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## General Formulation of Formal Grammars

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ABSTRACT

By extracting the basic properties common to the formal grammars appeared in existing literatures, we develop a general formulation of formal grammars. We define a pseudo grammar and derive from it the well-known probabilistic, fuzzy grammars and so on. Moreover, several interesting grammars such as $\sqcup *$ grammars, $\sqcup \sqcap$ grammars, $\sqcap \sqcup$ grammars, composite *B*-fuzzy grammars, and mixed fuzzy grammars, which have never appeared in any other papers before, are derived.

### 1. INTRODUCTION

By introducing the concept of randomness and fuzziness into the structure of formal grammars, some interesting grammars such as probabilistic (or stochastic) grammars and fuzzy grammars have been formulated [2–5, 13, 14].

In this paper, we develop a general formulation of formal grammars by extracting the basic properties common to the formal grammars appeared in the existing literature. By corresponding the element of the appropriate algebra, say, the complete distributive lattice, to each rule of a pseudo grammar, the evaluation (or weight) of the application of the rule is given. We evaluate a sentence by performing the operations of the corresponding algebra to the weight of the rules used in a generation of the sentence.

We derived from the pseudo grammars with various types of algebras the well-known phrase-structure grammars, probabilistic grammars, and fuzzy grammars. Still more, the grammars which have never appeared before, say, $\sqcup *$ grammars, $\sqcup \sqcap$ grammars, $\sqcap \sqcup$ grammars, composite B-fuzzy grammars, and mixed fuzzy grammars are derived.

It can be shown that there are max-weighted grammars and max-probabilistic grammars as special cases of $\sqcup *$ grammars, (pessimistic) fuzzy grammars and phrase structure grammars as special cases of $\sqcup \sqcap$ grammars, and optimistic fuzzy grammars as special cases of $\sqcap \sqcup$ grammars.

The pseudo grammar called a pseudo conditional grammar, whose weight of the application of a rule is conditioned by the rule used just before in a derivation, is also defined and from it a few interesting conditional grammars are derived in the same manner as pseudo grammars.

## 2. $L$-FUZZY SETS

We shall briefly review $L$-fuzzy sets by J. A. Goguen [6] and J. G. Brown [9] for the purpose of $\sqcup *$ grammars, $\sqcup\sqcap$ grammars, $\sqcap\sqcup$ grammars, and fuzzy grammars which will be defined later.

*L-Fuzzy Sets*

An *L-fuzzy set* $A$ in a space $X = \{x\}$ is characterized by a membership function $\mu_A$ as follows:

$$\mu_A: X \to L, \tag{1}$$

where $L$ is called a *membership space* and the value $\mu_A(x) \in L$ represents the "*grade of membership*" of $x$ in $A$.

A membership space $L$ may be assumed to be a partially ordered set or, more particularly, a lattice.

When $L$ is the unit interval $[0, 1]$, $A$ is a fuzzy set originated by L. A. Zadeh [7]. Moreover, when $L$ contains only two points 0 and 1, $A$ is a non-fuzzy set and its membership function $\mu_A$ reduces to the conventional characteristic function of a non-fuzzy set.

The notions of containment, equality, union, and intersection of $L$-fuzzy sets are defined as extensions of the corresponding notions in the ordinary non-fuzzy sets.

Let $A$ and $B$ be two $L$-fuzzy sets in $X$, and let $\mu_A$, $\mu_B$ be membership functions of $A$ and $B$, respectively; then, for all $x$ in $X$,

*Containment* $$A \subseteq B \Leftrightarrow \mu_A(x) \leqq \mu_B(x), \tag{2}$$

*Equality* $$A = B \Leftrightarrow \mu_A(x) = \mu_B(x), \tag{3}$$

*Union* $$A \cup B \Leftrightarrow \mu_{A \cup B}(x) = \mu_A(x) \sqcup \mu_B(x), \tag{4}$$

*Intersection* $$A \cap B \Leftrightarrow \mu_{A \cap B}(x) = \mu_A(x) \sqcap \mu_B(x), \tag{5}$$

where the operations $\leqq$, $\sqcup$, and $\sqcap$ represent an order relation, lub, and glb in $L$, respectively.

In the case of $L = [0, 1]$, that is, fuzzy sets by Zadeh, the operation $\sqcup$ reduces to max, and $\sqcap$ to min. In addition, the complement of a fuzzy set $A$ is defined as

*Complement* $$\bar{A} \Leftrightarrow \mu_{\bar{A}}(x) = 1 - \mu_A(x), \qquad \forall x \in X. \tag{6}$$

In this paper, the structure of the membership space $L$ is assumed to be the complete distributive lattice (or, more generally, the complete lattice ordered semigroup) on account of $L$-fuzzy relations denoted hereafter [6].†

*L-Fuzzy Relation*

An *L-fuzzy relation* $R$ in the product space $X \times Y = \{(x,y) | x \in X, y \in Y\}$ is a $L$-fuzzy set in $X \times Y$ characterized by a membership function $\mu_R$, i.e.,

$$\mu_R: X \times Y \to L. \tag{7}$$

*Product of L-Fuzzy Relations*

If $R_1$ and $R_2$ are two $L$-fuzzy relations in $X \times X$, then by the *product* (or *composition*) of $R_1$ and $R_2$ is meant a $L$-fuzzy relation in $X \times X$ which is denoted by $R_1 R_2$ and is defined as follows: If $L$ is a closg, then

$$\mu_{R_1 R_2}(x,z) = \bigsqcup_y [\mu_{R_1}(x,y) * \mu_{R_2}(y,z)], \tag{8}$$

where $\sqcup$ and $*$ are the operations of lub and semigroup in $L$, respectively. If $L$ is a complete distributive lattice, then

$$\mu_{R_1 R_2}(x,z) = \bigsqcup_y [\mu_{R_1}(x,y) \sqcap \mu_{R_2}(y,z)], \tag{9}$$

$$\mu_{R_1 R_2}(x,z) = \bigsqcap_y [\mu_{R_1}(x,y) \sqcup \mu_{R_2}(y,z)]. \tag{10}$$

If $L$-fuzzy relation $R$ is a fuzzy relation by Zadeh, that is, $R$ is characterized by a membership function,

$$\mu_R: X \times Y \to [0,1], \tag{11}$$

then the product of fuzzy relations $R_1$ and $R_2$ is defined as special cases of (9) and (10) [8, 12], that is,

$$\mu_{R_1 R_2}(x,z) = \sup_y \min [\mu_{R_1}(x,y), \mu_{R_2}(y,z)], \tag{12}$$

$$\mu_{R_1 R_2}(x,z) = \inf_y \max [\mu_{R_1}(x,y), \mu_{R_2}(y,z)]. \tag{13}$$

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† A complete lattice which is a semigroup with identity under $*$ and also satisfies the distributive law, for $x, y, x_i, y_i \in L$,

$$x * (\bigsqcup_i y_i) = \bigsqcup_i (x * y_i),$$

and

$$(\bigsqcup_i x_i) * y = \bigsqcup_i (x_i * y),$$

is a *complete lattice ordered semigroup* (=closg). Still more, if $*$ is replaced by $\sqcap$ in closg $L$, $L$ becomes a *complete distributive lattice*.

Note that the operation of product of ($L$-) fuzzy relations has the associative property, i.e.,

$$R_1(R_2 R_3) = (R_1 R_2) R_3. \tag{14}$$

Hence, let $R_1, R_2, \ldots, R_n$ be ($L$-) fuzzy relations on $X$, then the product $R_1 R_2 \cdots R_n$, say, in the case of (8), is defined as

$$\begin{aligned} &\mu_{R_1 R_2 \ldots R_n}(x_1, x_{n+1}) \\ &\quad = \bigsqcup_{x_2, \ldots, x_n} [\mu_{R_1}(x_1, x_2) * \mu_{R_2}(x_2, x_3) * \cdots * \mu_{R_n}(x_n, x_{n+1})]. \end{aligned} \tag{15}$$

Next, by using the concept of $L$-fuzzy sets, we shall define $L$-fuzzy languages. For simplicity, we denote $L$-fuzzy languages as fuzzy languages hereafter.

Let $\Sigma$ be a finite non-empty alphabet. The set of all finite strings over $\Sigma$ is denoted by $\Sigma^*$. The null string is denoted by $\Lambda$ and included in $\Sigma^*$.

*Fuzzy Languages*

A *fuzzy language* FL is a $L$-fuzzy set in $\Sigma^*$ characterized by a membership function such as $\mu_{\mathrm{FL}}: \Sigma^* \to L$.

The operations, containment, equality, union, and intersection of fuzzy languages are the same as those of $L$-fuzzy sets mentioned previously (see (2)–(5)). Moreover, the notions of concatenation and Kleene closure of ordinary languages can be extended to fuzzy languages by the following:

Let $L_1$ and $L_2$ be two fuzzy languages in $\Sigma^*$, and $\mu_{L_1}$ and $\mu_{L_2}$ be membership functions of $L_1$ and $L_2$, respectively.

*Concatenation*

The *concatenation* of $L_1$ and $L_2$ is a fuzzy language denoted by $L_1 \circ L_2$ or $L_1 \cdot L_2$ and defined as follows: Let a string $x$ in $\Sigma^*$ be expressed as a concatenation of a prefix string $u$ and a suffix string $v$, that is, $x = uv$. Then

$$\mu_{L_1 \circ L_2}(x) = \bigsqcup_u [\mu_{L_1}(u) \sqcap \mu_{L_2}(v)], \tag{16}$$

$$\mu_{L_1 \cdot L_2}(x) = \bigsqcap_u [\mu_{L_1}(u) \sqcup \mu_{L_2}(v)], \tag{17}$$

where $\sqcup$ in (16) and $\sqcap$ in (17) are taken over all prefixes $u$ of $x$.

Note that the concatenation $L_1 \circ L_2$ in (16) is related as $\sqcup\sqcap$ grammars, and $L_1 \cdot L_2$ in (17) is related as $\sqcap\sqcup$ grammars, which will be defined later.

*Kleene Closure*

By using the concatenation $L_1 \circ L_2$ or $L_1 \cdot L_2$, *Kleene closure* of a fuzzy language $L$ (written as $L^*$, or $\hat{L}$) is defined as

$$L^* = \Lambda \cup L \cup L \circ L \cup L \circ L \circ L \cup \cdots, \tag{18}$$

$$\hat{L} = \Lambda \cap L \cap L \cdot L \cap L \cdot L \cdot L \cap \cdots. \tag{19}$$

## 3. VARIOUS KINDS OF GRAMMARS

In this section we define a pseudo grammar, each production of which has a label, an ordinary rewriting rule, and weight $\mu(r)$ as in (21) and derive from it various kinds of grammars, which have or have not appeared in the existing literature, by employing an appropriate algebra system as a weighting space and performing the corresponding operations to weights $\mu(r)$'s.

DEFINITION. A *pseudo grammar* (PSG for short) is a system

$$\text{PSG} = (V_N, V_T, P, S, J, M, \mu), \tag{20}$$

where

(i) $V_N$ is a non-terminal vocabulary.

(ii) $V_T$ is a terminal vocabulary.

(iii) $S$ is an initial symbol in $V_N$.

(iv) $P$ is a finite set of productions such as

$$(r)\, u \rightarrow v\ \mu(r), \tag{21}$$

where $r \in J$, $u \rightarrow v$ is an ordinary rewriting rule with $u \in V_N{}^* - \{\Lambda\}$ and $v \in (V_N \cup V_T)^*$, and $\mu(r)$ is a weight of the application of the production $r$, which will be denoted in (vii).†

(v) $J$ is a set of (rewriting rule) labels as shown in (iv). $J = \{r\}$.

(vi) $M$ is a *weighting space*.

(vii) $\mu$ is a function such that

$$\mu : J \rightarrow M.$$

$\mu$ may be called a *weighting function* and the value $\mu(r)$ is a *weight* of the application of a production $r$.

Next, we shall briefly explain a derivation chain with weights (*weighted derivation chain*). However, the meaning of weight $\mu(r)$ denoted over $\rightarrow$ in a derivation chain will be stated in each grammar defined later.

If $(r)u \rightarrow v\ \mu(r)$ is in $P$, and $\alpha$ and $\beta$ are any strings in $(V_N \cup V_T)^*$, then

$$\alpha u \beta \xrightarrow[r]{\mu(r)} \alpha v \beta, \tag{22}$$

† In this paper, we often say label $r$ as production $r$ for convenience.

and $\alpha v \beta$ is said to be directly derivable from $\alpha u \beta$ by the production $r$. If $\alpha_1, \alpha_2, \ldots, \alpha_m$ are strings in $(V_N \cup V_T)^*$ and

$$\alpha_0 \xrightarrow[r_1]{\mu(r_1)} \alpha_1, \alpha_1 \xrightarrow[r_2]{\mu(r_2)} \alpha_2, \ldots, \alpha_{m-1} \xrightarrow[r_m]{\mu(r_m)} \alpha_m, \tag{23}$$

then $\alpha_m$ is said to be derivable from $\alpha_0$ by the productions $r_1, r_2, \ldots, r_m$. The expression

$$\alpha_0 \xrightarrow[r_1]{\mu(r_1)} \alpha_1 \xrightarrow[r_2]{\mu(r_2)} \alpha_2 \longrightarrow \cdots \xrightarrow[r_m]{\mu(r_m)} \alpha_m \tag{24}$$

will be referred to as a *weighted derivation chain* of length $m$ from $\alpha_0$ to $\alpha_m$ by the productions $r_1, r_2, \ldots, r_m$.

When $\alpha_0 = S$, $\alpha_m = x (\in V_T{}^*)$ in (24), i.e.,

$$S \xrightarrow[r_1]{\mu(r_1)} \alpha_1 \xrightarrow[r_2]{\mu(r_2)} \alpha_2 \longrightarrow \cdots \longrightarrow \alpha_{m-1} \xrightarrow[r_m]{\mu(r_m)} x, \tag{25}$$

$S$ is said to generate a terminal string $x$ by the productions $r_1, r_2, \ldots, r_m$. In general, there is more than one weighted derivation chain from $S$ to $x$.

Now, we shall obtain various kinds of grammars by adopting the appropriate algebra system as the weighting space $M$ of a weighting function $\mu$: $J \to M$ of a pseudo grammar PSG, and by performing the corresponding operations to $\mu(r)$'s.

[I] $\sqcup *$ *Grammar* ($=\sqcup *$G)†

(I-a): Let the weighting space $M$ in PSG be the complete lattice ordered semigroup $L$, namely, the weighting function $\mu$ is

$$\mu: J \to L.$$

In this case, $\mu$ can be regarded as the membership function of an $L$-fuzzy set in $J$.

(I-b): The grade of the generation of $x$ in $V_T{}^*$ by $\sqcup *$G, which is denoted as $\mu_{\sqcup * \mathrm{G}}(x)$, is given by using the concept of the product of $L$-fuzzy relations of (8) and by the weighted derivation chain from $S$ to $x$ of (25). Clearly, $\mu_{\sqcup * \mathrm{G}}(x)$ is in $M$ $(=L)$,

$$\mu_{\sqcup * \mathrm{G}}(x) = \sqcup\ [\mu(r_1) * \mu(r_2) * \cdots * \mu(r_m)], \tag{26}$$

where the lub $\sqcup$ is taken over all the weighted derivation chains from $S$ to $x$.

† As special cases of $\sqcup *$G, there are [II] $\sqcup\sqcap$Grammar, [XI] Max-Weighted Grammar, and [XII] Max-Probabilistic Grammar, which are defined later.

[II] $\sqcup\sqcap$ *Grammar* (=$\sqcup\sqcap$G)†

(II-a): The weighting space $M$ is the complete distributive lattice $L'$, that is, $\mu$ is

$$\mu: J \to L'.$$

(II-b): The grade $\mu_{\sqcup\sqcap G}(x)$ of the generation of $x$ in $V_T^*$ by $\sqcup\sqcap$G is given by using the product of $L$-fuzzy relations of (9):

$$\mu_{\sqcup\sqcap G}(x) = \sqcup\ [\mu(r_1) \sqcap \mu(r_2) \sqcap \cdots \sqcap \mu(r_m)], \tag{27}$$

where $\sqcup$ is taken over all the weighted derivation chains from $S$ to $x$.

[III] $\sqcap\sqcup$ *Grammar* (=$\sqcap\sqcup$G)‡

(III-a): This is the same as (II-a).

(III-b): The grade $\mu_{\sqcap\sqcup G}(x)$ of the generation of $x$ is given from the product of $L$-fuzzy relations of (10):

$$\mu_{\sqcap\sqcup G}(x) = \sqcap\ [\mu(r_1) \sqcup \mu(r_2) \sqcup \cdots \sqcup \mu(r_m)], \tag{28}$$

where $\sqcap$ is taken over all the weighted derivation chains from $S$ to $x$.

Let $L' = B$ (complete Boolean lattice) in (II-a) and (III-a), then we can define $\sqcup\sqcap$G and $\sqcap\sqcup$G on $B$, which may be written as $\sqcup\sqcap$BG and $\sqcap\sqcup$BG, respectively. We will denote the grades of the generation of $x$ by $\sqcup\sqcap$BG and $\sqcap\sqcup$BG as $\mu_{\sqcup\sqcap BG}(x)$ and $\mu_{\sqcap\sqcup BG}(x)$, respectively.

[IV] *Composite B-Fuzzy Grammar* (=CBFG)

(IV-a): The weighting space $M$ is the complete Boolean lattice $B$.

(IV-b): The grade $\mu_{CBFG}(x)$ of the generation of $x$ is defined as

$$\mu_{CBFG}(x) = (\alpha \sqcap \mu_{\sqcup\sqcap BG}(x)) \sqcup (\bar{\alpha} \sqcap \mu_{\sqcap\sqcup BG}(x)), \tag{29}$$

where $\alpha \in B$ and $\bar{\alpha}(\in B)$ is the complement of $\alpha$.

[V] *(Pessimistic) Fuzzy Grammar* (=PFG), or *Maximin Grammar* [5,13,14]

(V-a): Let $L' = [0,1]$ in (II-a).

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† [V] (Pessimistic) Fuzzy Grammar and [VIII] Ordinary Phrase Structure Grammar are considered as special cases of $\sqcup\sqcap$ Grammar.

‡ As a special case of $\sqcap\sqcup$ Grammar, there is [VI] Optimistic Fuzzy Grammar.

(V-b): The grade $\mu_{PFG}(x)$ of the generation of $x$ by PFG is given as follows by using the product of fuzzy relations of (12), in other words, by replacing $\sqcup$ by max and $\sqcap$ by min in (II-b):

$$\mu_{PFG}(x) = \max \min [\mu(r_1), \mu(r_2), \ldots, \mu(r_m)], \tag{30}$$

where maximum is taken over all the derivation chains from $S$ to $x$.

[VI] *Optimistic Fuzzy Grammar* (=OFG), or *Minimax Grammar*

(VI-a): This is the same as (V-a).

(VI-b): $\mu_{OFG}(x)$ is given as follows by using the product of fuzzy relations of (13), that is, by replacing $\sqcap$ by min and $\sqcup$ by max in (III-b):

$$\mu_{OFG}(x) = \min \max [\mu(r_1), \mu(r_2), \ldots, \mu(r_m)]. \tag{31}$$

[VII] *Mixed Fuzzy Grammar* (=MFG)

(VII-a): This is the same as (V-a).

(VII-b): $\mu_{MFG}(x)$ is given as follows:

$$\mu_{MFG}(x) = a\mu_{PFG}(x) + b\mu_{OFG}(x), \tag{32}$$

where $a$ and $b$ are real numbers such that $a + b = 1$ (cf. [1]), and the subscripts PFG and OFG denote [V] (Pessimistic) Fuzzy Grammar and [VI] Optimistic Fuzzy Grammar, respectively.

[VIII] *Phrase Structure Grammar* (=G)

(VIII-a): $L' = \{0, 1\}$ in (II-a) or (V-a).

(VIII-b): $\mu_G(x)$ is obtained in the same manner as $\mu_{PFG}(x)$ in (V-b).

*Note*: In this case the language $L(G)$ generated by G is defined as

$$L(G) = \{x \in V_T{}^* \mid \mu_G(x) = 1\}.$$

[IX] *Weighted Grammar* (=WG)

(IX-a): The weighting space $M$ is the set of non-negative real numbers.

(IX-b): $\mu_{WG}(x)$ is given as follows:

$$\mu_{WG}(x) = \Sigma \mu(r_1) \cdot \mu(r_2) \cdot \cdots \cdot \mu(r_m), \tag{33}$$

where the operations "$\Sigma$" and "$\cdot$" are sum and product in the ordinary sense, respectively.

[X] *Probabilistic* (or *Stochastic*) *Grammar* (=PG) [3,4]

(X-a): $M = [0,1]$, i.e., $\mu(r) \in [0,1]$, $r \in J$ and, in addition, $\mu(r)$ satisfies the following constraint; For each $J_u$,

$$\sum_{r \in J_u} \mu(r) = 1,$$

where $J_u$ is the set of all labels such that the left-hand side of the rewriting rule in the production of the pseudo grammar PSG is $u(\in V_N^* - \{\Lambda\})$.

(X-b): $\mu_{PG}(x)$ is defined in the same manner as $\mu_{WG}(x)$ in (IX-b) and can be regarded as the probability of the generation of $x$ by PG.

[XI] *Max-Weighted Grammar* (=MWG)

(XI-a): This is the same as (IX-a).

(XI-b): We take the maximum instead of taking $\Sigma$ in (IX-b), i.e.,

$$\mu_{MWG}(x) = \max\,[\mu(r_1)\cdot\mu(r_2)\cdot\cdots\cdot\mu(r_m)]. \tag{34}$$

It is noted that the expression above can be obtained by replacing $\sqcup$ by max and $*$ by $\cdot$ in $\sqcup*$G of [I].

[XII] *Max-Probabilistic Grammar* (=MPG)

(XII-a): This is the same as (X-a).

(XII-b): $\mu_{MPG}(x)$ is obtained in the same manner as $\mu_{MWG}(x)$ in (XI-b).

[XIII] *Label Sequence Grammar* (=LSG)

(XIII-a): The weighting space $M$ is $J^*$, where $J$ is the set of labels. The weight $\mu(r)$, $r \in J$, is defined as

$$\mu(r) = r, \qquad \text{for each } r \in J.$$

(XIII-b): $\mu_{LSG}(x)$, $x \in V_T^*$, is given as

$$\begin{aligned}\mu_{LSG}(x) &= \vee\,[\mu(r_1)\cdot\mu(r_2)\cdot\cdots\cdot\mu(r_m)]\\ &= \vee\,[r_1\cdot r_2\cdot\cdots\cdot r_m],\end{aligned} \tag{35}$$

where the operations "$\vee$" and "$\cdot$" are union and concatenation of (label) sequences, respectively. This expression (35) can be obtained by replacing $\sqcup$ by $\vee$ and $*$ by $\cdot$ in $\sqcup*$G of [I].

*Note:* We could regard $\mu_{LSG}(x)$ as the set of all the label sequences from $S$ to $x$. Let $C$ be the subset of $J^*$, then the language

$$L_C = \{x \in V_T^* \mid \mu_{LSG}(x) \cap C \neq \phi\}$$

can be regarded as the languages controlled by control language $C$ [10].

## 4. VARIOUS KINDS OF CONDITIONAL GRAMMARS

In this section we define a pseudo conditional grammar (PSCG for short) as an extension of a pseudo grammar PSG denoted in a previous section and derive from it several interesting conditional grammars which have or have not appeared in the existing papers, in the same way as we derived, in Section 3, various kinds of grammars from PSG.

DEFINITION. A *pseudo conditional grammar* (PSCG) is a system

$$\text{PSCG} = (V_N, V_T, P, S, J, M, \{\mu_1, \mu_2\}), \tag{36}$$

where $V_N$, $V_T$, $S$, $J$, and $M$ have essentially the same meanings as those for the PSG in the previous section. $P$ is a set of the rules with labels as follows:

$$(r)\; u \to v.$$

$\mu_1$ is a weighting function which is called an *initial rule designating function* such that

$$\mu_1 : J_S \to M, \tag{37–1}$$

where $J_S$ is the set of all labels whose rules are initial rules. $\mu_2$ is a conditional weighting function as follows:

$$\mu_2(r/r') \in M, \tag{37–2}$$

where $r$, $r' \in J$. $\mu_2(r/r')$ represents the weight of the application of the rule $r$ given the rule $r'$ used just before in a derivation.

It is noted that the notion of a conditional weighting function is similar to that of a conditional probability function.

We shall write $\mu$ for $\mu_1$ and $\mu_2$ if there occurs no confusion.

If the derivation chain from $S$ to $x$ ($\in V_T^*$) is

$$S \xrightarrow[r_1]{} \alpha_1 \xrightarrow[r_2]{} \alpha_2 \longrightarrow \cdots \longrightarrow \alpha_{m-1} \xrightarrow[r_m]{} x, \tag{38}$$

then the weights $\mu$ are put over the arrows $\to$ as follows:

$$S \xrightarrow[r_1]{\mu(r_1)} \alpha_1 \xrightarrow[r_2]{\mu(r_2/r_1)} \alpha_2 \longrightarrow \cdots \longrightarrow \alpha_{m-1} \xrightarrow[r_m]{\mu(r_m/r_{m-1})} x. \tag{39}$$

Now, let us define various kinds of conditional grammars.

[A] *Conditional* $\sqcup *$ *Grammar* (=C$\sqcup$*G)

(A-1): The weighting space $M$ in C$\sqcup$*G is the complete lattice ordered semigroup $L$.

(A-2): The grade of the generation of $x$ in $V_T{}^*$ by C$\sqcup$*G is given as follows by using the concept of the product of $L$-fuzzy relations of (8) and from the weighted derivation chain from $S$ to $x$ of (39):

$$\mu_{\mathrm{C}\sqcup *\mathrm{G}}(x) = \sqcup\, [\mu(r_1) * \mu(r_2/r_1) * \mu(r_3/r_2) * \cdots * \mu(r_m/r_{m-1})], \tag{40}$$

where the lub $\sqcup$ is taken over all the weighted derivation chains from $S$ to $x$.

[B] *Conditional* $\sqcup\sqcap$ *Grammar* (=C$\sqcup\sqcap$G)

(B-1): The weighting space $M$ is the complete distributive lattice $L'$.

(B-2): $\mu_{\mathrm{C}\sqcup\sqcap\mathrm{G}}(x)$, $x \in V_T{}^*$, is given from the product of $L$-fuzzy relations of (9):

$$\mu_{\mathrm{C}\sqcup\sqcap\mathrm{G}}(x) = \sqcup\, [\mu(r_1) \sqcap \mu(r_2/r_1) \sqcap \cdots \sqcap \mu(r_m/r_{m-1})]. \tag{41}$$

[C] *Conditional* $\sqcap\sqcup$ *Grammar* (=C$\sqcap\sqcup$G)

(C-1): This is the same as (B-1).

(C-2): $\mu_{\mathrm{C}\sqcap\sqcup\mathrm{G}}(x)$ is given from (10) as follows:

$$\mu_{\mathrm{C}\sqcap\sqcup\mathrm{G}}(x) = \sqcap\, [\mu(r_1) \sqcup \mu(r_2/r_1) \sqcup \cdots \sqcup \mu(r_m/r_{m-1})]. \tag{42}$$

[D] *Conditional Composite B-Fuzzy Grammar* (=CCBFG)

(D-1): The weighting space $M$ is the complete Boolean lattice $B$.

(D-2): $\mu_{\mathrm{CCBFG}}(x)$ is as follows:

$$\mu_{\mathrm{CCBFG}}(x) = (\alpha \sqcap \mu_{\mathrm{C}\sqcup\sqcap\mathrm{BG}}(x)) \sqcup (\bar{\alpha} \sqcap \mu_{\mathrm{C}\sqcap\sqcup\mathrm{BG}}(x)), \tag{43}$$

where $\alpha \in B$ and $\bar{\alpha} (\in B)$ is the complement of $\alpha$, and $\mu_{\mathrm{C}\sqcup\sqcap\mathrm{BG}}(x)$ and $\mu_{\mathrm{C}\sqcap\sqcup\mathrm{BG}}(x)$ are the grades of the generation of $x$ by C$\sqcup\sqcap$BG and C$\sqcap\sqcup$BG, which are grammars C$\sqcup\sqcap$G of [*B*] and C$\sqcap\sqcup$G of [*C*] on complete Boolean lattice $B$, respectively.

[E] *Conditional (Pessimistic) Fuzzy Grammar* (=CPFG), or *Conditional Maximin Grammar* [13]

(E-1): $L' = [0, 1]$ in (B-1).

(E-2): $\mu_{\mathrm{CPFG}}(x)$ is given from (12) as follows:

$$\mu_{\mathrm{CPFG}}(x) = \max\min\,[\mu(r_1), \mu(r_2/r_1), \ldots, \mu(r_m/r_{m-1})]. \tag{44}$$

[F] *Conditional Optimistic Fuzzy Grammar* (=COFG), or *Conditional Minimax Grammar*

(F-1): This is the same as (E-1).

(F-2): $\mu_{\mathrm{COFG}}(x)$ is given from (13) as follows:

$$\mu_{\mathrm{COFG}}(x) = \min\max\,[\mu(r_1), \mu(r_2/r_1), \ldots, \mu(r_m/r_{m-1})]. \tag{45}$$

[G] *Conditional Mixed Fuzzy Grammar* (=CMFG)

(G-1): This is the same as (E-1).

(G-2): $\mu_{\mathrm{CMFG}}(x)$ is as follows:

$$\mu_{\mathrm{CMFG}}(x) = a\mu_{\mathrm{CPFG}}(x) + b\mu_{\mathrm{COFG}}(x), \tag{46}$$

where $a$ and $b$ are real numbers such that $a + b = 1$.

[H] *Conditional Phrase Structure Grammar* (=CG)

(H-1): $L' = \{0, 1\}$ in (B-1) or (E-1).

(H-2): $\mu_{\mathrm{CG}}(x)$ is obtained in the same manner as $\mu_{\mathrm{CPFG}}(x)$ in (E-2).

*Note:* CG can be regarded as Programmed Grammars with success fields only defined by Rosenkranz [11].

[I] *Conditional Weighted Grammar* (=CWG) [2]

(I-1): $M$ is a set of non-negative real numbers.

(I-2): $\mu_{\mathrm{CWG}}(x)$ is given as:

$$\mu_{\mathrm{CWG}}(x) = \Sigma\mu(r_1)\cdot\mu(r_2/r_1)\cdot\cdots\cdot\mu(r_m/r_{m-1}). \tag{47}$$

[J] *Conditional Probabilistic Grammar* (=CPG) [2]

(J-1): $M = [0, 1]$ and, in addition, $\mu(r)$ and $\mu(r'/r)$ satisfy the following constraints, respectively:

$$\sum_{r \in J_S} \mu(r) = 1,$$

$$\sum_{r' \in J} \mu(r'/r) = 1,$$

where $J_S$ is the set of all labels whose rules are initial rules.

(J-2): $\mu_{CPG}(x)$ is given in the same manner as $\mu_{CWG}(x)$ in (I-2).

[K] *Conditional Max-Weighted Grammar* (=CMWG) [2]

(K-1): This is the same as (I-1).

(K-2): We take the maximum instead of taking $\Sigma$ in (I-2), i.e.,

$$\mu_{CMWG}(x) = \max\,[\mu(r_1)\cdot\mu(r_2/r_1)\cdot\cdots\cdot\mu(r_m/r_{m-1})]. \tag{48}$$

[L] *Conditional Max-Probabilistic Grammar* (=CMPG) [2]

(L-1): This is the same as (J-1).

(L-2): $\mu_{CMPG}(x)$ is defined in the same manner as $\mu_{CMWG}(x)$ in (J-2).

## 5. CONCLUSIONS AND REMARKS

We have derived various kinds of grammars and conditional grammars from a pseudo grammar and a pseudo conditional grammar. As an extension of the pseudo conditional grammar, we can consider the pseudo grammar whose weight of the application of the rule to be used next is conditioned by all the rules used in a derivation [14]. In this case, say, in the case of $\sqcup$*G, the grade of the generation of $x$ is given as

$$\sqcup\,[\mu(r_1) * \mu(r_2/r_1) * \mu(r_3/r_1, r_2) * \cdots * \mu(r_m/r_1, r_2, \ldots, r_{m-1})].$$

In the Weighted Grammar of [IX] in Section 3, we adopted the set of non-negative real numbers as the weighting space $M$, and the product and the sum as its operations. In this case, $M$ forms a semiring. Therefore, we hope that more interesting grammars will be formulated by adopting the appropriate algebras such as semiring, ring, and field.

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