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Some Considerations on Fuzzy Automata

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ABSTRACT

In this paper, we show: The threshold of fuzzy automata can be changed arbitrarily. The fuzzy sets of input sequences characterized by fuzzy automata constitute a distributive lattice, and the complement of the fuzzy set can be characterized by an optimistic fuzzy automaton.

INTRODUCTION

Among various types of automata, as is well-known, are deterministic, nondeterministic and probabilistic automata. Recently, W. G. Wee [*1*] proposed one another type of automata which he named fuzzy automata. The formulation of fuzzy automata is based on the concept of fuzzy sets and fuzzy systems defined by L. A. Zadeh [*5*, *6*]. Fuzzy automata include deterministic and nondeterministic finite automata as special cases and also have some properties similar to those of probabilistic automata.

In addition, fuzzy automata may be available, as those applications, to simulating learning systems such as pattern recognition and automatic control systems [*1*, *12*].

E. S. Santos [*4*] showed that the capability of a fuzzy automaton as an acceptor is equal to that of finite automaton.

In this paper, it is shown that the threshold of fuzzy automata can be changed arbitrarily by changing the values of each element of the fuzzy transition matrix and the initial state designator. Moreover, the family of the fuzzy sets of input sequences characterized by (pessimistic) fuzzy automata is closed under the operations of "union" and "intersection" in the sense of fuzzy set, and the complement of the fuzzy set is characterized by an optimistic fuzzy automaton.

We show that the similar properties to those mentioned above also hold for optimistic fuzzy automata.

1. FUZZY SETS

Fuzzy sets originated by L. A. Zadeh [*5*] are the classes of objects which do not have precisely defined criteria of membership.

A *fuzzy set* (*class*) A in space $X = \{x\}$ is characterized by a membership (characteristic) function $f_A(x)$, the value of which is in the interval [0, 1] and represents the "grade of membership" of x in A. When A is a set in the usual sense, $f_A(x)$ is 1 or 0 according as x does or does not belong to A. The notion of fuzzy is completely nonstatistical in nature.

The definitions of fuzzy sets below are natural extensions of those of the ordinary sets.

Union. $C = A \cup B \Leftrightarrow f_C(x) = \max[f_A(x), f_B(x)]$

Intersection. $C = A \cap B \Leftrightarrow f_C(x) = \min[f_A(x), f_B(x)]$

Inclusion. $A \subseteqq B \Leftrightarrow f_A(x) \leqslant f_B(x)$

Complement. $\bar{A} \Leftrightarrow f_{\bar{A}}(x) = 1 - f_A(x)$

Moreover, some properties such as De Morgan's law and the distributive law are also established. Therefore, fuzzy sets in X form a distributive lattice with a 0 and 1, but do not form a Boolean lattice, because $\bar{A}$ is not the complement of A in the lattice sense.

Fuzzy Relation. An n-ary fuzzy relation in X is a fuzzy set A in the product space $X \times X \times \cdots \times X$ and is characterized by the membership function $f_A(x_1, x_2, ..., x_n)$, where $x_i \in X$, $i = 1, ..., n$. In the case of binary fuzzy relations, the composition of two fuzzy relations A and B is denoted by $A \cdot B$ and is defined as a fuzzy relation in X whose membership function is related to those of A and B by

$$f_{A \cdot B}(x, y) = \sup_v \min[f_A(x, v), f_B(v, y)].$$

Note that the operation of composition has the associative property.

2. Fuzzy Automata

A fuzzy automaton was proposed by W. G. Wee as a model of pattern recognition and automatic control systems. An advantage of employing a fuzzy automaton as a learning model is its simplicity in design and computation. A learning fuzzy automaton is clearly nonstationary. In this paper, however, we assume a fuzzy automaton to be stationary and extend the definition by Wee as follows: In Wee's paper, the initial state of a fuzzy automaton is given in deterministic way. But we will introduce the fuzzy distribution, that is, the initial distribution.

Let Σ be a finite non-empty alphabet. The set of all finite sequences over Σ is denoted by Σ^*. The null sequence is denoted by Λ and included in Σ^*. $\#(S)$ is the number of elements in the set S.

DEFINITION 2.1. A *finite fuzzy automaton* over the alphabet Σ is a system

$$A = \langle S, \pi, \{F(\sigma) \mid \sigma \in \Sigma\}, \eta^G \rangle,$$

where

(1) $S = \{s_1, s_2, ..., s_n\}$ is a non-empty finite set of internal states.

(2) π is an n-dimensional fuzzy row vector, that is, $\pi = (\pi_{s_1}, \pi_{s_2}, ..., \pi_{s_n})$, where $0 \leqslant \pi_{s_i} \leqslant 1$, $1 \leqslant i \leqslant n$, and is called the *initial state designator.*

(3) G is a subset of S (the set of final states).

(4) $\eta^G = (\eta_{s_1}, \eta_{s_2}, ..., \eta_{s_n})'$ is an n-dimensional column vector whose i-th component equals 1 if $s_i \in G$ and 0 otherwise, and is called the *final state designator*.

(5) For each $\sigma \in \Sigma, F(\sigma)$ is a fuzzy matrix of order n (the *fuzzy transition matrix* of A) such that $F(\sigma) = \| f_{s_i, s_j}(\sigma) \|$ $1 \leqslant i \leqslant n$, $1 \leqslant j \leqslant n$.

Let element $f_{s_i, s_j}(\sigma)$ of $F(\sigma)$ be $f_A(s_i, \sigma, s_j)$, where $s_i, s_j \in S$ and $\sigma \in \Sigma$. The function f_A is a membership function of a fuzzy set in $S \times \Sigma \times S$; i.e., $f_A : S \times \Sigma \times S \to [0, 1]$. f_A may be called the *fuzzy transition function*. That is to say, for $s, t \in S$ and $\sigma \in \Sigma$, $f_A(s, \sigma, t) =$ the grade of transition from state s to state t when the input is σ.
The unity fuzzy transition function implies such a transition may exist definitely.

Remark. If f_A takes only two values 0 and 1, then a fuzzy automaton A is a nondeterministic finite automaton. In addition, only any one element of each row of matrix $F(\sigma)$, $\sigma \in \Sigma$ is "1" and the rest elements of each row are all equal to "0". Then a fuzzy automaton A is a deterministic finite automaton.

The grade of transition for an input sequence of length m is defined by an m-ary fuzzy relation. The fuzzy transition function is as follows: For input sequence $x = \sigma_1 \sigma_2 \cdots \sigma_m \in \Sigma^*$ and $s, t \in S$,

$$f_A(s, x, t) = \max_{q_1, q_2, ..., q_{m-1} \in S} \min [f_A(s, \sigma_1, q_1), f_A(q_1, \sigma_2, q_2), ..., f_A(q_{m-1}, \sigma_m, t)]$$

$=$ the grade of transition from state s to state t when the input sequence is $x = \sigma_1 \sigma_2 \cdots \sigma_m$.

DEFINITION 2.2. For $\Lambda, x, y \in \Sigma^*$ and $s, t \in S$,

$$f_A(s, \Lambda, t) = \begin{cases} 1 & \text{if } s = t \\ 0 & \text{if } s \neq t, \end{cases}$$

$$f_A(s, xy, t) = \max_{q \in S} \min [f_A(s, x, q), f_A(q, y, t)].$$

Especially, we call a fuzzy automaton with the grade of transition under the operation "max min" a *pessimistic fuzzy automaton* (pfa), and a fuzzy automaton under the operation "min max" an *optimistic fuzzy automaton* (ofa) [3].

DEFINITION 2.3. An *optimistic fuzzy automaton* over the alphabet Σ' is a system

$$B' = \langle S', \pi', \{F'(\sigma) \mid \sigma \in \Sigma'\}, \eta^{G'} \rangle,$$

where S' is a finite non-empty set (the internal state of B'), $\#(S') = n'$. π' is an n'-dimensional row vector (the initial state designator). A fuzzy transition function $f'_{B'}$ is defined as follows: For Λ, $x, y \in \Sigma'^*$ and $s, t \in S'$,

$$f'_{B'}(s, \Lambda, t) = \begin{cases} 0 & \text{if} \quad s = t \\ 1 & \text{if} \quad s \neq t, \end{cases}$$

$$f'_{B'}(s, xy, t) = \min_{q \in S'} \max[f'_{B'}(s, x, q), f'_{B'}(q, y, t)],$$

G' is a subset of S' (the set of final states), and an n'-dimensional column vector (the final state designator) $\eta^{G'} = (\eta'_{s_1}, \eta'_{s_2}, \ldots, \eta'_{s_{n'}})'$ is defined such that $\eta'_{s_i} = 0$ if $s_i \in G'$ and $\eta'_{s_i} = 1$ otherwise.

Note that a element of zero in π' means the definite existence of such a initial state.

In this paper, unless stated especially, by a "fuzzy automaton" we shall mean a pessimistic fuzzy automaton.

Let us show the fundamental properties of fuzzy matrices.

We denote by a_{ij} the (i, j)th entry of a fuzzy matrix A, where $0 \leqslant a_{ij} \leqslant 1$. We define:

$$A \prec B \Leftrightarrow a_{ij} \leqslant b_{ij}$$

$$0 = \| 0 \|, \qquad E = \| 1 \|$$

$$C = A \circ B \Leftrightarrow$$
$$c_{ij} = \max_k \min(a_{ik}, b_{kj})$$
$$I = \| m_{ij} \| \quad \text{where} \quad m_{ij} = \begin{cases} 1 \text{ if } i = j \\ 0 \text{ if } i \neq j \end{cases}$$
$$A^{m+1} = A^m \circ A, \qquad A^0 = I$$

$$C = A * B \Leftrightarrow$$
$$c_{ij} = \min_k \max(a_{ik}, b_{kj})$$
$$I' = \| m'_{ij} \| \quad \text{where} \quad m'_{ij} = \begin{cases} 0 \text{ if } i = j \\ 1 \text{ if } i \neq j \end{cases}$$
$$B^{m+1} = B^m * B, \qquad B^0 = I'$$

The following fundamental properties are derived immediately from these definitions:

Fundamental Properties

(1)	$0 \prec A \prec E$	(1)'	$0 \prec B \prec E$
(2)	$A \circ (B \circ C) = (A \circ B) \circ C$	(2)'	$A * (B * C) = (A * B) * C$
(3)	$A \circ I = I \circ A = A$	(3')	$B * I' = I' * B = B$
(4)	$A \circ 0 = 0 \circ A = 0$	(4)'	$B * E = E * B = E$
(5)	$A^p \circ A^q = A^{p+q}, \quad (A^p)^q = A^{pq}$	(5)'	$B^p * B^q = B^{p+q}, \quad (B^p)^q = B^{pq}$
(6)	if $A \prec B$ and $C \prec D$, then $A \circ C \prec B \circ D$.	(6)'	if $A \prec B$ and $C \prec D$, then $A * C \prec B * D$.

The definitions and the properties shown on the left side of the tables given above relate to the operation "$\circ$", and on the right side to the operation "$*$". Moreover, the operations "$\circ$" and "$*$" correspond to a pfa and an ofa, respectively.

The domain of the fuzzy transition matrix F of a fuzzy automaton A can be extended from Σ to Σ^* as follows:

DEFINITION 2.4. For $x = \sigma_1\sigma_2 \cdots \sigma_m \in \Sigma^*$, $\sigma_i \in \Sigma \cup \{\Lambda\}$ and $1 \leqslant i \leqslant m$, define $n \times n$ fuzzy transition matrices $F(x)$ by the following,

(1) $F(\Lambda) = I \qquad (n \times n$ identity matrix),

(2) $F(x) = F(\sigma_1) \circ F(\sigma_2) \circ \cdots \circ F(\sigma_m)$.

Let $F(x) = \|f_{s_i,s_j}(x)\|$ $1 \leqslant i \leqslant n$, $1 \leqslant j \leqslant n$, then obviously

$$f_{s_i,s_j}(x) = f_A(s_i, x, s_j).$$

Now, for $A = \langle S, \pi, \{F(\sigma) \mid \sigma \in \Sigma\}, \eta^G\rangle$, define

$$f_A(x) = \pi \circ F(x) \circ \eta^G, \qquad \text{for} \quad x \in \Sigma^*.$$

$f_A(x)$ is designated as the grade of transition of A, when started with initial distribution π over S to enter into a state in G after scanning the input sequence x. Then a input sequence x is said to be *accepted* by A with grade $f_A(x)$.

Now, by using the Fundamental Properties mentioned above, we have following theorems.

THEOREM 2.1. *For any $n \times n$ fuzzy transition matrix $F(\sigma)$, the sequence $F(\sigma), F(\sigma^2), F(\sigma^3), \ldots$ is ultimately periodic.*

Proof. Let $T = \{f_1, f_2, \ldots, f_l\}$ be the set of all the elements which occur in the matrix $F(\sigma)$, then the number of different matrices which can be obtained by multipling $F(\sigma)$ is at most l^{n^2}, that is, finite, THEOREM 2.2. *If $I \prec F(\sigma)$, then*

$$I \prec F(\sigma) \prec F(\sigma^2) \prec \cdots \prec F(\sigma^{n-1}) = F(\sigma^n) = F(\sigma^{n+1}) = \cdots.$$

Proof. We can prove our theorem in a similar way in a Boolean matrix [11].

3. FUZZY LANGUAGE

E. S. Santos [4] showed that the capability of fuzzy automata as acceptor is the same as that of finite automata, though fuzzy automata include the deterministic and non-deterministic finite automata as special cases.

We show that every fuzzy language can be represented in a fuzzy automaton with any threshold λ such that $0 \leqslant \lambda < 1$.

DEFINITION 3.1. Let $A = \langle S, \pi, \{F(\sigma) \mid \sigma \in \Sigma\}, \eta^G \rangle$ be a fuzzy automaton and λ a real number $0 \leqslant \lambda < 1$. The set of all input sequences accepted by A with parameter λ is defined as

$$L(A, \circ, \lambda) = \{x \mid f_A(x) > \lambda, x \in \Sigma^*\}.$$

λ is called a *threshold* of A and $L(A, \circ, \lambda)$ a *λ-fuzzy language*. For $0 \leqslant \lambda < 1$, a language L is λ-fuzzy if and only if there exists a A such that $L = L(A, \circ, \lambda)$. A language L is fuzzy if and only if, for some λ, it is λ-fuzzy.

THEOREM 3.1. (Santos, 1968) *λ-fuzzy language $L(A, \circ, \lambda)$ is a regular language.*

The same theorem also holds for an optimistic fuzzy automaton.

DEFINITION 3.2. For a fuzzy matrix $A = \| a_{ij} \|$, $0 \leqslant a_{ij} \leqslant 1$ and d a real number such that $0 \leqslant d \leqslant 1$, we define a fuzzy matrix $A' = \| a'_{ij} \|$ as follows:

$$a'_{ij} = \begin{cases} a_{ij} + d & \text{if} \quad a_{ij} \leqslant 1 - d \\ 1 & \text{otherwise.} \end{cases}$$

LEMMA 3.1. *For two fuzzy matrices U and V of same order, let the fuzzy matrices defined in Definition 3.2 be U' and V', respectively, then, for two fuzzy matrices $W = \| w_{ij} \|$ and $W' = \| w'_{ij} \|$ such that $W = U \circ V$ and $W' = U' \circ V'$, we have that*

$$w'_{ij} = \begin{cases} w_{ij} + d & \text{if} \quad w_{ij} \leqslant 1 - d \\ 1 & \text{otherwise.} \end{cases}$$

Proof. It is clear from the property of the operation "$\circ$".

Likewise, for a fuzzy matrix $A = \| a_{ij} \|$ and d' a real number $0 \leqslant d' \leqslant 1$, define a fuzzy matrix $A'' = \|a''_{ij}\|$ as follows:

$$a''_{ij} = \begin{cases} a_{ij} - d' & \text{if} \quad a_{ij} \geqslant d' \\ 0 & \text{otherwise.} \end{cases}$$

Then a similar result as in Lemma 3.1 holds.

THEOREM 3.2. *Every fuzzy language is λ-fuzzy, for any λ such that* $0 \leqslant \lambda < 1$.

Proof. Let $L = L(A, \circ, \mu)$ and let $A = \langle S, \pi, \{F(\sigma) \mid \sigma \in \Sigma\}, \eta^G \rangle$ be a fuzzy automaton, where $F(\sigma) = \| f_{s_i, s_j}(\sigma) \|$, $\pi = (\pi_{s_i})$, $s_i, s_j \in S$ and $\sigma \in \Sigma$. Omitting the trivial case $\lambda = \mu$, we can assume that $\lambda \neq \mu$.

(1) For the case of $\lambda > \mu$;

Consider the fuzzy automaton $A' = \langle S', \pi', \{F'(\sigma) \mid \sigma \in \Sigma'\}, \eta^{G'} \rangle$ where $S' = S$, $\Sigma' = \Sigma$, $\eta^{G'} = \eta^G$, and the fuzzy transition matrices $F'(\sigma) = \| f'_{s_i, s_j}(\sigma) \|$, $\sigma \in \Sigma'$ and $\pi' = (\pi'_{s_i})$ are defined as follows:

$$f'_{s_i, s_j}(\sigma) = \begin{cases} f_{s_i, s_j}(\sigma) + (\lambda - \mu) & \text{if} \quad f_{s_i, s_j}(\sigma) \leqslant 1 - \lambda + \mu, \\ 1 & \text{otherwise.} \end{cases}$$

$$\pi'_{s_i} = \begin{cases} \pi_{s_i} + (\lambda - \mu) & \text{if} \quad \pi_{s_i} \leqslant 1 - \lambda + \mu, \\ 1 & \text{otherwise.} \end{cases}$$

Thus, according to Lemma 3.1., for $x \in \Sigma^*$,

$$\pi' \circ F'(x) \circ \eta^{G'} = \begin{cases} \pi \circ F(x) \circ \eta^G + (\lambda - \mu) & \text{if} \quad \pi \circ F(x) \circ \eta^G \leqslant 1 - \lambda + \mu, \\ 1 & \text{otherwise.} \end{cases}$$

Therefore, $L(A', \circ, \lambda) = L(A, \circ, \mu)$ when $\lambda > \mu$.

(2) For the case of $\lambda < \mu$:

Consider the fuzzy automaton $A'' = \langle S'', \pi'', \{F''(\sigma) \mid \sigma \in \Sigma''\}, \eta^{G''} \rangle$ where $S'' = S$, $\Sigma'' = \Sigma$, $\eta^{G''} = \eta^G$, and the fuzzy transition matrices $F''(\sigma) = \| f''_{s_i, s_j}(\sigma) \|$ where $\sigma \in \Sigma''$, and $\pi'' = (\pi''_{s_i})$ are defined as follows:

$$f''_{s_i, s_j}(\sigma) = \begin{cases} f_{s_i, s_j}(\sigma) - (\mu - \lambda) & \text{if} \quad f_{s_i, s_j}(\sigma) \geqslant \mu - \lambda, \\ 0 & \text{otherwise.} \end{cases}$$

$$\pi''_{s_i} = \begin{cases} \pi_{s_i} - (\mu - \lambda) & \text{if} \quad \pi_{s_i} \geqslant \mu - \lambda, \\ 0 & \text{otherwise.} \end{cases}$$

Thus, for $x \in \Sigma^*$,

$$\pi'' \circ F''(x) \circ \eta^{G''} = \begin{cases} \pi \circ F(x) \circ \eta^G - (\mu - \lambda) & \text{if} \quad \pi \circ F(x) \circ \eta^G \geqslant \mu - \lambda, \\ 0 & \text{otherwise,} \end{cases}$$

Therefore, $L(A'', \circ, \lambda) = L(A, \circ, \mu)$ when $\lambda < \mu$. Hence, it follows that in both cases $L = L(A', \circ, \lambda)$ or $L(A'', \circ, \lambda)$, which implies our theorem.

It is easily shown that the same theorem holds for an optimistic fuzzy automaton.

4. Closure Properties of Fuzzy Automata

In this section, we use the concept of fuzzy sets instead of the set of input sequences with threshold λ.

It is shown that a family of fuzzy events characterized by not only pessimistic fuzzy automata (pfa for short) but also optimistic fuzzy automata (ofa for short) is closed under the operations of intersection and union in the fuzzy sense. And the complement of the fuzzy event by a pfa (an ofa) is characterized by an ofa (a pfa).

Definition 4.1. For a pfa $A = \langle S, \pi, \{F(\sigma) \mid \sigma \in \Sigma\}, \eta^G \rangle$, let a *fuzzy event* be the fuzzy set in Σ^* which is characterized by $f_A(x) = \pi \circ F(x) \circ \eta^G$, where $x \in \Sigma^*$.

We denote by $\tilde{L}(A, \circ)$ the fuzzy event by a pfa A and, similarly, by $\tilde{L}(B, *)$ the fuzzy event by an ofa B.

Definition 4.2. For two pfa A_1 and A_2

$$A_1 = \langle S_1, \pi_1, \{F_1(\sigma) \mid \sigma \in \Sigma\}, \eta^{G_1} \rangle$$

$$A_2 = \langle S_2, \pi_2, \{F_2(\sigma) \mid \sigma \in \Sigma\}, \eta^{G_2} \rangle,$$

define a *min pfa* $A_1 \otimes A_2$ as follows:

$$A_1 \otimes A_2 = \langle S, \pi, \{F(\sigma) \mid \sigma \in \Sigma\}, \eta^G \rangle,$$

where

$$S = S_1 \times S_2 = \{(s_i, t_j) \mid s_i \in S_1, t_j \in S_2, 1 \leqslant i \leqslant m, 1 \leqslant j \leqslant n\}$$

$G = G_1 \times G_2$, $m = \#(S_1)$ and $n = \#(S_2)$. As to the fuzzy transition function $f_{A_1 \otimes A_2}$ of a min pfa $A_1 \otimes A_2$, define

$$f_{A_1 \otimes A_2}((s, t), \sigma, (q, r)) = \min[f_{A_1}(s, \sigma, q), f_{A_2}(t, \sigma, r)]$$

for $(s, t), (q, r) \in S$ and $\sigma \in \Sigma$.

Moreover, the mn-dimensional row vector π is defined as follows: For $(s_i, t_j) \in S$, $1 \leqslant i \leqslant m$ and $1 \leqslant j \leqslant n$,

$$\pi = \pi_1 \otimes \pi_2 = (\xi_{(s_i, t_j)})$$

where

$$\xi_{(s_i,t_j)} = \min[\pi_{1s_i}, \pi_{2t_j}]$$

and

$$(s_i, t_j) = (s_1, t_1), (s_1, t_2),..., (s_1, t_n), (s_2, t_1),..., (s_m, t_n).$$

And the mn-dimensional column vector η^G is also $\eta^G = \eta^{G_1} \otimes \eta^{G_2}$.

Hence, the fuzzy transition matrices of order mn of $A_1 \otimes A_2$ is as follows: For two pfa A_1 and A_2, let $F_1(\sigma) = \| f_{s_i,s_j}(\sigma) \|$ and $F_2(\sigma) = \| f_{t_k,t_l}(\sigma) \|$ where $\sigma \in \Sigma$, be fuzzy transition matrices of A_1 and A_2, respectively, then fuzzy transition matrices $F(\sigma)$, $\sigma \in \Sigma$ of $A_1 \otimes A_2$ is defined by

$$F(\sigma) = F_1(\sigma) \otimes F_2(\sigma) = \| f_{(s_i,t_k),(s_j,t_l)}(\sigma) \|,$$

where

$$f_{(s_i,t_k),(s_j,t_l)}(\sigma) = \min[f_{s_i,s_j}(\sigma), f_{t_k,t_l}(\sigma)]$$
$$= f_{A_1 \otimes A_2}((s_i, t_k), \sigma, (s_j, t_l)).$$

Note that the operation $\otimes$ of fuzzy matrices corresponds to the tensor product of ordinary matrices.

LEMMA 4.1. *For fuzzy matrices A_1, A_2, B_1, B_2, A and B, for row vectors π_1 and π_2, and for column vectors η^{G_1} and η^{G_2}, we have that*

(1) $(A_1 \circ B_1) \otimes (A_2 \circ B_2) = (A_1 \otimes A_2) \circ (B_1 \otimes B_2)$

(2) $(\pi_1 \circ A \circ \eta^{G_1}) \otimes (\pi_2 \circ B \circ \eta^{G_2}) = (\pi_1 \otimes \pi_2) \circ (A \otimes B) \circ (\eta^{G_1} \otimes \eta^{G_2})$
$= \min[\pi_1 \circ A \circ \eta^{G_1}, \pi_2 \circ B \circ \eta^{G_2}]$

(3) $A_1 \otimes A_2$, $\pi_1 \otimes \pi_2$,..., *are fuzzy matrices.*

Proof. Obvious. See [*10*].

This enables us to prove the following closure theorem.

THEOREM 4.1. *Let A_1, A_2 and $A_1 \otimes A_2$ be* pfa *as in Definition* 4.2 *and $\tilde{L}(A_1, \circ)$, $\tilde{L}(A_2, \circ)$ and $\tilde{L}(A_1 \otimes A_2, \circ)$ be the fuzzy events characterized by A_1, A_2 and $A_1 \otimes A_2$, respectively. Then, in the fuzzy sense,*

$$\tilde{L}(A_1, \circ) \cap \tilde{L}(A_2, \circ) = \tilde{L}(A_1 \otimes A_2, \circ).$$

Proof. The membership functions of fuzzy events $\tilde{L}(A_1, \circ)$, $\tilde{L}(A_2, \circ)$ and $\tilde{L}(A_1 \otimes A_2, \circ)$ are

$$f_{A_1}(x) = \pi_1 \circ F_1(x) \circ \eta^{G_1}, \qquad f_{A_2}(x) = \pi_2 \circ F_2(x) \circ \eta^{G_2}$$

and

$$f_{A_1 \otimes A_2}(x) = \pi \circ F(x) \circ \eta^G,$$

respectively, where $x \in \Sigma^*$. By Lemma 4.1., for $x \in \Sigma^*$,

$$\begin{aligned} f_{A_1 \otimes A_2}(x) &= \pi \circ F(x) \circ \eta^G = (\pi_1 \otimes \pi_2) \circ (F_1(x) \otimes F_2(x)) \circ (\eta^{G_1} \otimes \eta^{G_2}) \\ &= \min[\pi_1 \circ F_1(x) \circ \eta^{G_1}, \pi_2 \circ F_2(x) \circ \eta^{G_2}] \\ &= \min[f_{A_1}(x), f_{A_2}(x)]. \end{aligned}$$

COROLLARLY 4.1. *For two* ofa B_1 *and* B_2, *and* $\tilde{L}(B_1, *)$ *and* $\tilde{L}(B_2, *)$ *be the fuzzy events by* B_1 *and* B_2, *respectively, then, in the fuzzy sense, there exists an* ofa B *such that*

$$\tilde{L}(B_1, *) \cup \tilde{L}(B_2, *) = \tilde{L}(B, *).$$

Proof. In Definition 4.2, by replacing the operation "min" by the operation "max" and defining a max ofa, we can easily prove Corollary 4.1.

We have shown that the family of fuzzy events by pfa is closed under intersection and the family by ofa is closed under union in the fuzzy sense.

Next, we will verify that the family of fuzzy events by pfa (ofa) is closed under union (intersection) in the fuzzy sense.

THEOREM 4.2. *For two* pfa A_1 *and* A_2, *let* $\tilde{L}(A_1, \circ)$ *and* $\tilde{L}(A_2, \circ)$ *be fuzzy events by* A_1 *and* A_2, *respectively, then, in the fuzzy sense, there exists a* pfa A *such that*

$$\tilde{L}(A_1, \circ) \cup \tilde{L}(A_2, \circ) = \tilde{L}(A, \circ).$$

Proof. Let A_1 and A_2 be two pfa as follows:

$$\begin{aligned} A_1 &= \langle \{s_1, s_2, \ldots, s_m\}, \pi_1, \{F_1(\sigma) \mid \sigma \in \Sigma\}, \eta^{G_1} \rangle \\ A_2 &= \langle \{t_1, t_2, \ldots, t_n\}, \pi_2, \{F_2(\sigma) \mid \sigma \in \Sigma\}, \eta^{G_2} \rangle \end{aligned}$$

Now, consider a pfa A, that is,

$$A = \langle \{s_1, \ldots, s_m, t_1, \ldots, t_n\}, \pi, \{F(\sigma) \mid \sigma \in \Sigma\}, \eta^G \rangle$$

where π, $F(\sigma)$ and η^G are given as follows;

If

$$\pi_1 = (\pi_{s_1}, \pi_{s_2}, \ldots, \pi_{s_m}) \quad \text{and} \quad \pi_2 = (\pi_{t_1}, \pi_{t_2}, \ldots, \pi_{t_n})$$

then

$$\pi = (\pi_{s_1}, \ldots, \pi_{s_m}, \pi_{t_1}, \ldots, \pi_{t_n}) = (\pi_1 \pi_2).$$

Moreover,

$$F(\sigma) = \begin{pmatrix} F_1(\sigma) & \mathbb{O} \\ \mathbb{O} & F_2(\sigma) \end{pmatrix}$$

and

$$\eta^G = \begin{pmatrix} \eta^{G_1} \\ \eta^{G_2} \end{pmatrix}.$$

In general, in fuzzy matrices, we have that

$$\text{(1)} \qquad \begin{pmatrix} A_1 & \mathbb{O} \\ \mathbb{O} & B_1 \end{pmatrix} \circ \begin{pmatrix} A_2 & \mathbb{O} \\ \mathbb{O} & B_2 \end{pmatrix} = \begin{pmatrix} A_1 \circ A_2 & \mathbb{O} \\ \mathbb{O} & B_1 \circ B_2 \end{pmatrix}$$

$$\text{(2)} \qquad (\pi_1 \pi_2) \circ \begin{pmatrix} A & \mathbb{O} \\ \mathbb{O} & B \end{pmatrix} \circ \begin{pmatrix} \eta^{G_1} \\ \eta^{G_2} \end{pmatrix} = \max[\pi_1 \circ A \circ \eta^{G_1}, \pi_2 \circ B \circ \eta^{G_2}]$$

Therefore, let

$$f_{A_1}(x) = \pi_1 \circ F_1(x) \circ \eta^{G_1}, \qquad f_{A_2}(x) = \pi_2 \circ F_2(x) \circ \eta^{G_2}$$

and $f_A(x) = \pi \circ F(x) \circ \eta^G$, where $x \in \Sigma^*$ be the membership functions which characterize fuzzy events $\tilde{L}(A_1, \circ)$, $\tilde{L}(A_2, \circ)$ and $\tilde{L}(A, \circ)$, respectively.

Then, for $x \in \Sigma^*$, we have;

$$\begin{aligned} f_A(x) &= \pi \circ F(x) \circ \eta^G = (\pi_1 \pi_2) \circ \begin{pmatrix} F_1(x) & \mathbb{O} \\ \mathbb{O} & F_2(x) \end{pmatrix} \circ \begin{pmatrix} \eta^{G_1} \\ \eta^{G_2} \end{pmatrix} \\ &= \max[\pi_1 \circ F_1(x) \circ \eta^{G_1}, \pi_2 \circ F_2(x) \circ \eta^{G_2}] \\ &= \max[f_{A_1}(x), f_{A_2}(x)]. \end{aligned}$$

COROLLARY 4.2. *For two* ofa B_1 *and* B_2, *let* $\tilde{L}(B_1, *)$ *and* $\tilde{L}(B_2, *)$ *be the fuzzy events by* B_1 *and* B_2, *respectively, then, in the fuzzy sense, there exists an* ofa B *such that*

$$\tilde{L}(B_1, *) \cap \tilde{L}(B_2, *) = \tilde{L}(B, *).$$

Proof. For two ofa B_1 and B_2, that is,

$$B_1 = \langle S_1, \pi_1, \{F_1(\sigma) \mid \sigma \in \Sigma\}, \eta^{G_1} \rangle$$

$$B_2 = \langle S_2, \pi_2, \{F_2(\sigma) \mid \sigma \in \Sigma\}, \eta^{G_2} \rangle,$$

let us define an ofa B as follows:

$$B = \langle S, \pi, \{F(\sigma) \mid \sigma \in \Sigma\}, \eta^G \rangle$$

where $S = S_1 \cup S_2$, $S_1 \cap S_2 = \phi$, $\pi = (\pi_1 \pi_2)$

$$F(\sigma) = \begin{pmatrix} F_1(\sigma) & \mathbb{1} \\ \mathbb{1} & F_2(\sigma) \end{pmatrix} \qquad \text{for all } \sigma \text{ in } \Sigma,$$

$$\eta^G = \begin{pmatrix} \eta^{G_1} \\ \eta^{G_2} \end{pmatrix}.$$

Then, we can prove our Corollary immediately in a similar way as in Theorem 4.2. We will show the inclusion property of pfa.

THEOREM 4.3. *Given two* pfa A_1 *and* A_2 *as follows:*

$$A_1 = \langle S_1, \pi_1, \{F_1(\sigma) \mid \sigma \in \Sigma\}, \eta^{G_1} \rangle$$
$$A_2 = \langle S_2, \pi_2, \{F_2(\sigma) \mid \sigma \in \Sigma\}, \eta^{G_2} \rangle.$$

If $\#(S_1) = \#(S_2)$, $F_1(\sigma) \prec F_2(\sigma)$ *for all* σ *in* Σ,

$$\pi_1 \prec \pi_2 \qquad \text{and} \qquad \eta^{G_1} \prec \eta^{G_2},$$

then, in the fuzzy sense,

$$\tilde{L}(A_1, \circ) \subseteqq \tilde{L}(A_2, \circ).$$

Proof. We can easily show that

$$f_{A_1}(x) \leqslant f_{A_2}(x) \qquad \text{for} \qquad x \in \Sigma^*$$

from the basic properties of fuzzy matrix described in Section 2.

Obviously, the same theorem also holds for ofa.

We will show the complement of fuzzy event by a pfa (an ofa) is characterized by an ofa (a pfa).

DEFINITION 4.3. If $A = \langle S, \pi, \{F(\sigma) \mid \sigma \in \Sigma\}, \eta^G \rangle$ is a pfa, the complementary ofa for A is defined as

$$\bar{A} = \langle \bar{S}, \bar{\pi}, \{\bar{F}(\sigma) \mid \sigma \in \Sigma\}, \eta^{\bar{G}} \rangle$$

where $\bar{S} = S$. As to fuzzy transition function $f_{\bar{A}}$ of $\bar{A}$, for $\sigma \in \Sigma$, $x, y \in \Sigma^*$, $s, t \in \bar{S}$ and f_A of A, we define,

$$f_{\bar{A}}(s, \sigma, t) = 1 - f_A(s, \sigma, t),$$
$$f_{\bar{A}}(s, xy, t) = \min_{l \in S} \max[f_{\bar{A}}(s, x, l), f_{\bar{A}}(l, y, t)]$$
$$= 1 - f_A(s, xy, t),$$

and the initial and final state vectors are

$$\bar{\pi} = (1, 1,..., 1) - \pi, \qquad \text{and} \qquad \eta^{\bar{G}} = (1, 1,..., 1)' - \eta^G.$$

Note that we can easily define a complementary pfa $\bar{B}$ for an ofa B in a similar way.

LEMMA 4.2. *For a fuzzy matrix $U = \| u_{ij} \|$, let $U' = \| u'_{ij} \|$ be a fuzzy matrix such that*

$$u'_{ij} = 1 - u_{ij}\,.$$

For fuzzy matrices $U_1, U_2,..., U_m$, let $U_1', U_2',..., U_m'$ be fuzzy matrices as defined above, respectively, then

$$U_1 \circ U_2 \circ \cdots \circ U_m + U_1' * U_2' * \cdots * U_m' = \begin{pmatrix} 1 & \cdots & 1 \\ \vdots & \cdots & \vdots \\ 1 & \cdots & 1 \end{pmatrix}$$

THEOREM 4.4. *Let A be a* pfa *and let $\bar{A}$ be a complementary* ofa *for A, then, in the fuzzy sense,*

$$\overline{\tilde{L}(A, \circ)} = \tilde{L}(\bar{A}, *).$$

Proof. Let $A = \langle S, \pi, \{F(\sigma) \mid \sigma \in \Sigma\}, \eta^G\rangle$ be a pfa and $\bar{A} = \langle S, \bar{\pi}, \{\bar{F}(\sigma) \mid \sigma \in \Sigma\}, \eta^{\bar{G}}\rangle$ be a complementary ofa for A, then by Lemma 4.2, for $x \in \Sigma^*$,

$$f_A(x) = \pi \circ F(x) \circ \eta^G = 1 - \bar{\pi} * \bar{F}(x) * \eta^{\bar{G}} = 1 - f_{\bar{A}}(x).$$

Therefore, we have $\overline{\tilde{L}(A, \circ)} = \tilde{L}(\bar{A}, *)$.

COROLLARY 4.3. *For an* ofa *B and a complementary* pfa *$\bar{B}$ for B, in the fuzzy sense,*

$$\overline{\tilde{L}(B, *)} = \tilde{L}(\bar{B}, \circ).$$

Proof. Immediately.

The family of fuzzy events characterized by pfa (ofa) constitutes a distributive lattice, but does not constitute a Boolean lattice clearly.

5. Conclusion

The threshold of fuzzy automata can be set arbitrarily by changing the value of each element of the fuzzy transition matrix and the initial state designator. Moreover, a family of fuzzy events characterized by pessimistic (optimistic) fuzzy automata is closed under the operations of union and intersection in the sense of fuzzy sets. And the complement of the fuzzy event by a pessimistic (an optimistic) fuzzy automaton is characterized by an optimistic (a pessimistic) fuzzy automaton.

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