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# Web Grammars and Several Graphs

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This paper is concerned with the class of web grammars introduced by Pfaltz, Rosenfeld and Montanari. In this paper, we show that context-sensitive web grammar cannot erase arcs, and monotone context-sensitive web grammar can erase arcs but cannot erase any vertices and they satisfy the condition $| \alpha | \leq | \beta |$ in the rules $\alpha \Rightarrow \beta$. Then some hierarchical results hold, when grammars are normal and nonnormal. Normal grammars have rules that each vertex to be rewritten has exactly one image in the right member of the rule; nonnormal ones have rules that some vertices have two more images. Also, it is shown that there exists a complete grammar which generates some types of Eulerian graphs, line graphs and 3-connected graphs.

## INTRODUCTION

There exists a variety of techniques whereby pictures may be interpreted or described in general ways. For example, Shaw's "formal picture description scheme" [1] and Dacey's [2] method are such techniques. One of the most recent and most interesting devices in picture processing has been introduced by Pfaltz and Rosenfeld [3] and Montanari [4]. These authors extend the concept of the one-dimensional grammars to graph theory, and apply the rewriting rule to general labeled graphs (called webs) in a natural way. But it is pointed out by Montanari that the embedding of the rewritten webs is the special problem associated with web grammars; in his paper, all the grammars are normal. In graph theory, however, it is often necessary to divide one vertex into two or more vertices to construct desired graph, as for examples, all complete graphs or all 3-connected graphs. In this paper, we prescribe that the embedding part satisfies the condition such that if there exists an image of a vertex of the left member of a rule in the right member, all vertices which have been adjacent to the vertex must be adjacent to the image after the application of the rule. And we consider grammars which have normal embedding and nonnormal embedding.

Now main results are summarized: First, we give the hierachy of the classes of webs generated by normal grammars and nonnormal grammars, respectively, and note that some hierachical results do not always hold good between classes of webs generated by normal context-free web grammars and those by nonnormal linear web grammars.

Second, the classes of webs indirectly generated by normal context-sensitive web grammars properly contain the classes of webs generated by normal ones.

Third, it is shown that there exists a complete web grammar which generates some types of Eulerian graphs and line graphs.

## Web Grammars

In this section, web grammars are introduced. Our definition is similar to that of Montanari, and Pfaltz and Rosenfeld.

Definition 2.1. Let $V$ be a finite set. A web $W$ over $V$ is a triple $(N_W, F_W, A_W)$, where

(1) $N_W$: a set of vertices;

(2) $F_W: N_W \rightarrow V$, i.e., labeling function;

(3) $A_W$: a set of binary relation on $N_W$ and its elements are called arcs.

Definition 2.2. A web grammar $G$ is a triple $(V, I, R)$ where

(1) $V$ is the vocabulary, and it consists of two disjoint parts, a nonterminal vocabulary $V_N$ and a terminal vocabulary $V_T$ $(V_N \neq \phi,\ V_T \neq \phi)$;

(2) $I$ is a finite set of initial webs over $V$;

(3) $R$ is a finite set of rewriting rules and its element, which is called rule, is formally described as a quadruple $(\alpha, C, \beta, E)$, where $\alpha, \beta$ are webs, and $C$ is a logical function called a contextual condition of the rule and prescribes the condition which web $\alpha$ and its adjacent vertices must satisfy. The logical functions $E$ specify the embedding of $\beta$ in $W - \alpha$. We prescribe that $E$ has a function that all vertices adjacent to the vertex to be rewritten in the host web must be adjacent to the image of that vertex; for a rule $(\alpha, C, \beta, E)$ of a given web grammar, if the image of $P$ in $N_\alpha$ is the set $\{Q_1, Q_2, ..., Q_n\}$ $(n \geqslant 1)$, then for any vertex $S$ in $N_{W-\alpha}$ ($W$ is a host web) if $(S, P) \in A_W$ then $(S, Q_i) \in A_{W-\alpha\cup\beta}$, and also $(S, P) \notin A_W$ then $(S, Q_i) \notin A_{W-\alpha\cup\beta}$ for $1 \leqslant i \leqslant n$.

Definition 2.3. Given a web $W = (N_W, F_W, A_W)$ over $V$, the web $S = (N_S, F_S, A_S)$ over the same $V$ is called a subweb of $W$ if

(1) $N_S \subseteq N_W$;

(2) If $X \in N_S$, $F_S(X) = F_W(X)$;

(3) For $P, Q \in N_S$, if $(P, Q) \in A_S$, $(P, Q) \in A_W$ and if $(P, Q) \notin A_S$, then $(P, Q) \notin A_W$.

DEFINITION 2.4. Given a web $W$, the rule $(\alpha, C, \beta, E)$ is applicable to the web $W$ if

(1) $\alpha$ is a subweb of $W$;

(2) $C$ is true.

DEFINITION 2.5. The language $L_G$ generated by a grammar $G$ consists of those webs on $V_T$ that can be derived from the initial webs by successively applying rules.

DEFINITION 2.6. In a given web grammar $G = (V, I, R)$ for any rewriting rules $(\alpha, C, \beta, E)$:

(a) If $C$ is always true and $|\alpha| \leqslant |\beta|$ and $N_\alpha \subsetneq N_\beta$, then the web grammar is called monotone context-sensitive web grammar (mcswg);

(b) In case (a), let a vertex $P$ have a label over $V_N$ in the web $\alpha$; then the given web grammar is called context-sensitive web grammar (cswg):

(i) if $X \in N_\alpha - \{P\}$, then $F_\alpha(X) = F_\beta(X)$;

(ii) if for $X, Y \in N_\alpha$, $(X, Y) \in A_\alpha$, then $(X', Y') \in A_\beta$ where $X', Y'$ are images of $X$ and $Y$, respectively.

In case (a), if the contextual condition $C$ is not always true, then, mcswg is called mcswg with applicability condition.

(c) In case (b), if the web $\alpha$ of each rule consists of only one vertex $P$, then the web grammar is called context-free web grammar (cfwg). Especially, if the webs in the initial webs and the right member of each rule consists of the web which has at most one vertex over $V_N$, the web grammar is called linear web grammar (lwg).

For example, the language generated by cswg is denoted as cswL and the family of cswL is written as $\text{csw}\overline{\text{L}}$.

Note that the context-sensitive web grammars defined by Montanari permit the insertion of vertices on an arc; but our context-sensitive web grammars (cswg) cannot do such an operation, fulfilled by our monotone context-sensitive web grammar (mcswg). Our mcswg is considered to concide with Montanari's monotone web grammar, except for the existence of images (Montanari's definition does not imply the existence of images of rewritten vertex, but all examples shown in his paper are represented by our mcswg).

DEFINITION 2.7. A rule $(\alpha, C, \beta, E)$ of a given web grammar is called normal if for any vertex $Q$ in $N_\alpha$, there exists exactly one vertex which is the image $Q$ in $N_\beta$. If the number of images is more than or equal to 2, the rule is called nonnormal. And if all the rules of a web grammar $G$ are normal (nonnormal), the web grammar is called normal (nonnormal), and normal cswg is denoted as ncswg and nonnormal cswg is denoted as ancswg.

DEFINITION 2.8. A web language $L_G$ is indirectly generated by a web grammar $G$, if

(a) The vocabulary $V_L$ is a proper subset of the terminal vocabulary $V_T$ of $G$;

(b) The language consists of just the subwebs of the terminal webs generated by $G$ whose vertices are labeled with symbols belonging to $V_L$;

(c) In any web generated by $G$, in which $N$ vertices are labeled with symbols of $V_L$, the number of vertices labeled with symbols of $V_T - V_L$ cannot exceed a fixed value $M_N$.

## CLASSES OF WEB GRAMMARS

In this section, the classes of webs generated by some types of web grammars are discussed.

THEOREM 1. *Given any web grammar $G$, a web grammar $H$ equivalent to $G$ can always be found, such that the initial web is a one-point web.*

*Proof.* Let $G = (V_G, I_G, R_G)$, $H = (V_H, I_H, R_H)$. If we construct

$$V_H = V_G \cup \{S\}, \qquad I_H = \{S\} \qquad \text{and} \qquad R_H = R_G \cup \{S \Rightarrow W_i \mid Wi \in I_G\},$$

then the equivalence between them is trivial. Q.E.D.

LEMMA 2. *There exists a* ncfw$\overline{\text{L}}$ *which does not belong to* nlw$\overline{\text{L}}$.

*Proof.* From the definition of nlwg, it is clear that nlwg cannot generate exactly all trees; that is, as the right member of any rules has at most one nonterminal symbol, the web derived from an initial web cannot expand to arbitrary directions. The ncfwg of Fig. 1 is equivalent to the grammar of Pfaltz and Rosenfeld (Theorem 1) except for directedness. Therefore, the results are evident. Q.E.D.

Some connected graphs can be disconnected by the removal of a single vertex, called a cutpoint. If a connected graph has a cutpoint, it is called separable; otherwise it is called nonseparable. In general, a block of a graph is a maximal nonseparable subgraph of the graph, but a graph $G$ itself is called a block if it is nonseparable.

LEMMA 3. *The blocks of* ncfwL $L_G$ *generated by* ncfwg *whose initial webs consist of only one-point webs consist of only webs $\beta$ of the rules of $G$ and the blocks of $\beta$.*

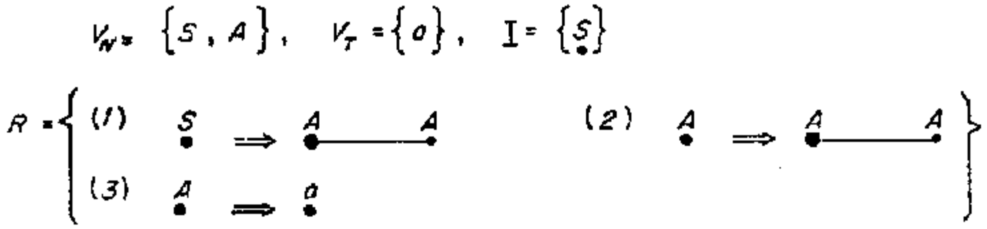


FIG. 1. This ncfwg generates a set of all trees. The image of $S$ of rule (1) is a leftmost vertex of the right member of the rule. For a ncfwg (nlwg), the image of the left member of rules is denoted by a dot.

*Proof.* Let the right member $\beta$ of any rules be connected webs. On a derivation of a web $W$ in $L_G$, let $\{\underset{.}{S}\} \overset{*}{\Rightarrow} W' \Rightarrow W'' \overset{*}{\Rightarrow} W$ be the derivation chain. Now consider that one vertex $\alpha$ in the web $W'$ are rewritten to web $W''$ by the application of a rule $\alpha \Rightarrow \beta$. Here let an image of vertex $\alpha$ be $\alpha_i$, i.e., $\alpha_i = \text{im}(\alpha)$. It is very easy to see that the web $W'' - \{\alpha\}$ is partitioned into at least two parts, $W' - \{\alpha\}$ and $W'' - W'$, and that for $u \in W' - \alpha$ and $v \in W'' - W'$, all $u - v$ paths pass the vertex $\alpha_i$ in the web $W''$. Therefore $\alpha_i$ is a cutpoint. If the web $W'$ is a block, then $W'$ is a block of $W''$, and if the web $\beta$ is a block, then $\beta$ is a block of $W''$; otherwise, block of $\beta$ is a block of $W''$. Similar considerations are repeated with respect to the web $W'$. Therefore the proof is done. Q.E.D.

A graph is called acyclic if it has no cycles (cycle is often called circuit and its definition is well known). If a connected graph is acyclic, it is called a tree.

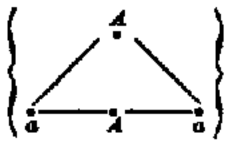
COROLLARY 4. *If the right member $\beta$ of any rules of a given* ncfwg *$G$ are acyclic, then any member of* ncfwL *$L_G$ is also acyclic.*

LEMMA 5. *It is not always true that the blocks of* ancfwL *$L_G$ consist of only web $\beta$ of the rules of $G$ and the blocks of $\beta$.*

*Proof.* Consider an ancfwg $G$ of Fig. 2. On a derivation of a web $W$ in $L_G$ from the initial web, let rule (2) be applied to a web

$$W' = \{\overset{a}{\bullet}\!\!-\!\!-\!\!\overset{A}{\bullet}\!\!-\!\!-\!\!\overset{a}{\bullet}\}$$

derived from $\{\underset{.}{S}\}$. Since rule (2) is a nonnormal rule and an image of the left member $A$ of rule (2) is two vertex $A$ of the right member of it, all vertices which have been adjacent to $A$ in the web $W'$ must be also adjacent to two vertices $A$ after the application of rule (2). Therefore the derivation chain is described as

$$\{\underset{.}{S}\} \Rightarrow W' \Rightarrow$$  $$\overset{*}{\Rightarrow} W.$$

As the derivation process of $W$ from $W''$ (above a four-point web) preserves the adjacency of any points in a derived web, web $W''$ is a subweb of the web $W$. Clearly the web $W$ is itself a block and it is not isomorphic to any $\beta$ and its blocks. Q.E.D.

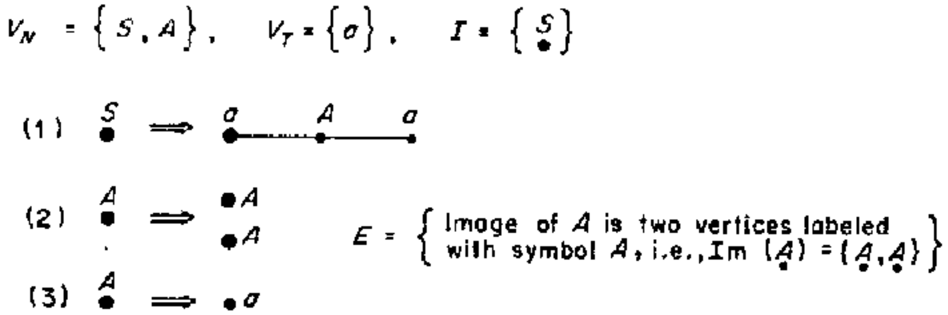

$V_N = \{S, A\}$, $V_T = \{a\}$, $I = \{S\}$
(1) S ⟹ a A a
(2) A ⟹ A A
E = { Image of A is two vertices labeled with symbol A, i.e., Im (A) = {A, A} }
(3) A ⟹ a

FIG. 2. This ancfwg generates a set of webs that cannot be generated by any ncfwg. Embedding of rule (2) specifies that any vertices adjacent to $A$-vertex in the host web are adjacent to both the vertices labeled with symbol $A$ in the right member of the rule. (Hereafter, the embedding part of nonnormal grammars is denoted as in this figure.)

THEOREM 6. $\mathrm{ncfw}\overline{\mathrm{L}} \subsetneq \mathrm{ancfw}\overline{\mathrm{L}}$.

*Proof.* The relation $\mathrm{ncfw}\overline{\mathrm{L}} \subseteq \mathrm{ancfw}\overline{\mathrm{L}}$ is a direct consequence of the definition. Now we show that there exists an ancfwL which ncfwg is unable to generate. According to Lemma 5, the number of types of blocks of ancfwL is not always finite, but according to Lemma 3, one of ncfwL is always finite. It must be concluded that if a language whose number of types of blocks is infinite can be generated by ncfwg, its set of rules must be infinite, but it contradicts with the definition of the grammar. Q.E.D.

COROLLARY 7. $\mathrm{nlw}\overline{\mathrm{L}} \subsetneq \mathrm{anlw}\overline{\mathrm{L}}$.

LEMMA 8. *An* ncswg *of Fig.* 3 *generates exactly all connected nonseparable webs.*

*Proof. Only nonseparable webs.* As only nonterminal symbol $B$ is converted into the terminal symbol $a$ by the application of rule (16), vertices labeled with nonterminal symbols $A$, $C$, $D$ must be rewritten to $B$-vertex. If rule (4) is applied to the web derived from the initial web by the application of rules (2) and (3), a circuit is generated. Unless rule (4) is applied to the host web, rule (5) is not applicable to the web. And also if rule (8) is applied to the web over symbol $B$, rules (10)–(13) or (14) and (15) must always be applied to the web derived from that web by application of rules (9) and (10). To terminate the derivation, if rule (4) is used, rule (5) or (6) must also be used; and if rule (8) is used, then rules (10)–(13) or (14) and (15) must be applied. For the cases described above, the derivation is able to terminate.

*All nonseparable webs.* A trivial nonseparable web is generated by the application of rule (1). Whiteney has shown that it is possible to build up any connected nonseparable

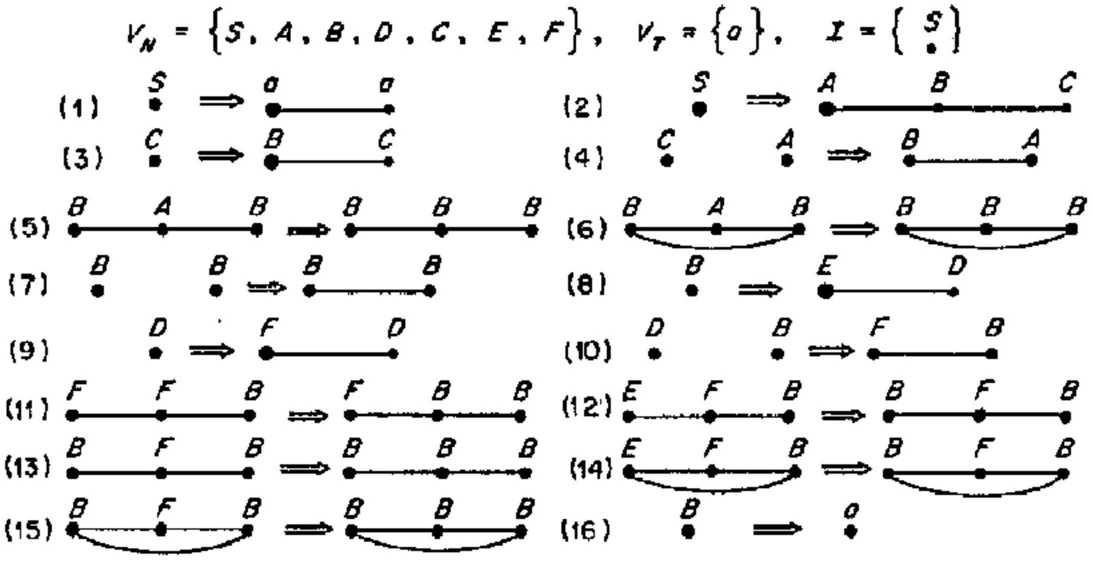


FIG. 3. This ncswg generates a set of all nonseparable webs. The image of the left member of rules except for one-point web is written at the place corresponding to the rewritten vertex. For example, $C$-vertex and $A$-vertex correspond to $B$-vertex and $A$-vertex, respectively in rule (4).

webs containing more than two arcs by starting with a circuit and adding to it arcs or chains of arcs. After the recursive application of rule (3), by the application of rules (4) and (5) or (6) in order, arbitrary circuits can be constructed. The addition of an arc is performed by applying rule (7), and the addition of a chain of any length can be obtained by first application of rule (8) and the repetitive application of rule (9) and last application of rules (10)–(13) or (14) and (15). As described before, if any derivation from the initial web terminates, the above procedure can be always done. Q.E.D.

THEOREM 9. ncfw$\overline{\mathrm{L}} \subsetneq$ ncsw$\overline{\mathrm{L}}$.

*Proof.* This theorem follows directly from Lemma 3 and 8.

COROLLARY 10. anlw$\overline{\mathrm{L}} \subsetneq$ ancfw$\overline{\mathrm{L}}$.

*Proof.* Let $SK$-graph be a separable graph whose blocks consist of only complete

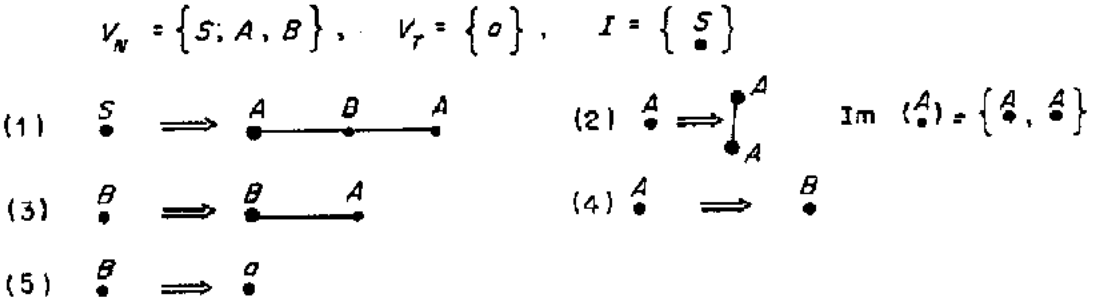


FIG. 4. This ancfwg generates a set of all $SK$-webs.

blocks except as one point graph. It is clear that ancfwg of Fig. 4 generates all $SK$-graphs. Here, note that a vertex labeled with a symbol (for example, $B$-vertex in the figure) belonging to $V_N$ can be a cutpoint of an arbitrary number of complete blocks. Since any webs derived by anlwg can have at most one vertex to be rewritten, it is clear that any anlwg cannot generate a set of all $SK$-graphs, but generate only a subset of them. Q.E.D.

COROLLARY 11. *There exists* anlwL *which is not a member of* ncfw$\overline{\text{L}}$.

*Proof.* Let $L_G$ be a set of all complete webs $K_P$, where $P$ is a number of vertices. It is evident that an anlwg of Fig. 5 generates any members of $L_G$, that is, a new vertex added to a web derived from an initial web should be adjacent to every vertex of the web.

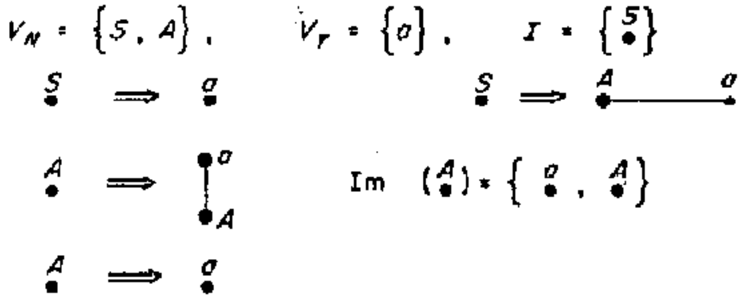


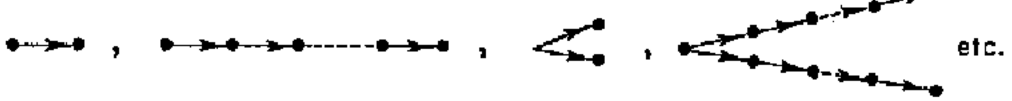
FIG. 5. This anlwg generates a set of all complete webs $K_G$.

Now we show that $L_G$ cannot be generated by any ncfwg. From Lemma 3, every vertex rewritten by the application of rule (except that the right member $\beta$ is a one-point web) becomes a cutpoint of more than or equal to two blocks. A ncfwg which should generate $L_G$, therefore, must have only a set of intial webs consisting of only complete webs and a set of rules of which types are $S' \Rightarrow K_n$ ($n \geqslant 1$ and their vertices must be labeled with terminal symbol). Then the set of initial webs $I$ or the set of rules $R$ are not finite. Q.E.D.

COROLLARY 12. anlwL $\underset{\supset}{\subsetneqq}$ ncfw$\overline{\text{L}}$.

THEOREM 13. *A set of webs $L_G$ of Fig.* 6 *is not a* ncfwL.

*Proof.* Assume that ncfwg $G$ generates $L_G$ of Fig. 6. From Lemma 3, blocks of $L_G$ consist of only web $\beta$ of a rule of the given web grammar $G$. As kinds of blocks of the given language are



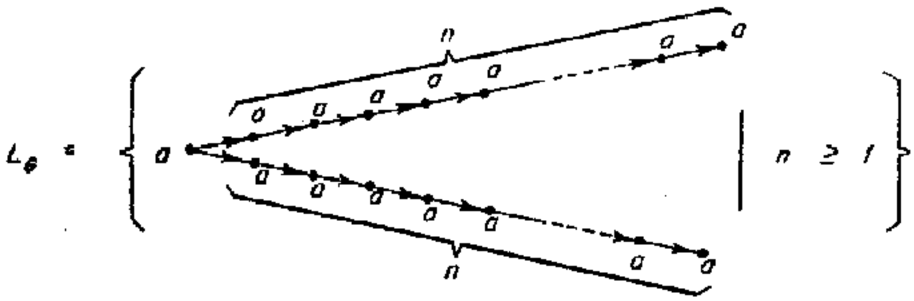


FIG. 6. This set of webs is not a ncfwL. The web language corresponding to $\{a^n b^n \mid n \geqslant 1\}$ in the 1-dimensional grammar is ncswL.

the right member of any rule of the grammar must be isomorphic to one of the above webs. For example, suppose that the right member of the rules consists of only minimal block, i.e., •——• From the consideration of the combination of rewritten vertex and its image, one of the possible grammars expected to generate $L_G$ must be the grammar of Fig. 6a. It is easy to see that this grammar cannot control the

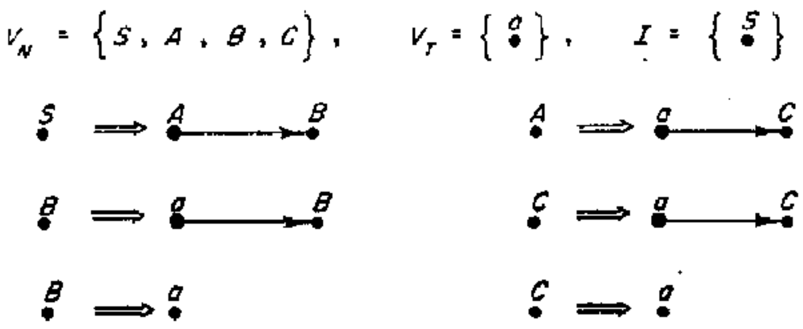


FIG. 6a. One of the possible ncfwg which are considered to generate a set of all webs shown in Fig. 6.

number of vertices on two different branches; that is, this grammar generates not only $L_G$ but also a set of webs, branches of which have different numbers of vertices. For other cases, similar discussions can be considered. Q.E.D.

Comparing ncfwg with a context-free string grammar, we can see that the latter is strong in generation because any sequences can be inserted into strings. Thus the web language which corresponds to cfL$\{a^n b^n \mid n \geqslant 1\}$ is not ncfwL. The language of Theorem 13 can be constructed by ncswg, but we abbreviate the details here.

The degree of a vertex $P$ in a graph (web) $G$, denoted deg $P$, is the number of arcs incident with $P$.

THEOREM 14. *An* ncswg *cannot generate a set of all separable webs.*

*Proof.* Assume that the number of vertices on a block in a web derived from the initial web of a given ncswg is $n$, and that $n$ is sufficiently large in comparison with the cardinality of the vocabulary, denoted as $|V|$. Consider the web of all vertices which are cutpoints of other blocks. To obtain such a web, every vertex on that block should be rewritten to a cutpoint of a block $B_i$ by the application of a rule $A_j \Rightarrow C_i B_i$, where $A_j$ is a vertex which is to be rewritten and $C_i$ is the image of $A_j$ and is a cutpoint of block $B_i$. Now consider the case that one vertex of the web is a cutpoint of $m$ blocks, $B_{1i}$, $B_{2i}$,..., $B_{|V|i}$,..., $B_{mi}$, where $m$ is sufficiently large in comparison with $|V|$. Then there exist many points labeled with similar symbols in the blocks $B_{ji}$ and $B_{ki}$ ($k \neq j$) without limit, because $|V|$ must be finite. As arbitrary blocks are able to be constructed by that grammar, the grammar should have a rule which adds any vertices, arcs and chain of arcs to the block. Since these rules are, however, applicable to vertices of both blocks $B_{ji}$ and $B_{ki}$ if they are applied to the vertex of $B_{ji}$ and $B_{ki}$, it happens that two different blocks $B_{ji}$ and $B_{ki}$ are rewritten as one block; that is, there exists a case where one vertex of a given block cannot be a cutpoint of an arbitrary number of other blocks (see Fig. 7a). Also there exist many vertices labeled with similar symbols in the given web without any limitation, because $n$ is sufficently large. Let such two vertices be $X$, $Y$ and let $X$ be a cutpoint of any number of blocks $B_1, B_2, \ldots, B_m, \ldots$, and let $Y$ be a cutpoint of blocks $B_1', B_2', \ldots, B_m', \ldots$. Then, there exist many points labeled with similar symbols on the different blocks without any limitation. In order to construct all possible blocks, it is necessary for the grammar to have a rule which adds an arc between a vertex $X(Y)$ and a vertex $P_k(P_k')$ on the block $B_k(B_k')$. A rule which is applicable to vertices $X$ and $P_k$ is also applicable to vertices $Y$ and $P_k$ and to vertices $X$, $P_K'$ (see Fig. 7b). Then it is easy to see that two different blocks—the given block and block $B_k$—are rewritten to one block.

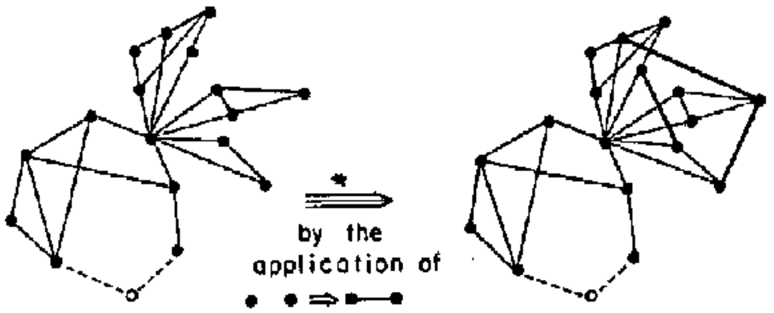


FIG. 7a. The process that several blocks are converted into one block. The line written in blod strokes indicates the arc added to two different blocks.

In general, if the same idea is applied to the vertices on the given block, we can conclude that it happens that nonseparable webs are generated by ncswg as there exists a case where the generated webs themselves become a block.

To generate only separable webs, it is necessary to attach a marked vertex to each block; this marked vertex must be adjacent to all vertices on the same block (see Fig. 7c). If the addition of arcs is done by referring to the marked vertex, no arcs are

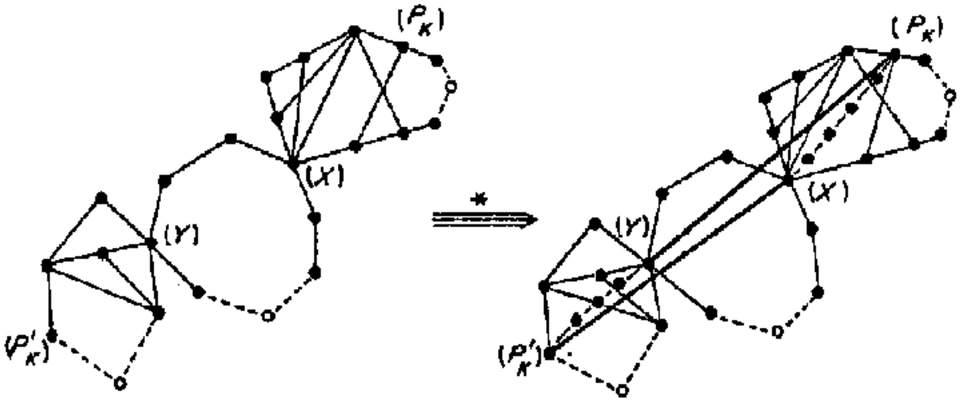


FIG. 7b. The case where new blocks and original block are converted into one block. The symbols $P_K$ and $P_K{}'$ are identical and so are the symbols $X$ and $Y$.

added between two vertices on the different blocks, but this marked vertex is a special vertex, that is, it is adjacent to all vertices on the same block. Since any ncswg cannot erase any arcs, this marked vertex cannot be converted into a vertex of any possible blocks. All separable webs, therefore, cannot be generated by this method. Q.E.D.

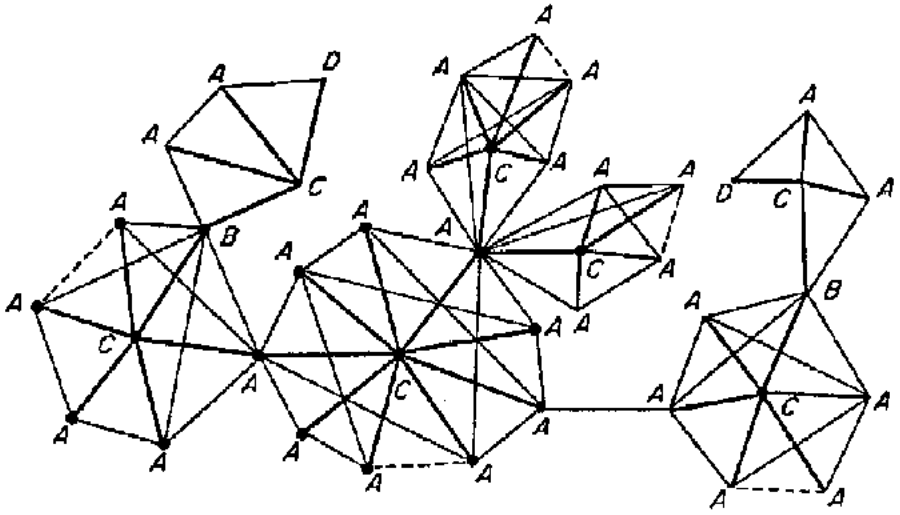


FIG. 7c. An example of a web which has a marked vertex on each block except one-arc block. A marked vertex is shown as a vertex labeled with $C$.

THEOREM 15. ncsw$\overline{\mathrm{L}}$ $\subsetneq$ I-ncsw$\overline{\mathrm{L}}$.

*Proof.* Let $G_1 = (V_1, I_1, R_1)$ be a ncswg which generates ncswL $L_{G_1}$, and $V_1 = V_{T_1} \cup V_{N_1}$, $V_{T_1} \cap V_{N_1} = \varnothing$, $V_{T_1} \neq \varnothing$, $V_{N_1} \neq \varnothing$. We construct the ncswg $G_2$ which indirectly generates $L_{G_1}$. Let $G_2 = (V_2, I_2, R_2)$.

(1) A vocabulary of $G_2$ can be obtained from one of $G_1$ by the following procedure: New vocabulary $V_S(V_S \cap V_1 = \varnothing)$ is added to ones of $G_1$. And $V_{T_2} = V_{T_1} \cup V_S$, $V_{N_2} = V_{N_1}$, $V_2 = V_{T_2} \cup V_{N_2}$.

(2) For the initial webs, $I_1 = I_2$,

(3) A set of rules can be obtained from one of $G_1$ by the following procedure: For any rules $(\alpha, C, \beta, E)$ in $R_1$, a web $\alpha'$ over $V_S$ is added to both $\alpha$ and $\beta$ so that the rule is normal (this procedure can always be carried out).

If we specify the terminal vocaburary $V_{T_1}$ as $V_L$, it is easy to see that the web over $V_{T_1}$ generated by $G_2$ is isomorphic to the web of $L_{G_1}$. (This procedure is the immediate consequence of Montanari.)

Now, we show that there exists an example that is never able to be generated by any ncswg. Let us consider the set of all separable webs. As shown in Theorem 14, every ncswg cannot generate only such webs. From Lemma 8, ncswg, however, can generate a set of all nonseparable webs. And as shown in the Fig. 7c of Theorem 14, ncswg which attaches a marked vertex to each block can generate only separable webs (note that the concept of indirect generation is necessary). The proof of this theorem, therefore, is clear from these facts. Q.E.D.

If we use nmcswg to generate a set of all separable webs, it seems possible that a marked vertex is converted into a vertex of arbitrary block by erasing arcs from the marked vertex. We show that this method gives us a construction of all separable webs by nmcswg.

First, another form of Whitney's theorems describing the method of the construction of all nonseparable graphs is shown, and a nmcswg generating all separable webs is also shown.

A nullity of a connected graph having $V$ vertices and $E$ arcs is given by the equation

$$N = E - V + 1.$$

Consider graphs $G_1$ and $G_2$ having disjoint vertex sets $N_1$ and $N_2$ and arc sets $A_1$ and $A_2$, respectively. Their union $G = G_1 \cup G_2$ has $N = N_1 \cup N_2$ and $A = A_1 \cup A_2$. Their join is denoted $G_1 + G_2$ and consists of $G_1 \cup G_2$ and all arcs joining $N_1$ with $N_2$.

In order to give another form of Whitney's results, the "wheel" invented by W. T. Tutte is needed. For $n \geqslant 4$, the wheel $W_n$ is defined to be the graph $K_1 + C_{n-1}$, and this graph is clearly nonseparable. To generate all separable webs, a marked vertex attached to each block is necessary. Let us consider the center of the wheel, which is a vertex $K_1$ and is denoted hereafter as $C$-vertex as shown in Fig. 7c, as a marked vertex.

Now the serise of theorems on the connected nonseparable graphs must be brought to mind:

THEOREM I. *A nonseparable graph $G$ containing at least two arcs contains no loops and is of nullity $> 0$. Eeach vertex is on at least two arcs.*

THEOREM II. *A nonseparable graph of nullity 1 is a circuit $C_n$ ($n \geqslant 3$).*

THEOREM III. *If $G$ is a nonseparable graph of nullity $N > 1$, we can remove an arc or a chain of arcs from $G$, leaving a nonseparable graph $G'$ of nullity $N - 1$.*

THEOREM IV. *Any nonseparable graph containing at least two arcs can be build up by taking first a circuit, then adding successively arcs or a chain of arcs, so that at any stage of the construction a nonseparable graph can be obtained.*

Based on these theorems, let us consider a method that a circuit is constructed from a wheel—a wheel $W_n$ containing $n$ ($n \geqslant 4$) vertices. Its nullity $N$ is equal to $n - 1$. From Theorem II, it is possible to construct a circuit $C_n$ by removing $n - 2$ arcs from the wheel $W_n$. In this case, a number of arcs which can be removed from a $C$-vertex (the center of the wheel) is at most $n - 3$, because if $n - 2$ arcs are removed, the degree of the center of the wheel is equal to 1 and this is contradictory to Theorem I. Consequently, just one arc must be removed from other arcs. When $n - 3$ arcs are removed from the $C$-vertex, there exist just two vertices $P, Q$ having deg 3 in the resulting graph $G$. Only when the two vertices $P, Q$ are adjacent to each other can a circuit $C_n$ be constructed by removing an arc $(P, Q)$ from the graph $G$. Unless the two vertices $P, Q$ are adjacent to each other, the circuit cannot be constructed; but it is recognized that there exists at least one process constructing a circuit from the wheel by erasing arcs.

LEMMA 16. *Any nonseparable web having $n$ ($n \geqslant 4$) vertices can be obtained by a sequence of operations of the following types:*

(I) *The construction of a wheel $W_n$ ($n \geqslant 4$).*

(II) *The construction of $G$-I by adding arcs or a chain of arcs to $W_n$; in this case, all new vertices are adjacent to $C$-vertex.*

(III) *The construction of $G$-II by removing some arcs from $W_n$, $G$-I if the degree of $C$-vertex is more than $K$ ($K \geqslant 2$).*

(IV) *The construction of $G$-III by removing just one arc $(P, Q)$ from $W_n$, $G$-I or $G$-II if three vertices $P, Q$ and $C$-vertex are adjacent to each other.*

*The obtained graphs $W_n$, $G$-I, $G$-II and $G$-III are nonseparable and all such graphs can be obtained at any stage of the operations mentioned above.*

*Proof.* In the proof of this lemma, the following lemma is used.

LEMMA 17. *An* nmcswg *of Fig.* 8 indirectly *generates a set of all separable webs.*

The proof of this lemma is trival from Montanari's Theorem 2. In a web derived by this grammar, if a $C$-vertex is removed by the indirect generation, the web is a nonseparable web having no marked vertex. Similar to this case, if a $C$-vertex is removed from $W_n$, then we can obtain a circuit $C_n$ and by adding arcs or chains to

$C_n$, any nonseparable web $G'$ can be obtained. The graph $G$-I $(W_n)$ can be considered as a graph which is, therefore, constructed by adding $C$-vertex to $G'$ $(C_n)$. In this case, the addition of $C$-vertex can be done by the following three operations:

(a) a length of circuit or a chain of arcs is extended by 1,

(b) a new chain of arcs whose length is equal to 2 is added,

(c) some arcs between a new vertex ($C$-vertex) and all other vertices are added after the operation (a) or (b).

When the operation (b) is applied to the two adjacent vertices $P$ and $Q$, a nonseparable web can be obtained even if an arc $(P, Q)$ is removed (see Fig. 8a). This corresponds to the operation (IV), and operation (III) corresponds to the reverse operation of (c). Especially, if deg $C = 2$ and the application of (IV) is possible, circuits can be obtained.

Q.E.D.

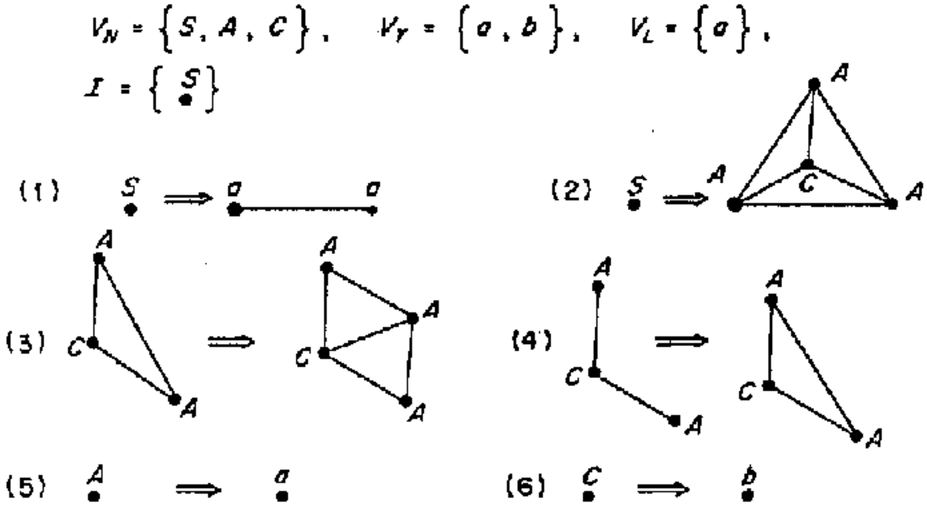

$V_N = \{S, A, C\}$, $V_T = \{a, b\}$, $V_L = \{a\}$,
$I = \{\overset{S}{\bullet}\}$
(1)
S
a
a
(2)
S
A
A
C
A
(3)
A
C
A
A
C
A
A
(4)
A
C
A
A
C
A
(5)
A
a
(6)
C
b

FIG. 8. This nmcswg can indirectly generate a set of all separable webs. After $m$ ($m \geqslant 0$) applications of rule (3), a wheel $W_{m+4}$ can be derived.

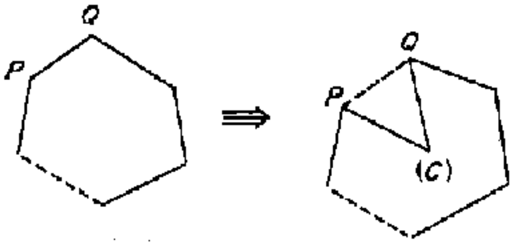


FIG. 8a. When a new chain $\overset{P}{\bullet}\!-\!\overset{(C)}{\bullet}\!-\!\overset{Q}{\bullet}$ is added, the arc $(P, Q)$ can be erased.

THEOREM 18. *An* nmcswg *of Fg. 9 generates a set of all separable webs.*

*Proof.* It is clear that all combinations of $K_2$, $K_3$ and $W_4$ by applying a set of rules

(1)–(9). As rule (7), (8) or (9) must be applied to $B$-vertex and no arcs are ever added to two vertices on the different blocks, the separability of the derived web is evident.

Now, we show that any blocks having $n$ ($n \geqslant 4$) vertices can be derived from $W_4$ by applying rules. A set of rules (10), (11) corresponds to the operations (I) and (II) shown in Lemma 16. If rule (12) is applied, then the operations (I) and (II) can never be performed. By applying a set of rules (13)–(18), the operation corresponding to the operation (III) can be performed, and a rule (19) corresponds to the operation (IV). Consequently, all blocks can be generated by the given nmcswg. Q.E.D.

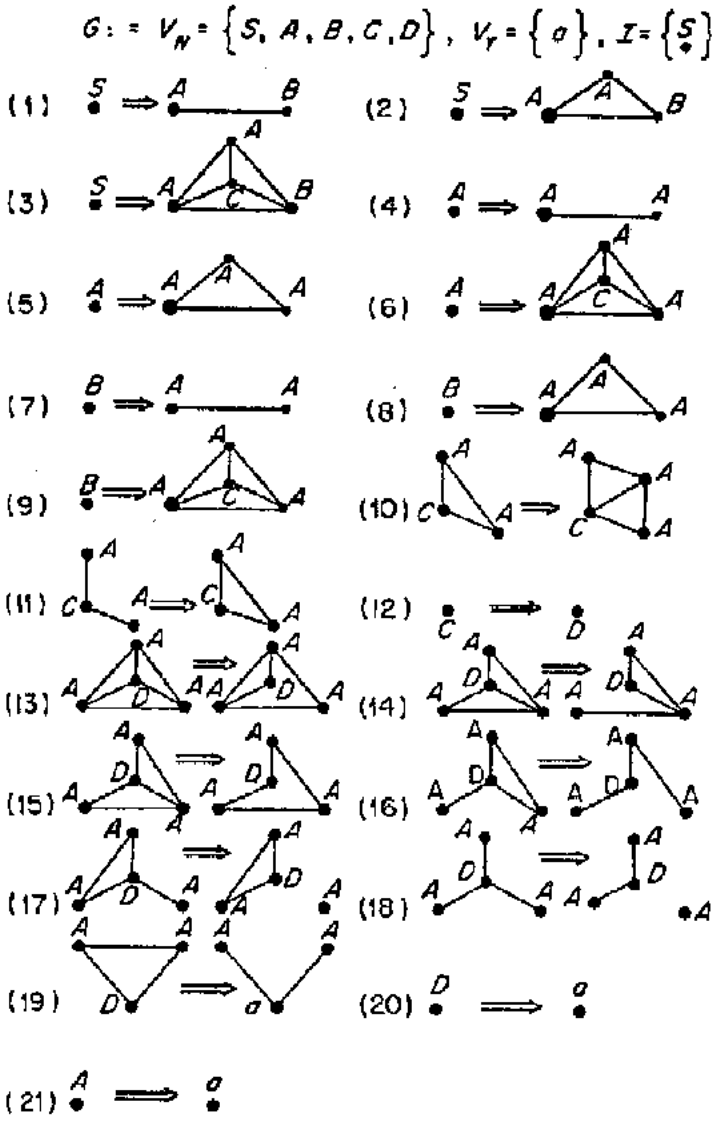


FIG. 9. This nmcswg generates a set of all separable webs.

THEOREM 19. ncsw$\overline{\mathrm{L}} \subsetneq$ nmcsw$\overline{\mathrm{L}}$.

*Proof.* From the definition of web grammars, the relation ncsw$\overline{\mathrm{L}} \subseteq$ nmcsw$\overline{\mathrm{L}}$ is trivial. Referring to Theorems 14 and 18, the proof of this theorem is evident. Q.E.D.

THEOREM 20. *An* ancfwg *cannot generate a set of all circuits.*

*Proof.* A degree of each vertex on a web never decreases after an application of nonnormal rules. Further, a degree of all vertices on $\beta$ of any rules considered to generate circuits is less than or equal to 2, since all vertices of circuits are on just two arcs. Any one of circuits having $n$ ($n \geqslant 3$) vertices must be derived from one web $W$ having $k$ ($k \leqslant n - 1$) vertices whose degree is less than or equal to 2 (note that, considering a vertex of zero degree, $W$ is a one-point web. Let us denote it as $W_0$). Let a vertex, to be rewritten by a rule of a given ancfwg, be $S$ and a collection of vertices adjacent to $S$ be $S(a)$. Then, it is clear that the following equation follows:

$$0 \leqslant | S(a)| \leqslant 2.$$

Let us consider the following three cases:

(1) If $| S(a)| = 1$, then $S(a)$ can be denoted as $S(a) = \{S_b\}$. A vertex $S_b$ is adjacent to at most one vertex except for $S$, since $1 \leqslant \deg S_b \leqslant 2$. A web $W$ is, therefore, one-arc web or a simple path whose length is finite. Now let a collection of vertices in the web $\beta$ specified to be images of the vertex $S$ be $Q(\beta)$. A degree of any vertex $Q_\beta$ in $Q(\beta)$ in the web $\beta$ is 1 or 0 (otherwise, there exists a vertex having deg 3 in the derived web). Here, let $Q(\beta)$ be a set of vertices $\{Q_{\beta 1}, Q_{\beta 2}, ..., Q_{\beta m}\}$ ($m \geqslant 1$).

(i) In case $m \geqslant 2$, the type of this rule is nonnormal, and on a web derived from the web by applying this rule, the degree of $S_b$ is more than or equal to 2. Then the value of $m$ must be equal to 2, and the degree of $S_b$ on the web $W$ must be equal to 1, that is, $W$ must be one-arc web. Here, the following three possibilities are worth considering:

(a) If $\deg Q_{\beta 1} = 0$ and $\deg Q_{\beta 2} = 1$, then there must exist a vertex $P_0$ or a simple path consisting of $Q_{\beta 2}$, $P_1$, $P_2$, ..., $P_i$ ($i \geqslant 2$) and $P_1$ (or $P_i$) is adjacent to $Q_{\beta 2}$. In this case, a web derived by applying this rule is a simple path whose length is more than or equal to 3 (see Fig. 10a).

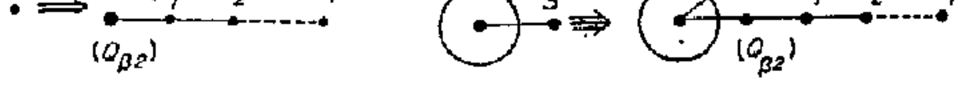


FIG. 10a. $| S(a)| = 1$, $\deg Q_{\beta 1} = 0$ and $\deg Q_{\beta 2} = 2$. In this case, only an open path can be derived; circuits cannot be derived.

(b) If $\deg Q_{\beta 1} = \deg Q_{\beta 2} = 1$, then a web $\beta$ is a one-arc web or a simple path, beginning with $Q_{\beta 1}$ ($Q_{\beta 2}$) and ending with $Q_{\beta 2}$ ($Q_{\beta 1}$), of which length is $j - 2$ ($j \geqslant 4$). Then a $K_3$ or a circuit having $j$ length is derived (see Fig. 10b).

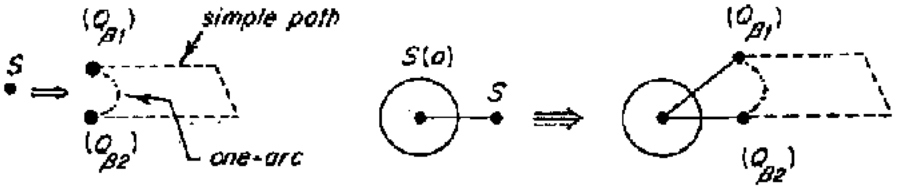


FIG. 10b. $|S(a)| = 1$, $\deg Q_{\beta 1} = \deg Q_{\beta 2} = 1$. In this case, circuits can be derived, but the left member of the rules must describe all the possibilities satisfying the above conditions.

(c) If $\deg Q_{\beta 1} = \deg Q_{\beta 2} = 0$, a web $\beta$ consists of only these two vertices (otherwise, any web derived by applying this rule is disconnected). In this case, a derived web is a simple path having three vertices.

(ii) In case $m = 1$, the type of this rule is normal. $Q(\beta)$ can be denoted as $Q_{\beta 1}$. Then it is clear that a degree of $Q_{\beta 1}$ is equal to 1 (if a degree of $Q_{\beta 1}$ is 0, it is meaningless), and that a web $\beta$ must be one-arc web or a simple path. A web derived by applying this rule is a simple path having more than or equal to 3.

(2) If $|S(a)| = 2$, then $|Q(\beta)| \leqslant 2$ by a reason similar to case (1). Assume that there exists at least one vertex $Q_{\beta 1}$ in $Q(\beta)$ which satisfies $\deg Q_{\beta 1} \geqslant 1$; any webs derived by applying this rule have at least one vertex whose degree is more than 2 whether this rule is normal or not. Consequently, for all vertices in $Q(\beta)$, their degree must be equal to 0. Here, if this rule is normal it is meaningless, since the number of vertices of a derived web does not increase. This rule can be considered as a non-normal rule, and a web $\beta$ consists of only these two vertices. A circuit having 4 vertices can be derived by the application of this rule, but any other circuits having $m$ ($m \geqslant 5$) vertices cannot be derived by applying any rules since the degree of all these vertices is equal to 2 and all arcs cannot be erased by ancfwg (see Fig. 10c).

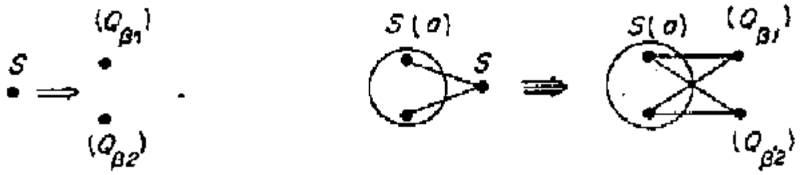


FIG. 10c. $|S(a)| = 2$, $\deg Q_{\beta 1} = \deg Q_{\beta 2} = 0$. In this case, a circuit $C_4$ can be derived.

Except for the case (b), simple paths with any length can be obtained, but these simple paths having more than four vertices are never converted into a circuit, because, from the above discussions, simple paths which can be converted into a circuit are only 2-length simple paths.

(3) If $|S(a)| = 0$, for each rule $\alpha \Rightarrow \beta$, $W$ is a one-point web $W_0$ and $\beta$ is a circuit having $n$ ($n \geqslant 3$) vertices.

Consequently, for the case (b) or (3), the circuits with any length can be derived. But the web $\beta$ of the rules must describe all patterns of simple paths or circuits; that is, the set of rules $R$ or the set of intial webs $I$ is not finite. Q.E.D.

This theorem is intuitively evident from the definition of ancfwg and the embedding $E$. Nevertheless, we give a precise proof for this theorem.

Corollary 21. *Any* ancfwg *cannot generate a set of all nonseparable webs.*

*Proof.* A circuit is a most simple nonseparable web. From Theorem 18, the proof is trivial. Q.E.D.

In conclusion of this section, we summarize the results mentioned above.

$$\mathrm{nlw}\overline{\mathrm{L}} \subsetneq \mathrm{ncfw}\overline{\mathrm{L}} \subsetneq \mathrm{ncsw}\overline{\mathrm{L}} \subsetneq \mathrm{nmcsw}\overline{\mathrm{L}},$$

$$\mathrm{nlw}\overline{\mathrm{L}} \subsetneq \mathrm{anlw}\overline{\mathrm{L}} \subsetneq \mathrm{ancfw}\overline{\mathrm{L}},$$

$$\mathrm{ncsw}\overline{\mathrm{L}} \subsetneq \text{I-}\mathrm{ncsw}\overline{\mathrm{L}}, \qquad \mathrm{ncfw}\overline{\mathrm{L}} \overset{\subset}{\underset{\supset}{\neq}} \mathrm{anlw}\overline{\mathrm{L}}.$$

From this section, some typical examples of web grammars which generate a set of webs whose structures are specified are given. For a simple example, consider a grammar that generates exactly all bipartite graphs. A bipartite graph is a graph whose vertex set $V$ can be partitioned into two subsets $V_1$ and $V_2$ such that every arc of $G$ joins $V_1$ with $V_2$. Consequently, it is clear that a graph is bipartite if and only if all its its cycles, if they exist, are even. If a grammar can control a length of any cycles, it generates all bipartite graphs. It is easy to see that the ncswg of Fig. 11 generates a set of all connected bipartite graphs.

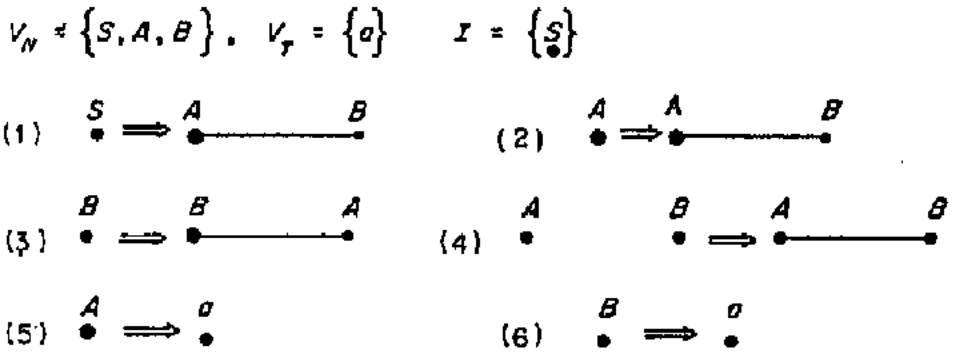

$V_N = \{S, A, B\}$, $V_T = \{a\}$ $I = \{\overset{S}{\bullet}\}$
(1) S ⟹ A B
(2) A ⟹ A B
(3) B ⟹ B A
(4) A B ⟹ A B
(5) A ⟹ a
(6) B ⟹ a

Fig. 11. This ncswg generates a set of all connected bipartite webs.

## A Grammar which Generates Eulerian Graphs

In this section, we show some examples of web grammars generating Eulerian graphs. It is well known that a graph is Eulerian graph if and only if the degree of all its vertices is even. By extending the method shown by Montanari, a nmcswg can describe some types of Eulerian graphs.

THEOREM 22. *A* nmcswg *of Fig.* 12 *generates the set of all connected nonseparable eulerian webs.*

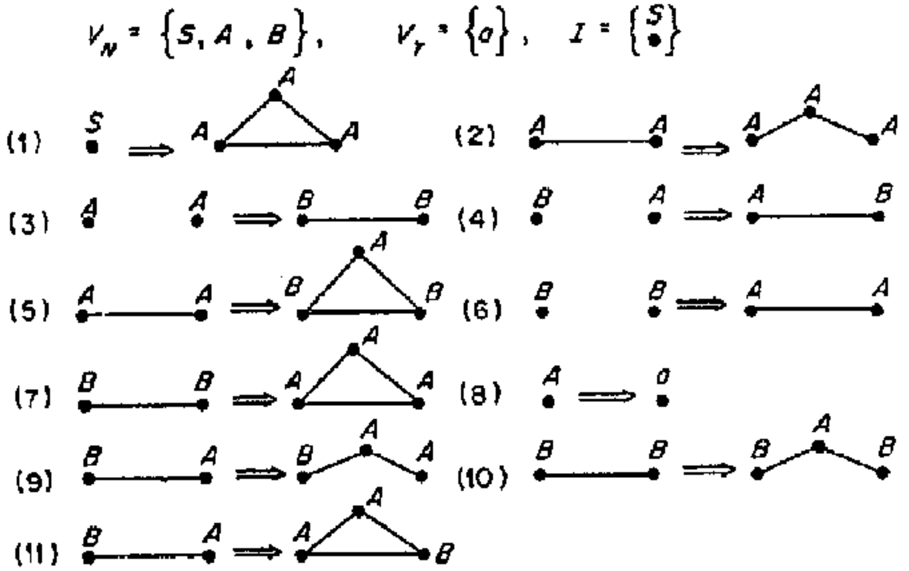


FIG. 12. This nmcswg generates a set of all nonseparsble eulerian webs. *A*-vertex has even degree and *B*-vertex has odd degree.

*Proof. Only a set of nonseparable eulerian webs.* To an initial web, only rule (1) can be applied, and its consequence is a minimal eulerian graph, except for webs with multiple paths and loops, which is also nonseparable. As the application of any rules cannot erase any possible paths, any webs derived from the above nonseparable webs are also nonseparable. On the step of any derivations, a degree of vertices labeled with *A* symbol (*A*-vertex) is even and a degree of *B*-vertex is odd and a number of *B*-vertices is even and since a symbol that can be rewritten to a terminal symbol *a* is only *A*, the degree of each vertex of the web is even after the derivation has terminated. Then only nonseparable eulerian webs can be generated.

*All nonseparable eulerian webs.* By the iterative application of rule (2) to a web derived from an initial one by applying rule (1), arbitrary circuits can be derived. In order to obtain all eulerian webs, it is necessary to prepare rules which add an arc or a chain of arcs to the two vertices labeled with the above symbols *A*, *B*. In case where two vertices are not adjacent each other, an arc is added by applying rules (3), (4) or (6), and a chain of arcs is added by appropriate applications of rules (5), (7) or (11) and (2), (7) or (10). Then the given grammar has a sufficient set of rules that corresponds to the above procedure to obtain arbitrary eulerian webs. Q.E.D.

THEOREM 23. *An* nmcswg *G of Fig.* 13 *indirectly generates the set of all separable eulerian webs.*

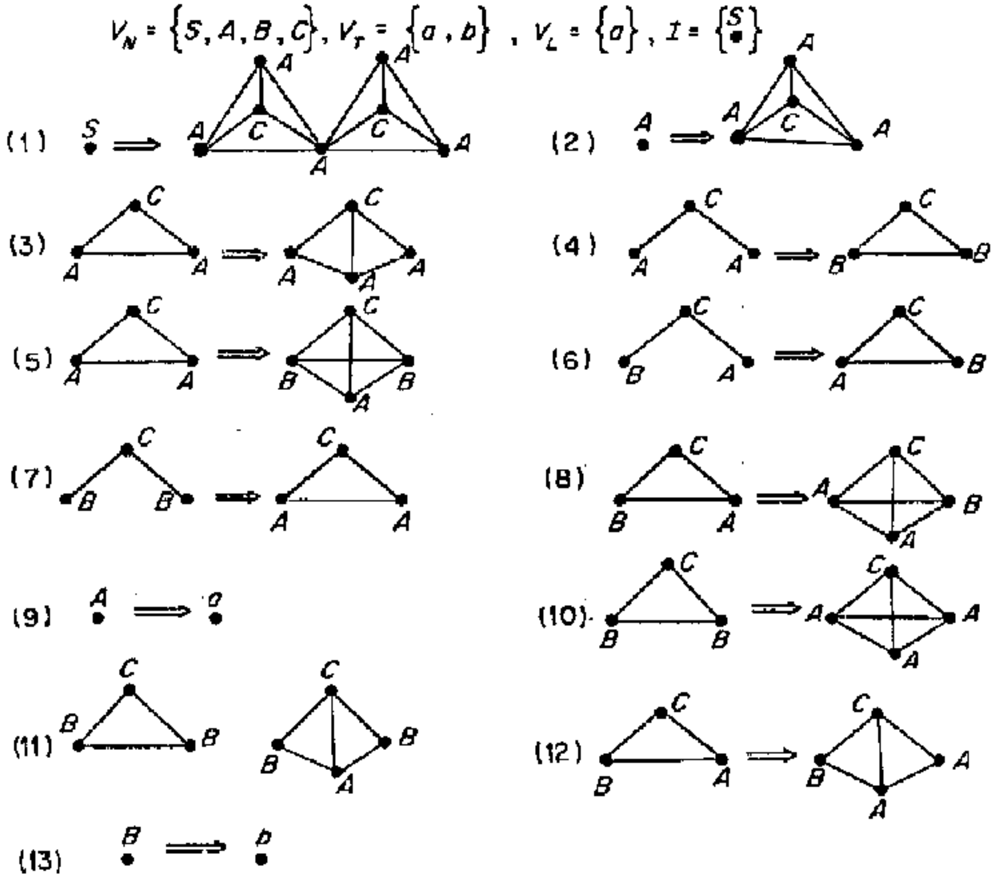


FIG. 13. This nmcswg indirectly generates a set of all separable eulerian webs.

*Proof.* It is clear that each block derived by the given grammar is a eulerian block from the previous Theorem 22. If only separable webs can be derived by the given grammar, the proof will be done. Since a web derived from the initial web by applying rule (1) is separable, and no arcs and chain of arcs can be added to two vertices of different blokes, that is, it is only the case where a vertex $C$ exists which is adjacent to both vertices that an arc or a chain of arcs is added to the two vertices. At any steps of the derivation, the obtained web is always separable. To prove that all separable eulerian webs can be generated from the intial webs, it is sufficient to show that all combinations of any blocks can be realized by the given grammar. Let the number of blocks be $n$. If $n$ equals 2, the result is evident by applying rule (1). Now assume that all combinations of $n$ blocks can be realized, and hereafter it is shown that all combinations of $n + 1$ blocks is obtained by the grammar. Given a eulerian web $B_0$ of its number of blocks is $n + 1$. If one block $B_1$ of the given web is separated at its cutpoint $P$, a remaining web $B_2$ has $n$ blocks and can be generated by the grammar. On the derivation of the web $B_2$, instead of applying rule (9) to the vertex $P$, a block isomorphic to the block $B_1$ can be obtained by applying rule (2) to a vertex $P$ and rules (3)–(8) in sucession. And then, by applying rule (9) to $P$ and each vertex of $B_1$, the web obtained is clearly isomorphic to the given web which has $n + 1$ blocks. Q.E.D.

THEOREM 24. *An* nmcswg $G$ *of Fg.* 14 *indirectly generates a set of all eulerian webs.*

*Proof.* This theorem follows immediately from the previous Theorems 22 and 23. Q.E.D.

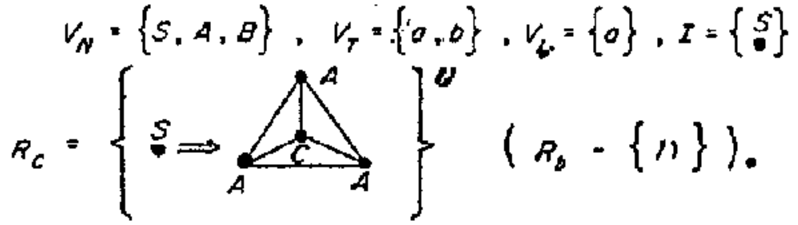

$V_N = \{S, A, B\}$, $V_T = \{a, b\}$, $V_L = \{a\}$, $I = \{S\}$
$R_C =$
S
A
A
A
C
U
$(R_b - \{n\})$.

FIG. 14. This nmcswg indirectly generates a set of all eulerian webs.

## A GRAMMAR WHICH GENERATES LINE GRAPHS

The concept of the line graphs associated with given graphs is natural, and has been studied by many mathematicians.

Let $S$ be a set and $F = \{S_1, S_2, ..., S_p\}$ a family of distinct nonempty subsets of $S$ whose union is $S$. The intersection graph of $F$ is denoted $\Omega(F)$ and defined by $F = V(\Omega(F))$ with $S_i$ and $S_j$ adjacent whenever $i \neq j$ and $S_i \cap S_j = \varnothing$. If the blocks of graph $G$ is taken as the family $F$ of sets, then the intersection graph $\Omega(F)$ is the block graph of $G$, denoted by $B(G)$. Now consider the set $X$ of arcs of a graph $G$ as a family of 2-vertices subsets of $V(G)$. The line graph of $G$, denoted $L(G)$, is the intersection graph $\Omega(X)$, that is, the vertices of $L(G)$ are the arcs of $G$, with two vertices of $L(G)$ adjacent whenever the corresponding arcs of $G$ are.

For a connected separable graph $G$ with blocks $\{B_i\}$ and cutpoint $\{C_j\}$, the block-cutpoint graph of $G$, denoted by bc($G$), is defined as the graph having vertices set $\{B_i\} \cup \{C_j\}$, with the vertices adjacent if one corresponds to a block $B_i$ and the other to a cutpoint $C_j$, and $C_j$ is in $B_i$.

Hereafter a collection of all trees is denoted as $T$.

THEOREM 25. *An* ncfwg $G$ *of Fig.* 15 *generates* $L_G = \{\text{bc}(G) \mid G = L(T)\}$.

*Proof. Only the block-cutpoint graph bc*($G$). It is known that a graph is the line graph of a tree if and only if it is a connected block graph in which each cutpoint lies in exactly two blocks and that graph is the block graph of some graphs if and only if every block of such graphs is complete. Consequently, $C$-vertex corresponding to a

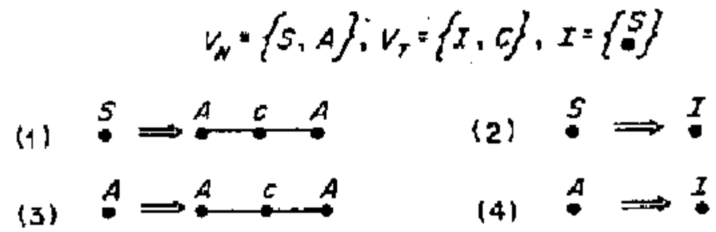


FIG. 15. This ncfwg generates a set of all block-curpoint graphs of line graphs of arbitrary trees.

cutpoint should be adjacent to exactly two $I$-vertices corresponding to a block and its bc($G$) ought to be a tree. Over the grammar of Fig. 15, it is easily seen that $C$-vertex is adjacent to only two $A$-vertices and that only all trees can be generated from Corollary 5. Since any endpoints of the generated trees are labeled by $I$, and $I$-vertex and $C$-vertex alternately lie on any branches of trees, webs generated by the grammar clearly satisfy the condition of Theorem 3(a) of Montanari.

*All block-cutpoint graphs* bc($G$). It is possible to generate all trees of its two $A$-vertices are adjacent to only a common $C$-vertex and of which $A$-vertex has an arbitrary degree. Since any symbol, except $S$, that can be rewritten is only symbol $A$, if we compare this grammar with the grammar of Lemma 1 in Fig. 1, it is easily seen that this grammar generates the set of all trees. Q.E.D.

THEOREM 26. An mcswg $G_b$ *with applicability condition of Fig.* 16 *indirectly generates* $L_{G_b} = \{L(T)\}$.

*Proof.* By applying rules (1) and (2) iteratively, arbitrary bc($G$) can be derived from the initial web. To a one-point web corresponding to nonseprable line graphs, one of the rules (13)–(15) is applied at first.

First, we show that all separable line graphs $L(T)$ can be generated by a given grammar. To derive bc($G$) web, if one wants to construct a block which consists of only a one-arc web, one can apply rule (4). When the degree of all vertices of bc($G$) except endpoints is 2, if one wants to construct a web whose block consists of only a one-arc web, by applying rule (6) to the $I$-vertex (except endpoints) and rule (4) to the others, one can construct the desired web because the applicability condition of rule (12) holds and that by applying first rule (12) and then rule (11), the derivation can terminate.

In case of deg $I \geqslant 2$, one of the rules (3)–(7) can be applied to the $I$-vertex. If the isomorphic to the rules, derived by applying rule (5) or (7) can be derived. An rules (8)–(10) are applied to the webs derived by applying rules (4) or (6), webs application of rules (8)–(10) corresponds to an operation of adding a vertex or an arc to the complete block (in the grammar of Theorem 4 [Montanari], his rule (7) corresponds to the operation of adding a chain of arcs, but our grammar need not have such a rule because each block is a complete block).

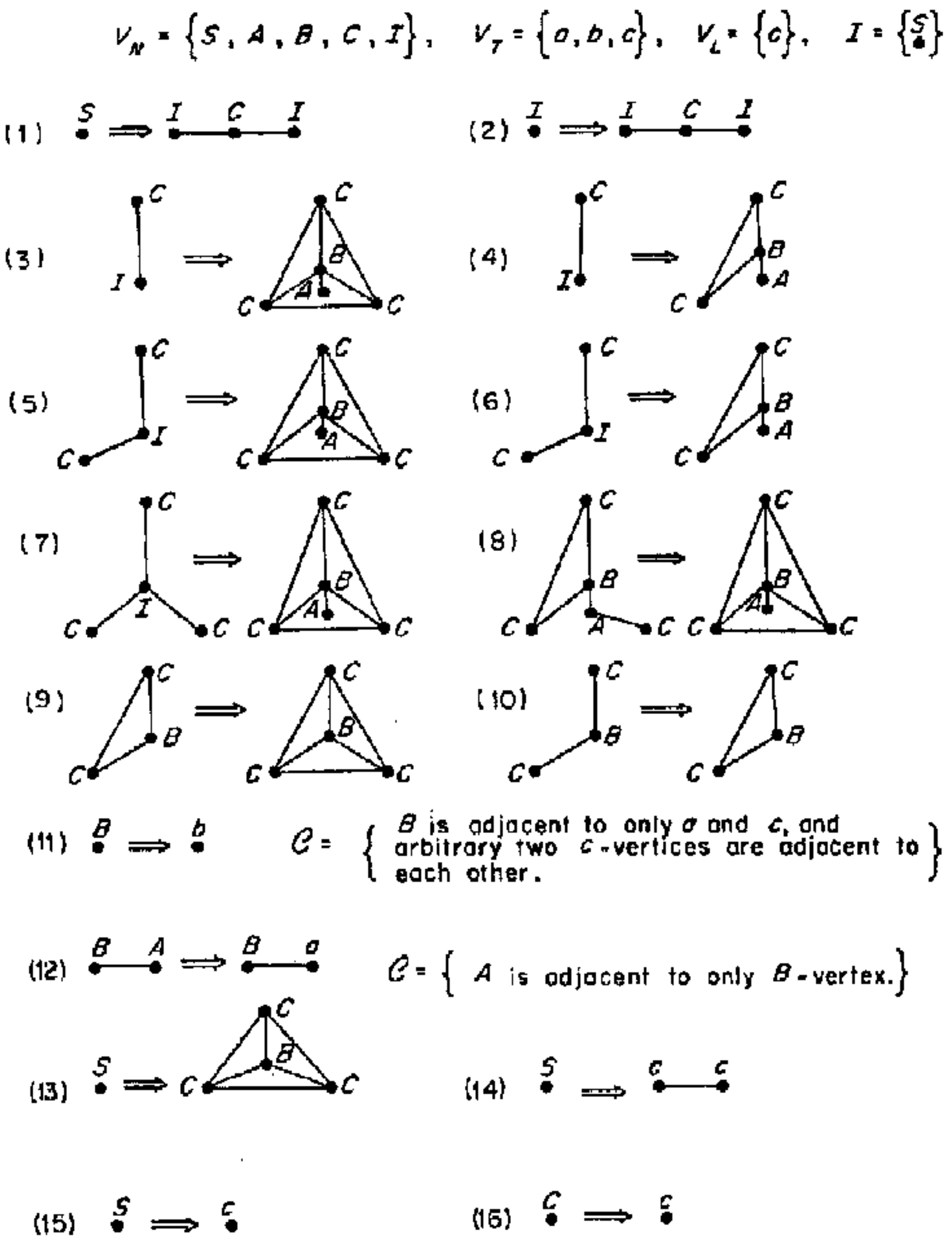

$V_N = \{S, A, B, C, I\}$, $V_T = \{a, b, c\}$, $V_L = \{c\}$, $I = \{S\}$
(1)
(2)
(3)
(4)
(5)
(6)
(7)
(8)
(9)
(10)
(11)
$\mathcal{C}$ = { B is adjacent to only a and c, and arbitrary two c-vertices are adjacent to each other. }
(12)
$\mathcal{C}$ = { A is adjacent to only B-vertex. }
(13)
(14)
(15)
(16)
S
I
C
A
B
a
b
c

FIG. 16. This nmcswg with the applicability condition indirectly generates a set of all line graphs of trees $T$.

From the previous result, since it is difficult for nmcswg to generate complete graphs, arbitrary complete blocks are generated by use of the applicability condition of rule (11). As rule (10) must be applied iteratively to the nonadjacent vertices on a same block until the applicability condition of rule (11) is satisfied, each block becomes a complete block.

Second, we show that all nonseparable line graphs can be generated by the given grammar. This web corresponds to a one-point web and if rule (15) is applied to this web, trivial line graph–one-point web, i.e., a line graph of a one-arc graph, is generated;

and by applying rule (14) to an initial web, nontrivial minimal line graph is generated. By applying rule (13) to the initial web, $K_3$ is generated, and by iterative application of rules (9) and (10), arbitrary complete blocks can be generated, for the same reason as described above. Consequently, all $L(T)$ can be generated by the grammar.

*Only line graphs of trees.* This part is a direct consequence of the necessary and sufficient condition of $L(T)$ and of the proof given by Montanari. Q.E.D.

In the grammar given by Montanari which generates all separable webs, if his rule (2) is applied to an $I$-vertex whose degree is more than 2 and rule (3) is applied to an $I$-vertex the degree of which is more than 3, there exist no rules applicable to a derived web, and the derivation, therefore, does not terminate. In this grammar, the derivation always terminate. But we do not discuss a general problems here.

It is easily seen that there exists a grammar that does not make use of block-cutpoint graph and that is equivalent to the above- mentioned grammar. In this paper, we adopt Montanari's method to make clear the justification of our insistence.

Now, consider the grammar that generates a set of all line graphs. The necessary and sufficient condition for a graph to be a line graph is that its arcs can be partitioned into complete subgraphs in such a way that no vertex lies in more than two of the subgraphs. Given a line graph $L(G)$, there exists at least a family of partition $F = \{B_1, B_2, \ldots, B_n\}$, where $B_i$ is a block. For $i \neq j$ $(1 \leqslant i, j \leqslant n)$, let $C_{ij}$ be a collection of vertices which lies in both $B_i$ and $B_j$. From the definition of the web, i.e., all webs do not have loops and multiple paths, the following lemma holds.

LEMMA 27. $| C_{ij} | = 1$.

*Proof.* Assume that $| C_{ij}| = n$ $(n > 1)$ and $C_{ij} = \{C_1, C_2, \ldots, C_n\}$, where $C_i$ is a common vertex of $B_i$ and $B_j$. Since each block $B_i$ and $B_j$ is complete, $C_k$ and $C_m$ are adjacent in both $B_i$ and $B_j$ and this implies that'there exist at least $n$ multiple paths. This is a contradiction. Q.E.D.

Now, let a set of $C_{ij}$ be $\{C_{ij}\}$. $BD$-graph, denoted by bd($G$), of a line graph $G$ is defined as a graph having vertices set $\{B_i\} \cup \{C_{ij}\}$ with two vertices adjacent if one corresponds to a block $B_i$ and the other to $C_{ij}$, and $C_{ij}$ is in $B_i$ and $B_j$. Note that if a graph is bc($G$), then it is also a bd($G$).

When a line graph $G$ is given and if one can construct a bd($G$) of $G$, it is always possible to generate an infinite set of line graphs which contains a given $G$ by corresponding $B_i$-vertex to complete blocks and $C_{ij}$-vertex to a common vertex between two blocks. For a given line graph $G$, of course, the partition of $G$ is not always unique. But if all possible bd($G$) can be constructed, it is evident that all line graphs can be generated by use of the constructed bd($G$). From the above-mentioned discussions, the following two theorems hold.

THEOREM 28. *An* mcswg *G with applicability condition of Fig.* 17 *generates a set of all* bd($G$).

*Proof. Only BD-graph of line graphs.* From Lemma 27, any $C$-vertex corresponding to $C_{ij}$ should be adjacent to only two $I$-vertices corresponding to $B_i$. Since $C$-vertex is adjacent to just two $A$-vertices and nonadjacent to any other vertices, and also by the applicability condition of rule (6), different $C$-vertices are forbidden to be adjacent to the same more $A$-vertices, it is clear that no webs except bd($G$) can be generated by the grammar.

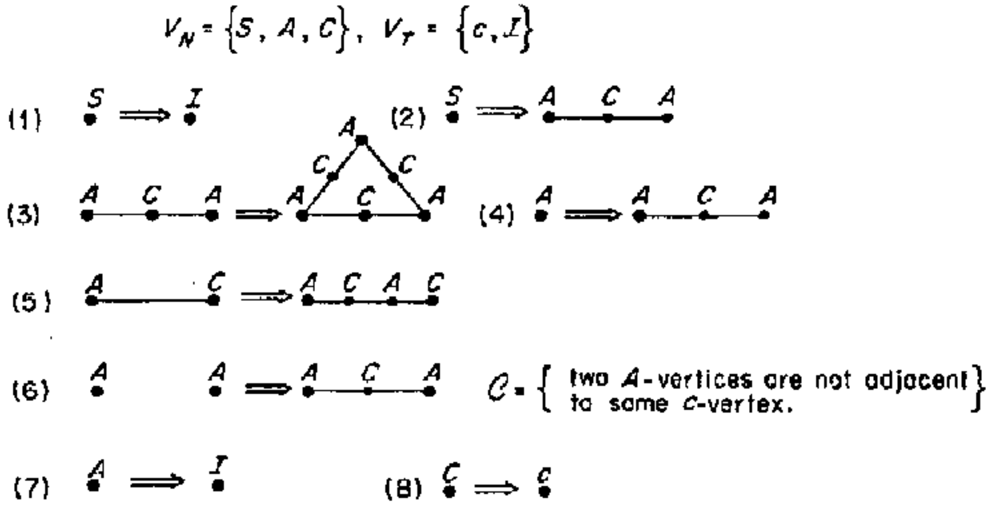

$V_N = \{S, A, C\}$, $V_T = \{c, I\}$
(1)
(2)
(3)
(4)
(5)
(6)
(7)
(8)
S
I
A
C
c
$\mathcal{C}$ = { two A-vertices are not adjacent to same C-vertex. }

FIG. 17. This nmcswg with the applicability condition generates a set of all $BD$-graphs.

*All BD-graphs.* If two $C$-vertices are adjacent or there is an arc between $C$-vertex and $I$-vertex, webs which do not belong to bd($G$) are generated because every $C$-vertex can be adjacent to only two $I$-vertices. Consequently, the possible pairs of vertices which can be rewritten by the rule are the following:

(1) To the nonadjacent two $A$-vertices which are not adjacent to the same $C$-vertex, an arc on which just one new $C$-vertex lies is added [this is done by rule (6)];

(2) To the two $A$-vertices adjacent to the same $C$-vertex, a chain of arcs the length of which is 4 is added [this is done by rule (3)];

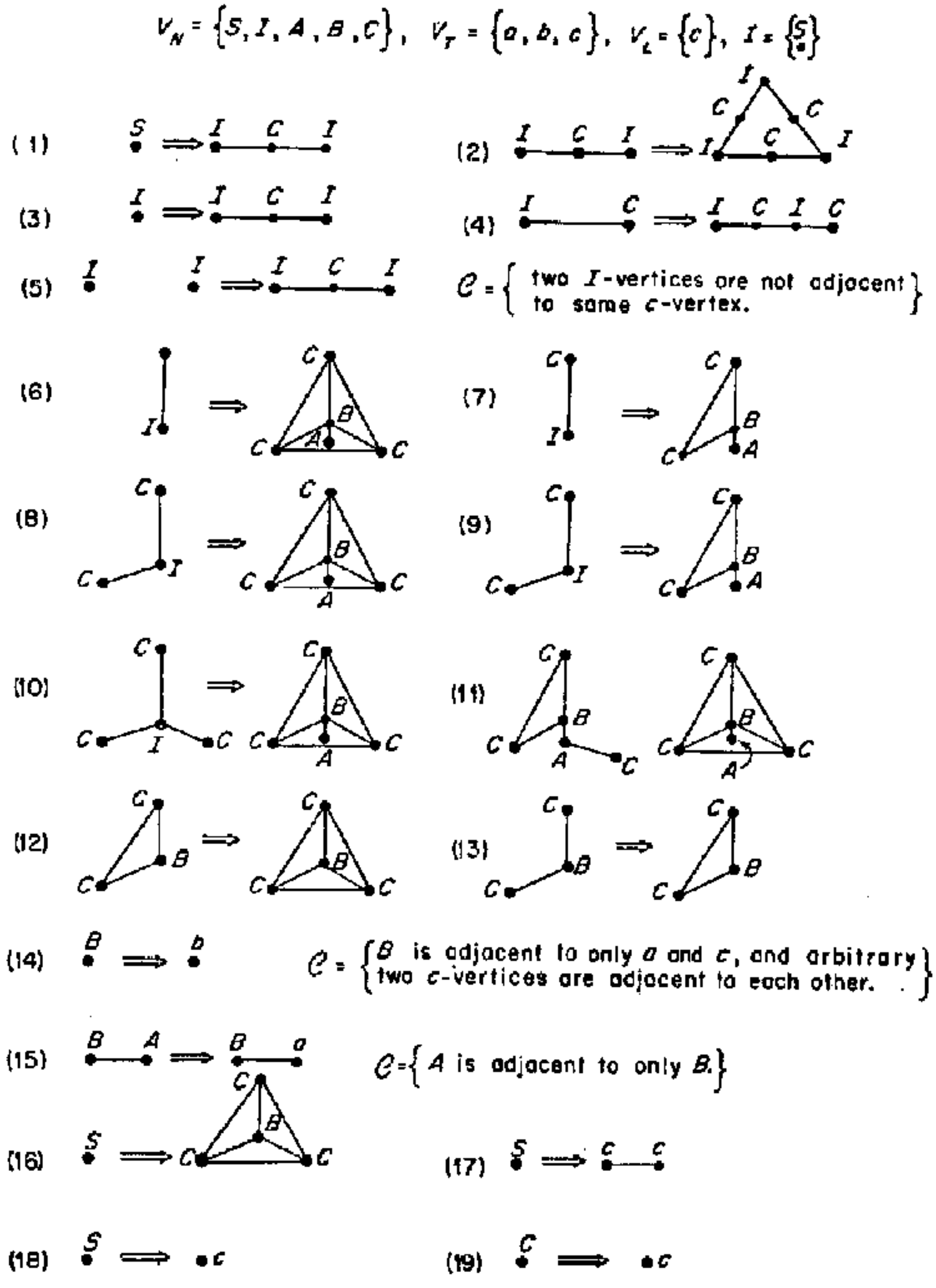
(3) To the $A$-vertex and $C$-vertex which are adjacent to each other, new two vertices $A$-vertex and $C$-vertex are added on the arcs [this is done by rule (5)].

And the other rules are similar to the ones of Theorem 26—that is, by applying the rules corresponding to the above-mentioned 3 cases and rule (1), nonseparable bd($G$) are derived, and by applying the rules except rule (7), separable bd($G$) can be derived. [Of course, it happens that separable blocks are converted into nonseparable ones by applying rule (6)]. From the above discussion it is clear that all bd($G$) can be generated by the given grammar. Q.E.D.

THEOREM 29. *An* nmcswg *G of Fig.* 18 *indirectly generates a set of all line graphs.*

*Proof.* This theorem immediately follows from Theorem 26. Q.E.D.

Finally we show the grammar that generates that set of all 3-connected graphs. A set of all 3-connected graphs is the special case of a set of all nonseparable ones, that



$V_N = \{S, I, A, B, C\}$, $V_T = \{a, b, c\}$, $V_L = \{c\}$, $I = \{\overset{S}{\bullet}\}$

(5) $\mathcal{C} = \{$ two $I$-vertices are not adjacent to same $c$-vertex. $\}$

(14) $\mathcal{C} = \{$ $B$ is adjacent to only $a$ and $c$, and arbitrary two $c$-vertices are adjacent to each other. . $\}$

(15) $\mathcal{C} = \{$ $A$ is adjacent to only $B$. $\}$

FIG. 18. This nmcswg with the applicability condition generates a set of all line graphs.

is, nonseparable graphs is 2-connected. For the construction of 2-connected graphs, the addition of an arc and a chain of arcs to the $K_3$ is necessary as described before, and this procedure can be done by the normal grammar. But, for the construction of all 3-connected graphs, the division of a vertex to at least two vertices is necessary and it is difficult to realize such an operation by the normal grammar because all vertices adjacent to the former vertex must be adjacent to the exact one of the two rewritten vertices and such a collection of vertices can exist in graphs without limits. But by nonnormal grammars, such an operation can easily be realized.

THEOREM 30. *An* anmcswg *of Fig.* 19 *generates a set of all* 3-*connected graphs.*

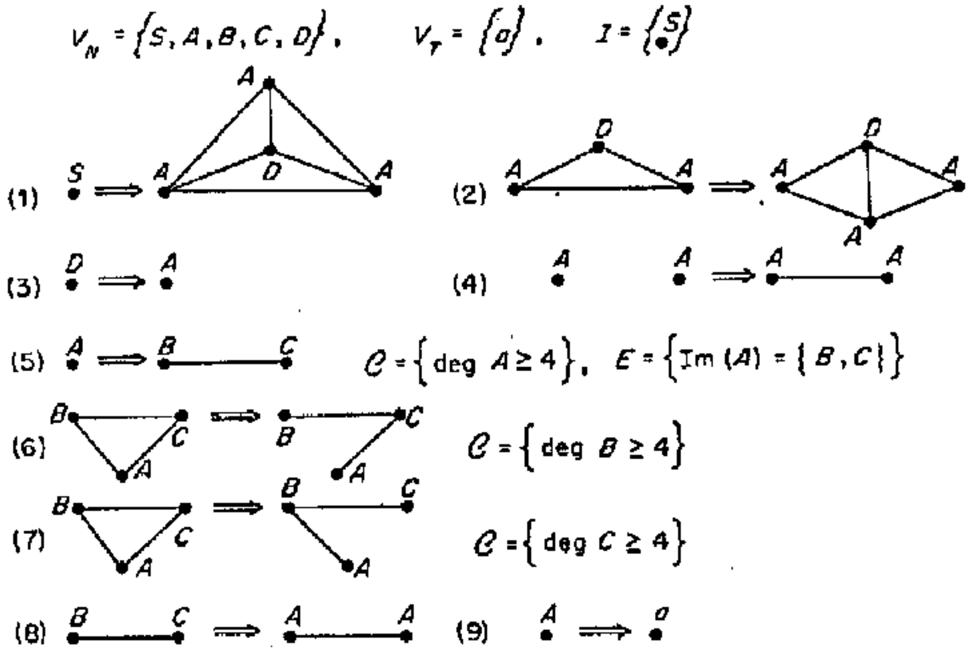

$V_N = \{S, A, B, C, D\}$, $V_T = \{a\}$, $I = \{S\}$
(1) S ⟹ A A A D
(2) A D A ⟹ A D A A
(3) D ⟹ A
(4) A A ⟹ A A
(5) A ⟹ B C $\mathcal{C} = \{\deg A \geq 4\}$, $E = \{\text{Im}(A) = \{B, C\}\}$
(6) B C A ⟹ B C A $\mathcal{C} = \{\deg B \geq 4\}$
(7) B C A ⟹ B C A $\mathcal{C} = \{\deg C \geq 4\}$
(8) B C ⟹ A A
(9) A ⟹ a

FIG. 19. This anmcswg generates a set of all 3-connected graphs. Note that the applicability condition of rules (5)–(7) can be described in their context, and that rule (5) is nonnormal.

*Outline of proof.* This theorem immediately follows from Tutte's theorem [Theorem 13]. He has shown that all exact 3-connected graphs can be obtained by the following procedures:

(1) All wheels are 3-connected.

(2) All graphs that can be obtained from a wheel by a sequence of operations of the following two types;

(i) the addition a new arc,

(ii) the replacement of a vertex $v$ having degree at least 4 by two adjacent vertices $v_1$, $v_2$ such that each vertex formerly joined to $v$ is joined to exactly one of $v_1$

and $v_2$ so that in the resulting graph, $\deg v_1 \geqslant 3$ and $\deg v_2 \geqslant 3$. Rules (1)–(3) correspond to the case (1) and rule (4) to (i) and rules (5)–(8) to (ii). The nonnormal rule (5) divides an $A$-vertex into $B$, $C$-vertex. If rule (6) or (7) is applied repetitively until the contextual condition results in a failure, a minimal 3-connected graph derived from the host 3-connected graph can be obtained. Consequently, the completeness of this grammar is clear.

Q.E.D.

## Conclusions

The classes of web grammars and the languages of various types of web grammars have been investigated. The normal context-free web grammars can generate the proper subsets of the webs whose number of types of blocks are finite. The nonnormal context-free web grammars can generate proper subsets, the number of types of blocks of which is not always finite. And the indirect generation by normal context-senseitive web grammars is a strong generating device in comparison with the generation by ones.

Finally, various types of web grammars that generate some interesting graphs are shown. According to the results in this paper and in Montanari's, almost all graphs can be described by web grammars. However, we could not solve the relation between nmcsw$\overline{\text{L}}$ and I-nmcsw$\overline{\text{L}}$.

For a future study, the type of applicability condition and a reasonably different definition of an embedding part will be the topics.

In this paper, we take the attitude that belongs more to the areas of description of graphs than to automata theory. We think that it is not unreasonable to predicate that we can conceive of machines which accept or recognize various types of graphs. We want to concentrate on this point from now on.

## Acknowledgments

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