ELSEVIER

Fuzzy Sets and Systems 87 (1997) 47–56

FUZZY sets and systems

Reasoning conditions on Kóczy's interpolative reasoning method in sparse fuzzy rule bases. Part II

Yan Shi*, Masaharu Mizumoto

Division of Information and Computer Sciences, Osaka Electro-Communication University, Neyagawa, Osaka 572, Japan

Received January 1995; revised January 1996

Abstract

In our pre-work, the two sufficient and necessary conditions have been given on Kóczy's interpolative reasoning method in sparse fuzzy rule bases, to guarantee that the reasoning consequence is of triangular-type if the fuzzy rules and an observation are defined by triangular-type membership functions. However, the two conditions are too strong to use the reasoning method on practical applications. In this paper, we analyze the properties of the reasoning method in detail and give several applicable sufficient conditions on it, in order to make the reasoning consequence always a normal and convex fuzzy set. © 1997 Elsevier Science B.V.

Keywords: Interpolative reasoning; Sparse fuzzy rule bases; Convexity; Monotone increasing function

1. Introduction

Most of the fuzzy reasoning methods used in fuzzy control and fuzzy expert systems are based on the compositional rule of inference [9]. In these fuzzy reasoning methods the input universe of discourse is covered with the rule bases completely, and when an observation is given, a consequence can be calculated by some proper rules of inference [5, 4]. However, when the fuzzy rule base is sparse, i.e., the input universe of discourse is covered incompletely with the rule base, there will be an empty space between two neighboring antecedents, and thus the conventional fuzzy reasoning methods do not work well. Namely, if an observation comes in the empty space, no rule will be fired and no consequence is derived. To attack this problem, Kóczy and Hirota have proposed a fuzzy reasoning method called a linear interpolative reasoning in [2, 3]. Using the method, we can obtain the reasoning consequence for an observation flanked by two disjoint antecedents. However, it was shown in [7, 6] that the reasoning consequences by the method are sometimes funny fuzzy sets. In some cases, the membership functions of the consequences lose the convexity, and so it is very difficult to use the linear interpolative reasoning method for the practical applications.

In this paper, we analyze the properties of the linear interpolative reasoning method, and give several applicable conditions on it, to show that if fuzzy rules $A_1 \Rightarrow B_1$, $A_2 \Rightarrow B_2$ and an observation A^* are defined

*Corresponding author.

0165-0114/97/$17.00
PII S0165-0114(96)00031-0

by triangular (or trapezoidal) membership functions, the interpolated conclusion B^* will preserve the convexity. That is, B^* is always a normal and convex fuzzy set under the conditions. Furthermore, we can show that two kinds of applicable conditions are special cases of the reasoning conditions given by Shi et al. [7, 6].

2. The linear interpolative reasoning in sparse fuzzy rule bases

In "If A_i then B_i" ($i = 1, 2, \ldots, n$) fuzzy rule models, a reasoning consequence can be calculated by means of the compositional rule of inference [5, 4] for a given observation A^* in the following:

$$B^* = A^* \circ R,$$

where $\circ$ denotes a relational composition, and R is a fuzzy relation representing the set of fuzzy rules.

Usually, in the case of the compositional rule of inference, the input universe of discourse X is supposed to be covered completely with the antecedents A_i ($i = 1, 2, \ldots, n$), i.e. the following relation:

$$X = \bigcup \mathrm{supp}(A_i)$$

always holds, where $\mathrm{supp}(A_i)$ stands for the support of A_i. In other words, $\mathrm{supp}(A^*) \subseteq \bigcup \mathrm{supp}(A_i)$ is always true. We call such fuzzy rule bases as the dense rule bases [1]. In this case, for any observation A^*, we can obtain the consequence B^* using a proper fuzzy relation such as the arithmetic rule by Zadeh [9].

However, if the fuzzy rule base is sparse, i.e. there exists at least a subset X' of X such that

$$X' \subset X - \bigcup \mathrm{supp}(A_i),$$

then the conventional fuzzy reasoning methods do not work well when the following relation holds for an observation A^*:

$$X' = \mathrm{supp}(A^*).$$

As seen in Fig. 1, when an observation A^* is flanked by two disjoint antecedents A_1 and A_2, the consequence B^* would be nothing by the conventional fuzzy reasoning methods.

For such problem, Kóczy and Hirota [2, 3] pointed out that the consequence B^* should also be flanked by two consequences B_1 and B_2, and proposed a reasoning method named a linear interpolate reasoning, which is introduced briefly as follows.

Definition 1. Denote the set of all normal and convex fuzzy sets in the universe X_i by $P(X_i)$. Then for $A_1, A_2 \in P(X_i)$, if the following conditions hold:

$$\inf\{A_{1\alpha}\} < \inf\{A_{2\alpha}\}, \qquad \sup\{A_{1\alpha}\} < \sup\{A_{2\alpha}\} \quad \forall \alpha \in (0, 1], \tag{1}$$

then it is said that A_1 is less than A_2, i.e. $A_1 < A_2$, where $A_{1\alpha}$ and $A_{2\alpha}$ are α-cut sets of A_1 and A_2, respectively, and $\inf\{A_{i\alpha}\}$ is the infimum of $A_{i\alpha}$ and $\sup\{A_{i\alpha}\}$ is the supremum of $A_{i\alpha}$ ($i = 1, 2$).

Definition 2. Let A_1 and A_2 be fuzzy sets on the universe of discourse X with $|X| < \infty$, and $A_1 < A_2$, then the lower and the upper distances between α-cut sets $A_{1\alpha}$ and $A_{2\alpha}$ are defined as

$$d_L(A_{1\alpha}, A_{2\alpha}) = d(\inf\{A_{1\alpha}\}, \inf\{A_{2\alpha}\}), \tag{2}$$

$$d_U(A_{1\alpha}, A_{2\alpha}) = d(\sup\{A_{1\alpha}\}, \sup\{A_{2\alpha}\}), \tag{3}$$

where d is Euclidean distance or, more generally, Minkowski distance.

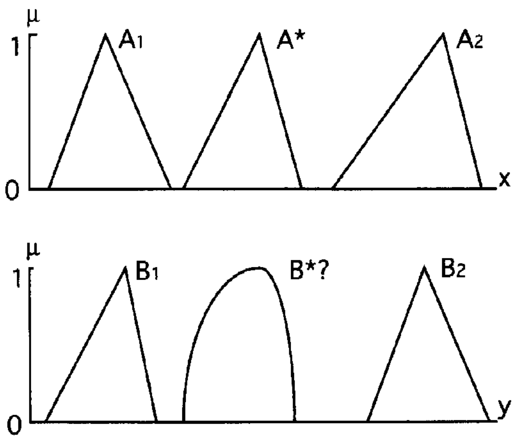

Fig. 1. The fuzzy reasoning in the sparse rule base.

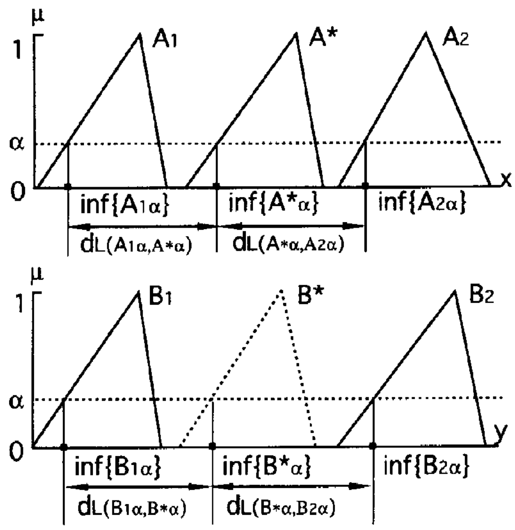

Fig. 2. Linear interpolative reasoning method.

Definition 3. Let $A_1 \Rightarrow B_1, A_2 \Rightarrow B_2$ be disjoint fuzzy rules on the universe of discourse $X \times Y$, and A_1, A_2 and B_1, B_2 be fuzzy sets on X and Y, respectively. Assume that A^* is an observation of the input universe X. If $A_1 < A^* < A_2$ then the linear fuzzy rule interpolation between two fuzzy rules is defined as

$$d_L(A_{1\alpha}, A_\alpha^*) : d_L(A_\alpha^*, A_{2\alpha}) = d_L(B_{1\alpha}, B_\alpha^*) : d_L(B_\alpha^*, B_{2\alpha}), \tag{4}$$

$$d_U(A_{1\alpha}, A_\alpha^*) : d_U(A_\alpha^*, A_{2\alpha}) = d_U(B_{1\alpha}, B_\alpha^*) : d_U(B_\alpha^*, B_{2\alpha}), \tag{5}$$

where $\alpha \in [0, 1]$.

Solving Eqs. (4) and (5) using Eqs. (2) and (3), we obtain $\inf\{B_\alpha^*\}$ and $\sup\{B_\alpha^*\}$ as follows:

$$\inf\{B_\alpha^*\} = \frac{d_L(A_{1\alpha}, A_\alpha^*) \inf\{B_{2\alpha}\} + d_L(A_\alpha^*, A_{2\alpha}) \inf\{B_{1\alpha}\}}{d_L(A_{1\alpha}, A_\alpha^*) + d_L(A_\alpha^*, A_{2\alpha})}, \tag{6}$$

$$\sup\{B_\alpha^*\} = \frac{d_U(A_{1\alpha}, A_\alpha^*) \sup\{B_{2\alpha}\} + d_U(A_\alpha^*, A_{2\alpha}) \sup\{B_{1\alpha}\}}{d_U(A_{1\alpha}, A_\alpha^*) + d_U(A_\alpha^*, A_{2\alpha})}. \tag{7}$$

Fig. 2 shows a simple explanation of the linear interpolative reasoning method for (4) with the fuzzy rules and the observation arranged by triangular-type membership functions.

Using the resolution principle [8] for $\inf\{B_\alpha^*\}$ and $\sup\{B_\alpha^*\}$ with $\alpha \in [0, 1]$ given in (6) and (7), we can obtain the reasoning consequence B^* for an observation A^* flanked by two antecedents A_1 and A_2 under which the disjoint fuzzy rules $A_1 \Rightarrow B_1, A_2 \Rightarrow B_2$, and the observation A^* are arranged by any type of membership functions. Especially, when the disjoint fuzzy rules $A_1 \Rightarrow B_1$, $A_2 \Rightarrow B_2$, and the observation A^* are defined by triangular-type (or trapezoidal-type) membership functions, the computation size becomes very small [2, 3]. In this case, we can express Eqs. (6) and (7) in the simple forms.

For example, assume that the left slopes of A_1, A^* and A_2 are denoted as $1/k_1$, $1/k$ and $1/k_2$, and the left slopes of B_1 and B_2 are as $1/h_1$ and $1/h_2$, and $a_1 = \inf\{A_{1\alpha}\}$, $a = \inf\{A_\alpha^*\}$, $a_2 = \inf\{A_{2\alpha}\}$, $b_1 = \inf\{B_{1\alpha}\}$,

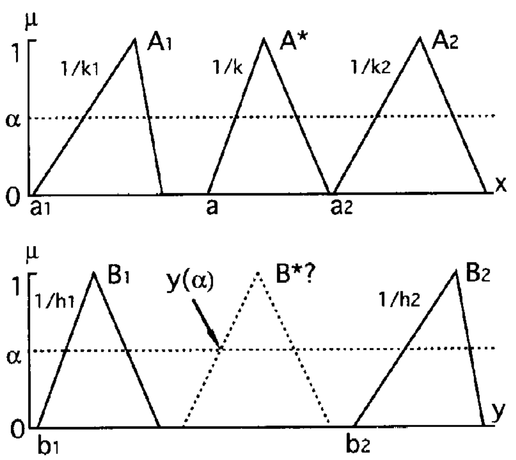

Fig. 3. Interpolation with triangular membership functions.

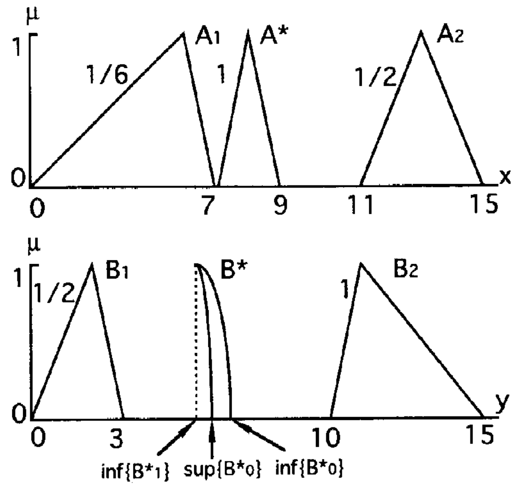

Fig. 4. Interpolation by the linear interpolative reasoning method.

$b_2 = \inf\{B_{2\alpha}\}$ at $\alpha = 0$, respectively, as shown in Fig. 3. Then, Eq. (6) can be written as follows:

$$\inf\{B^*_\alpha\} = y(\alpha) = \frac{[(k-k_1)\alpha + (a-a_1)](h_2\alpha + b_2) + [(k_2-k)\alpha + (a_2-a)](h_1\alpha + b_1)}{(k_2-k_1)\alpha + (a_2-a_1)}$$

$$= \{[h_1(k_2-k) + h_2(k-k_1)]\alpha^2 + [b_1(k_2-k) + b_2(k-k_1) + h_1(a_2-a) + h_2(a-a_1)]\alpha$$
$$+ [b_1(a_2-a) + b_2(a-a_1)]\}/\{(k_2-k_1)\alpha + (a_2-a_1)\}. \quad (8)$$

As seen in Eq. (8), $\inf\{B^*_\alpha\}$ is a hyperbolic function of $\alpha \in [0, 1]$ in general. That is to say, the interpolated conclusion B^* would not be a triangular-type membership function even though the rules $A_1 \Rightarrow B_1$ and $A_2 \Rightarrow B_2$, and the observation A^* are all defined by triangular-type membership functions. And furthermore, B^* does not sometimes preserve convexity since there occurs such an unusual case that $\inf\{B^*_1\} < \inf\{B^*_0\}$ (or $\sup\{B^*_1\} > \sup\{B^*_0\}$), or more $\inf\{B^*_\alpha\} > \sup\{B^*_\alpha\}$ as shown in Fig. 4 in which the membership function of B^* is a funny fuzzy set.

In order to make the membership function of B^* to be of triangular-type, Shi et al. [7] have given two sufficient conditions for the linear interpolative reasoning method under which the membership function of the consequence B^* can be guaranteed to be of triangular-type, i.e. two sufficient conditions to guarantee the linearity and monotonicity of $\inf\{B^*_\alpha\}$. Furthermore, Shi and Mizumoto [6] have proved that the two conditions are also necessary conditions which are simply described as follows.

Condition I [7, 6]. If two disjoint fuzzy rules $A_1 \Rightarrow B_1$, $A_2 \Rightarrow B_2$ and an observation A^* ($A_1 < A^* < A_2$) are defined by triangular-type membership functions, and $\inf\{A_{1\alpha}\} = k_1\alpha + a_1$, $\inf\{A^*_\alpha\} = k\alpha + a$, $\inf\{A_{2\alpha}\} = k_2\alpha + a_2$, $\inf\{B_{1\alpha}\} = h_1\alpha + b_1$, $\inf\{B_{2\alpha}\} = h_2\alpha + b_2$, respectively, then the interpolated conclusion $\inf\{B^*_\alpha\}$ is a linear monotone increasing function if and only if the following conditions hold:

$$k = \beta k_2 + (1-\beta)k_1, \quad (9)$$

$$a = \beta a_2 + (1-\beta)a_1, \quad (10)$$

where $\beta \in [0, 1]$ is a constant.

Condition II [7, 6]. If two disjoint fuzzy rules $A_1 \Rightarrow B_1$, $A_2 \Rightarrow B_2$ and an observation A^* ($A_1 < A^* < A_2$) are defined by triangular-type membership functions, and $\inf\{A_{1\alpha}\} = k_1\alpha + a_1$, $\inf\{A^*_\alpha\} = k\alpha + a$, $\inf\{A_{2\alpha}\} =$

$k_2\alpha + a_2$, $\inf\{B_{1\alpha}\} = h_1\alpha + b_1$, $\inf\{B_{2\alpha}\} = h_2\alpha + b_2$, respectively, then the interpolated conclusion $\inf\{B^*_\alpha\}$ is a linear monotone increasing function if and only if the following conditions hold:

$$(b_2 - b_1) = \gamma(a_2 - a_1), \tag{11}$$

$$(h_2 - h_1) = \gamma(k_2 - k_1), \tag{12}$$

$$k \geqslant k_1 - h_1/\gamma, \tag{13}$$

where $\gamma > 0$ is a constant.

3. Several applicable conditions on the linear interpolative reasoning method

As seen in Fig. 4, the interpolated conclusion B^* by using the reasoning method in (6) and (7) does not always preserve linearity and convexity in general. Especially, it would become an important problem on the practical applications if the membership function of the consequence B^* had lost the convexity. Clearly, Conditions I and II mentioned above are too strong to use them well on the practical applications. In order to use the reasoning method to the practical problems, it will be necessary that the interpolated conclusion B^* should preserve at least convexity under some conditions. For this purpose, we shall discuss several cases on the applicable conditions of the reasoning method in detail. The following discussions will be focused on the left sides of the membership functions of the antecedents, consequences and an observation. That is, we only need to discuss their monotonicity. If the left piece of the consequence B^* is monotone increasing and its right piece is monotone decreasing, then we can say that the consequence B^* preserves convexity.

Assume $k \neq k_1 = k_2$, and $h_1 \neq h_2$, Eq. (8) is written as follows:

$$y(\alpha) = \{[(h_2 - h_1)(k - k_1)]\alpha^2 + [(b_2 - b_1)(k - k_1) + h_1(a_2 - a) + h_2(a - a_1)]\alpha + [b_1(a_2 - a) + b_2(a - a_1)]\}/(a_2 - a_1). \tag{14}$$

Hence, $y(\alpha)$ becomes a quadratic polynomial. We analyze the monotonicity of function $y(\alpha)$ using the slopes of antecedents and consequences of fuzzy rules and the slope of an observation.

In order to study the monotonicity of $y(\alpha)$ of Eq. (14), the first-order differential and the quadratic differential of $y(\alpha)$ are given, respectively, as follows:

$$y'(\alpha) = [2(h_2 - h_1)(k - k_1)\alpha + (b_2 - b_1)(k - k_1) + h_1(a_2 - a) + h_2(a - a_1)]/(a_2 - a_1), \tag{15}$$

$$y''(\alpha) = 2(h_2 - h_1)(k - k_1)/(a_2 - a_1). \tag{16}$$

It is noticed that $\mathrm{sgn}(y'') = \mathrm{sgn}((h_2 - h_1)(k - k_1))$ because of $(a_2 - a_1) > 0$. That is to say, the sign of $y''(\alpha)$ depends on the relation of the sign of $(h_2 - h_1)$ and that of $(k - k_1)$. In the following, we shall discuss several different cases about the signs of $(h_2 - h_1)$ and $(k - k_1)$, respectively.

Case 1: $h_2 - h_1 > 0$, $k - k_1 > 0$

In this case, according to Eq. (16), we have a result

$$y''(\alpha) > 0,$$

for any $\alpha \in [0, 1]$. This means that $y'(\alpha)$ is always a monotone increasing function of $\alpha(\in [0, 1])$. In Eq. (15), taking $\alpha = 0$, we get

$$y'(0) = [(b_2 - b_1)(k - k_1) + h_1(a_2 - a) + h_2(a - a_1)]/(a_2 - a_1) > 0,$$

because all of the factors satisfy $(b_2 - b_1) > 0$, $(k - k_1) > 0$, $h_1(a_2 - a) > 0$, $h_2(a - a_1) > 0$, $(a_2 - a_1) > 0$. Thus, for any $\alpha \in [0, 1]$, $y'(\alpha) > 0$ because of its monotonicity. So we have:

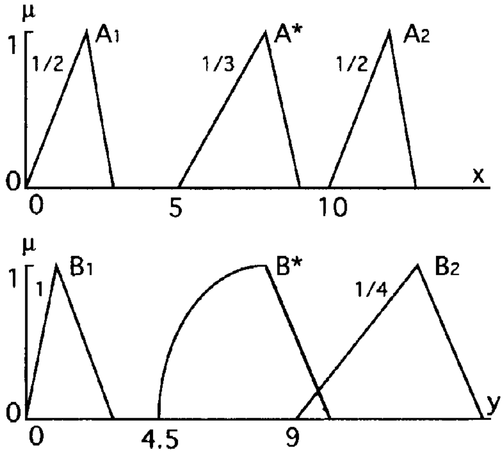

Fig. 5. Interpolation under condition (17).

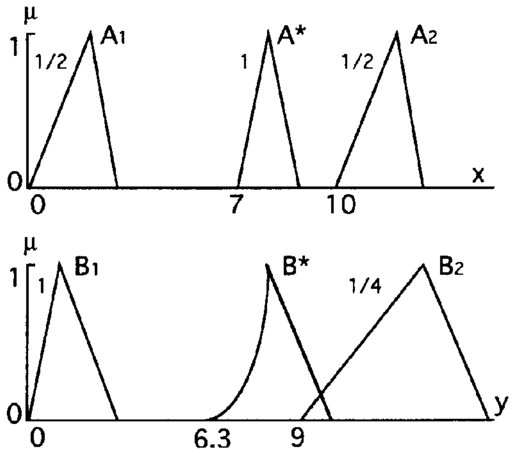

Fig. 6. Interpolation under conditions (18) and (19).

Statement 1: $\inf\{B_\alpha^*\}$ is a monotone increasing function about $\alpha(\in[0,1])$ if the following condition is satisfied:

$$h_2 - h_1 > 0,\ k - k_1 > 0. \tag{17}$$

Fig. 5 shows an example under the above reasoning condition, where $k_1 = 2$, $k = 3$, $k_2 = 2$, $a_1 = 0$, $a = 5$, $a_2 = 10$; $h_1 = 1$, $h_2 = 4$, $b_1 = 0$, $b_2 = 9$.

Case 2: $h_2 - h_1 > 0,\ k - k_1 < 0$

In this case, according to Eq. (16), we have

$$y''(\alpha) < 0.$$

This means that $y'(\alpha)$ is a monotone decreasing function. In order to make $y(\alpha)$ a monotone increasing function on the interval $[0,1]$, we give a restriction as follows.

First, let $y'(1) \geqslant 0$, then $y'(\alpha) \geqslant 0$ for any $\alpha \in [0,1]$ because of the monotonicity of $y'(\alpha)$, which shows that $y(\alpha)$ is monotone increasing on the interval $[0,1]$. And then, in Eq. (15), taking $\alpha = 1$, we get

$$y'(1) = [2(h_2 - h_1)(k - k_1) + (b_2 - b_1)(k - k_1) + h_1(a_2 - a) + h_2(a - a_1)]/(a_2 - a_1) \geqslant 0,$$

i.e.,

$$2(h_2 - h_1)(k - k_1) + (b_2 - b_1)(k - k_1) + h_1(a_2 - a) + h_2(a - a_1) \geqslant 0.$$

Rearranging the above inequality, we have

$$k - k_1 \geqslant -[h_1(a_2 - a) + h_2(a - a_1)]/\{[(h_2 + b_2) - (h_1 + b_1)] + (h_2 - h_1)\}.$$

We notice here that $[h_1(a_2 - a) + h_2(a - a_1)] > 0$, $\{[(h_2 + b_2) - (h_1 + b_1)] + (h_2 - h_1)\} > 0$. Therefore, we get

Statement 2: $\inf\{B_\alpha^*\}$ is a monotone increasing function about $\alpha(\in[0,1])$ if the following conditions are satisfied:

$$h_2 - h_1 > 0, \tag{18}$$

$$0 > k - k_1 \geqslant \frac{-[h_1(a_2 - a) + h_2(a - a_1)]}{[(h_2 + b_2) - (h_1 + b_1)] + (h_2 - h_1)}. \tag{19}$$

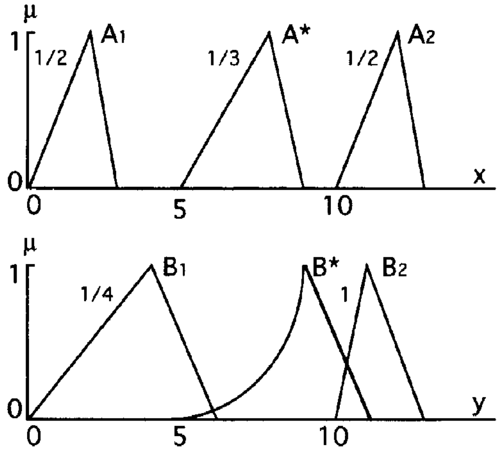

Fig. 7. Interpolation under conditions (20)–(22).

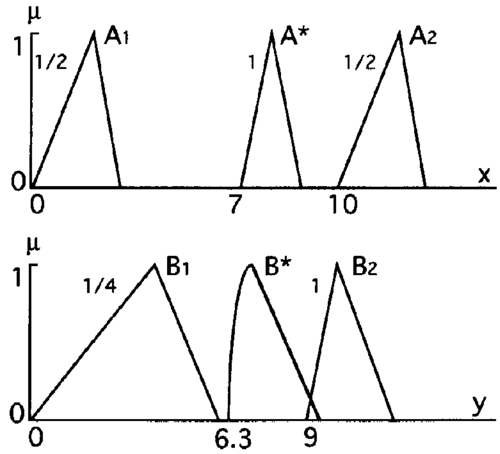

Fig. 8. Interpolation under conditions (23) and (24).

Fig. 6 shows an example under the above reasoning conditions, where $k_1 = 2, k = 1, k_2 = 2, a_1 = 0, a = 7, a_2 = 10; h_1 = 1, h_2 = 4, b_1 = 0, b_2 = 9$.

Case 3: $h_2 - h_1 < 0, k - k_1 > 0$

In this case, $y''(\alpha) < 0$. As discussed in Case 2, first it is necessary to add the condition:

$$k - k_1 \geqslant \frac{-[h_1(a_2 - a) + h_2(a - a_1)]}{[(h_2 + b_2) - (h_1 + b_1)] + (h_2 - h_1)}.$$

Here, $[h_1(a_2 - a) + h_2(a - a_1)] > 0$ is always true, and if $\{[(h_2 + b_2) - (h_1 + b_1)] + (h_2 - h_1)\} > 0$ is also true, then, of course, the condition mentioned above is not necessary. The problem is that if $2(h_1 - h_2) \geqslant (b_2 - b_1)$, then $\{[(h_2 + b_2) - (h_1 + b_1)] + (h_2 - h_1)\} \leqslant 0$. Thus, we should restrict $2(h_1 - h_2) \neq (b_2 - b_1)$, with the condition

$$k - k_1 \geqslant \frac{h_1(a_2 - a) + h_2(a - a_1)}{(h_1 - h_2) - [(h_2 + b_2) - (h_1 + b_1)]}.$$

Hence, we have:

Statement 3: $\inf\{B^*_\alpha\}$ is a monotone increasing function about $\alpha(\in [0, 1])$ if the following conditions are satisfied:

$$h_2 - h_1 < 0, \tag{20}$$

$$k - k_1 \geqslant \frac{h_1(a_2 - a) + h_2(a - a_1)}{(h_1 - h_2) - [(h_2 + b_2) - (h_1 + b_1)]}, \tag{21}$$

$$2(h_1 - h_2) \neq b_2 - b_1. \tag{22}$$

Fig. 7 shows an example under the above reasoning conditions, where $k_1 = 2, k = 3, k_2 = 2, a_1 = 0, a = 5, a_2 = 10; h_1 = 4, h_2 = 1, b_1 = 0, b_2 = 10$.

Case 4: $h_2 - h_1 < 0, k - k_1 < 0$

In this case, we have from Eq. (16)

$$y''(\alpha) > 0.$$

As in the Case 1, this means that $y'(\alpha)$ is a monotone increasing function about $\alpha(\in[0,1])$. From Eq. (15), we have

$$y'(0)=[(b_2-b_1)(k-k_1)+h_1(a_2-a)+h_2(a-a_1)]/(a_2-a_1).$$

Unlike Case 1, $y'(0)$ cannot be guaranteed to be more than or equal to 0, because of $(k-k_1)<0$. In order to guarantee that $y(\alpha)$ is a monotone increasing function about $\alpha(\in[0,1])$, let $y'(0)\geqslant 0$, i.e.

$$[(b_2-b_1)(k-k_1)+h_1(a_2-a)+h_2(a-a_1)]/(a_2-a_1)\geqslant 0.$$

Rearranging the above inequality, we get the condition as follows:

$$k\geqslant k_1-[h_1(a_2-a)+h_2(a-a_1)]/(b_2-b_1).$$

Therefore, for any $\alpha\in[0,1]$, $y'(\alpha)>0$ because of its monotonicity. We get

Statement 4: $\inf\{B^*_\alpha\}$ is a monotone increasing function about $\alpha(\alpha\in[0,1])$ if the following conditions are satisfied:

$$h_2-h_1>0, \tag{23}$$

$$0>k-k_1\geqslant -[h_1(a_2-a)+h_2(a-a_1)]/(b_2-b_1). \tag{24}$$

Fig. 8 shows an example under the above reasoning conditions, where $k_1=2$, $k=1$, $k_2=2$, $a_1=0$, $a=7$, $a_2=10$; $h_1=4$, $h_2=1$, $b_1=0$, $b_2=9$.

Until now, we have discussed and given four kinds of conditions in order to make the quadratic polynomial $\inf\{B^*_\alpha\}$ a monotone increasing function of $\alpha(\in[0,1])$. The similar discussion is possible to the case of the right sides of the membership functions. For the right sides, if similar conditions as (17)–(24) are given, then $\sup\{B^*_\alpha\}$ becomes a monotone decreasing function about $\alpha(\in[0,1])$, and the membership function of the interpolated conclusion B^* preserves convexity.

In the following, we shall discuss two kinds of special cases which are different from the conditions mentioned above.

4. Two special applicable conditions on the linear interpolative reasoning method

In many cases of the practical application we will hope the fact that if fuzzy rules $A_1\Rightarrow B_1$, $A_2\Rightarrow B_2$ and the observation A^* are defined by triangular (or trapezoidal) membership functions, then the interpolated conclusion B^* given in (6) and (7) is also triangular-type (or trapezoidal-type), because it is very convenient for an operator to calculate the membership function of B^* [2,3]. Unfortunately, we cannot get the above conclusion immediately without any additional conditions as discussed in the previous discussions. In order to use (6) and (7), and to make the membership function of B^* a triangular-type (or trapezoidal-type), we shall give two special conditions, one of which is performed by means of restricting the antecedents of fuzzy rules and the observation, and another is performed by means of restricting the rules.

Case 5: $k=k_1=k_2$

The condition $k=k_1=k_2$ means that all of the slopes of the antecedents and the observation are the same. According to Eq. (14), we get easily

$$y(\alpha)=[(a-a_1)(h_2\alpha+b_2)+(a_2-a)(h_1\alpha+b_1)]/(a_2-a_1)$$

and

$$y'(\alpha)=[(a-a_1)h_2+(a_2-a)h_1]/(a_2-a_1)>0.$$

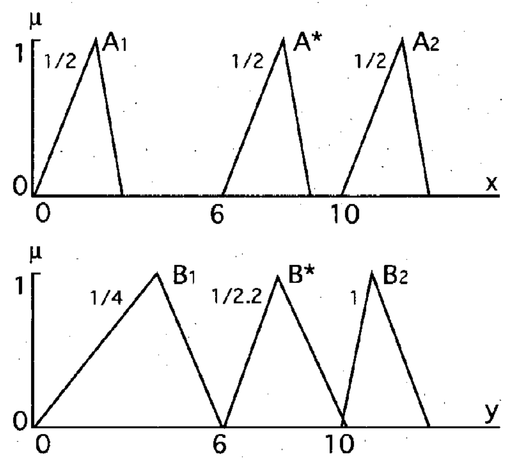

Fig. 9. Interpolation under condition (25).

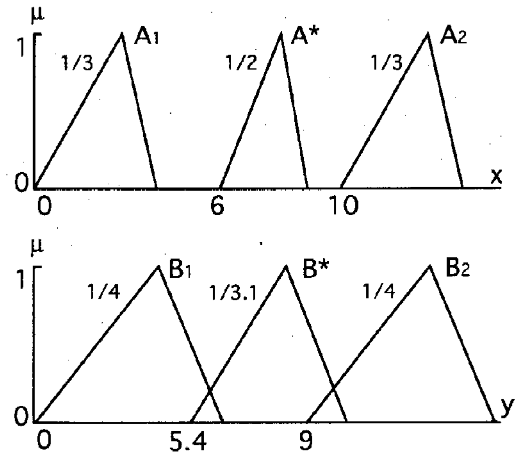

Fig. 10. Interpolation under conditions (26)–(28).

Obviously, $y(\alpha)$ is a monotone increasing linear function about $\alpha(\in[0,1])$. This is a special case of Condition I when $k_2 = k_1$, because in Condition I, $k = \beta k_2 + (1-\beta)k_1$ is always true for any $\beta(\in[0,1])$ under the condition $k = k_1 = k_2$. Thus, the condition $a = \beta a_2 + (1-\beta)a_1$ is always met only if a is between a_1 and a_2. We get

Statement 5: $\inf\{B^*_\alpha\}$ is a linear increasing function on the interval $[0,1]$ if the condition

$$k = k_1 = k_2 \tag{25}$$

holds.

Fig. 9 shows an interpolation result under the above reasoning condition, where $k_1 = 2$, $k = 2$, $k_2 = 2$, $a_1 = 0$, $a_2 = 10$; $h_1 = 4$, $h_2 = 1$, $b_1 = 0$, $b_2 = 10$.

Case 6: $k_1 = k_2$, $h_2 = h_1$

In this case, the slopes of the antecedents are the same, and those of the consequences are so in the fuzzy rules. According to Eq. (14), we have

$$y(\alpha) = \frac{[(b_2-b_1)(k-k_1) + h_1(a_2-a) + h_2(a-a_1)]\alpha + b_1(a_2-a) + b_2(a-a_1)}{a_2-a_1}$$

$$= \frac{[(b_2-b_1)(k-k_1) + h_1(a_2-a_1)]\alpha + b_1(a_2-a) + b_2(a-a_1)}{a_2-a_1}$$

and

$$y'(\alpha) = (b_2-b_1)(k-k_1)/(a_2-a_1) + h_1.$$

Clearly, $y(\alpha)$ is a linear function on the interval $[0,1]$. To make $y(\alpha)$ a monotone increasing, we assume $y'(\alpha) \geqslant 0$, i.e.

$$(b_2-b_1)(k-k_1)/(a_2-a_1) + h_1 \geqslant 0.$$

Rearranging the above inequality, we get the condition

$$k \geqslant k_1 - h_1(b_2-b_1)/(a_2-a_1).$$

Let $(b_2-b_1)/(a_2-a_1) = \gamma$, then the above inequation becomes

$$k \geqslant k_1 - h_1/\gamma \quad (\gamma > 0).$$

This is a special case of Condition II when $h_2 - h_1 = k_2 - k_1 = 0$. In Condition II, $h_2 - h_1 = \gamma(k_2 - k_1)$ is always true for any $\gamma(>0)$ under the condition $k_1 = k_2$ and $h_1 = h_2$. Thus, of course, there is a γ that meets equation $b_2 - b_1 = \gamma(a_2 - a_1)$.

Statement 6: $\inf\{B_\alpha^*\}$ is a linear increasing function on the interval $[0, 1]$ if the conditions

$$k_1 = k_2, \tag{26}$$

$$h_2 = h_1, \tag{27}$$

$$k \geqslant k_1 - h_1/\gamma \quad (\gamma > 0) \tag{28}$$

hold.

For example, let $k_1 = 3$, $k_2 = 3$, $a_1 = 0$, $a_2 = 10$; $h_1 = 4$, $h_2 = 4$, $b_1 = 0$, $b_2 = 9$ and $k = 2$, $a = 6$, then the conclusion of the interpolation under the above conditions is shown in Fig. 10.

It is noted that the similar discussion holds for the right sides of the membership functions. If similar conditions as (25) or (26), (27) and (28) are given for the right sides, the membership function of the interpolated conclusion B^* will be of triangular-type (or trapezoidal-type).

In Sections 3 and 4, the discussions are made on the left sides of the membership functions and six different cases are given. The discussion on the right sides of the membership functions has the similar conclusions.

5. Conclusion

In this paper, we have given several conditions that the membership function of the interpolated conclusion B^* obtained by the interpolative reasoning method can preserve convexity or be triangular-type (or trapezoidal-type) when fuzzy rules $A_1 \Rightarrow B_1$, $A_2 \Rightarrow B_2$ and an observation A^* are given by triangular (or trapezoidal) membership functions.

Though there are such sufficient conditions, it is still limited in the practical application of the above interpolative reasoning method. It will be necessary to find out new interpolative reasoning methods which can guarantee that for a triangular-type (or trapezoidal-type) observation, the interpolated conclusion will also be a triangular-type (or trapezoidal-type). Further, it will be interested in searching a new fuzzy reasoning method in which if fuzzy rules $A_1 \Rightarrow B_1$, $A_2 \Rightarrow B_2$ and the observation A^* are defined by normal "convex" fuzzy sets, the interpolated conclusion B^* will also be normal "convex" fuzzy set.

References

[1] L.T. Kóczy, Approximation models and reasoning, *Proc. 3rd Internat. Conf. on Fuzzy Logic, Neural Nets and Soft Computing* (1994) 257–258.

[2] L.T. Kóczy and K. Hirota, Interpolative reasoning with insufficient evidence in sparse fuzzy rules bases, *Inform. Sci.* **71** (1993) 169–201.

[3] L.T. Kóczy and K. Hirota, Approximate reasoning by linear rule interpolation and general approximation, *Int. J. Approximate Reasoning* **9** (1993) 197–225.

[4] M. Mizumoto, Multifold fuzzy reasoning as interpolative reasoning, in: R.R. Yager and L.A. Zadeh, Eds., *Fuzzy Sets, Neural Networks and Soft Computing* (Van Nostrand Reinhold, New York, 1994) 188–193.

[5] M. Mizumoto and H.-J. Zimmermann, Comparison of fuzzy reasoning methods, *Fuzzy Sets and Systems* **8** (1982) 253–283.

[6] Y. Shi and M. Mizumoto, Some considerations on Kóczy's interpolative reasoning method, *J. Japan Soc. Fuzzy Theory Systems* **8** (1996) 147–157.

[7] Y. Shi, M. Mizumoto and Z.Q. Wu, The reasoning conditions on Kóczy's interpolative reasoning method in sparse fuzzy rule bases, *Fuzzy Sets and Systems* **75** (1995) 63–71.

[8] L.A. Zadeh, Similarity relations and fuzzy orderings, *Inform. Sci.* **3** (1971) 177–200.

[9] L.A. Zadeh, Calculus of fuzzy restriction, in: L.A. Zadeh et al., Eds., *Fuzzy Sets and Their Applications to Cognitive and Decision Processes* (Academic Press, New York, 1975) 1–39.