The labels sets $\{k, m\}$, $J^{(1)}$, and $J^{(2)}$ are mutually disjoint and the label set J of 2-BG is as follows.

$$J = J^{(1)} \cup J^{(2)} \cup \{k, m\}.$$

Finally, the fuzzy function μ of 2-BG is

$$\mu(r) = \begin{cases} \mu^{(1)}(r) \ldots r \in J^{(1)} \\ \mu^{(2)}(r) \ldots r \in J^{(2)} \\ I \ldots r \in \{k, m\} \end{cases}$$

where I is the greatest element of Boolean lattice B.

THEOREM 4.8 *Type 2 B-fuzzy languages by type 2 B-fuzzy grammars are not closed under the intersection.*

Proof Let the Boolean lattice B be $\{0, I\}$ which consists of two elements 0 and I, then type 2 B-fuzzy grammars become ordinary type 2 grammars. It is well-known that type 2 languages by type 2 grammars are not closed under the intersection. Therefore the theorem holds.

THEOREM 4.9 *Type 2 B-fuzzy languages by type 2 B-fuzzy grammars are not closed under the complement.*

Proof It is obvious from the theorem 4.5.

THEOREM 4.10 *Given two type 2 B-fuzzy grammars 2-$BG^{(1)}$ and 2-$BG^{(2)}$, then there exists a 2-BG which satisfies the concatenation of L(2-$BG^{(1)}$) and L(2-$BG^{(2)}$), that is,*

$$L(2\text{-}BG) = L(2\text{-}BG^{(1)}) \circ L(2\text{-}BG^{(2)}). \tag{60}$$

Proof Let 2-$BG^{(1)}$ and 2-$BG^{(2)}$ be

$$2\text{-}BG^{(1)} = (V_N^{(1)}, V_T^{(1)}, P^{(1)}, S_1, J^{(1)}, \mu^{(1)}, B)$$

$$2\text{-}BG^{(2)} = (V_N^{(2)}, V_T^{(2)}, P^{(2)}, S_2, J^{(2)}, \mu^{(2)}, B)$$

where $V_N^{(1)} \cap V_N^{(2)} = \phi$ and $J^{(1)} \cap J^{(2)} = \phi$. Then 2-$BG$ is given as

$$2\text{-}BG = (V_N, V_T, P, S, J, \mu, B),$$

where $V_N = V_N^{(1)} \cup V_N^{(2)} \cup \{S\}$, $S \notin V_N^{(1)} \cup V_N^{(2)}$. And $V_T = V_T^{(1)} \cup V_T^{(2)}$. The productions P of 2-BG is

$$P = P^{(1)} \cup P^{(2)} \cup \{(k)\, S \to S_1 S_2\}$$

and label set J is $J^{(1)} \cup J^{(2)} \cup \{k\}$. The fuzzy function is defined as

$$\mu(r) = \begin{cases} \mu^{(1)}(r) \ldots r \in J^{(1)} \\ \mu^{(2)}(r) \ldots r \in J^{(2)} \\ I \ldots r = k \end{cases}$$

where I is the greatest element of B.

THEOREM 4.11 *Given a type 2 B-fuzzy grammar 2-BG, there is a type 2 B-fuzzy grammar 2-BG' which realizes Kleene closure such as*

$$L(2\text{-}BG') = L(2\text{-}BG)^*. \tag{61}$$

Proof Let $2\text{-}BG$ be $(V_N, V_T, P, S, J, \mu, B)$, then $2\text{-}BG'$ is given as $2\text{-}BG' = (V'_N, V_T, P', S', J', \mu', B)$, where $V'_N = V_N \cup \{S'\}, S' \in V_N$. $P' = P \cup \{(k)\ S' \to SS', (m)\ S' \to \varepsilon\}$. $J' = J \cup \{k, m\}$. The fuzzy function μ' is

$$\mu'(r) = \begin{cases} \mu(r) \ldots r \in J \\ I \ldots r \in \{k, m\}. \end{cases}$$

5. CONCLUSIONS

We have defined fuzzy grammars on Boolean lattices (B-fuzzy grammars) and studied several fundamental properties of B-fuzzy languages characterized by B-fuzzy grammars. It is founded that the generative power of B-fuzzy grammars can be enhanced although fuzzy grammars cannot enhance the generative power, and the basic concepts and results in the formal languages can be extended to B-fuzzy languages. The proofs, however, are generally somewhat longer since they involve the membership functions on Boolean lattice rather than on $\{0, 1\}$.

We hope that many interesting formal grammars on the particular algebras such as distributive lattice, lattice ordered semigroup, ring, and, more closely, the representative Boolean lattice as given in Figure 1 will be formulated and a number of interesting results will be obtained from these grammars.

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