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Reasoning conditions on Kóczy's interpolative reasoning method in sparse fuzzy rule bases

Shi Yan*, Masaharu Mizumoto, Wu Zhi Qiao

Department of Management Engineering, Osaka Electro-Communication University, Neyagawa, Osaka 572, Japan

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Abstract

In the sparse fuzzy rule bases, conventional fuzzy reasoning methods encounter difficulties because of the lack of inference evidence. To tackle this problem, Kóczy and Hirota have proposed a fuzzy reasoning method called a linear interpolative reasoning method. In this paper, we analyze the Kóczy and Hirota's reasoning method and find that the reasoning consequences by this method sometimes become abnormal fuzzy sets. The conditions of the reasoning method are also discussed analytically.

Keywords: Fuzzy reasoning; Interpolative reasoning; Linear interpolation; Sparse fuzzy rule bases

1. Introduction

All of the present methods of fuzzy reasoning are based on the compact rule bases [5, 6, 8–10, 11]. In these fuzzy reasoning methods the input universe of discourse is covered by the rule bases completely, and when an observation occurs, a consequence can be derived by some proper rules of inference. When the fuzzy rule bases are sparse, that is, the input universe of discourse is covered incompletely by the rule bases, there is an empty space between two membership functions of antecedents, and thus the conventional fuzzy reasoning methods encounter difficulties because of the lack of inference evidence. If an observation occurs in an empty space, no rule will be fired and thus no consequence is derived. To tackle this problem, Kóczy and Hirota [3,4] have proposed a fuzzy reasoning method called a linear interpolative reasoning.

In this paper, we analyze the property of their method and show that the reasoning consequences by the method sometimes become abnormal fuzzy sets. Especially, we prove that the statement "If fuzzy rules $A_1 \Rightarrow B_1, A_2 \Rightarrow B_2$ and the observation A^* are defined by triangular membership functions, the interpolated conclusion B^* will also be triangular-type" stated in [3,4] is improper. Two kinds of reasoning conditions of

* Corresponding author.

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the method are given in order to show that the interpolated conclusions are given as triangular-type membership functions.

2. Fuzzy reasoning problem on sparse fuzzy rule bases

As for a fuzzy reasoning problem, we have the following Tomato Problem proposed by Mizumoto and Zimmerman [10].

We can have a proper reasoning for the fuzzy reasoning:

Antecedent 1: If a tomato is red then the tomato is ripe.
Antecedent 2: This tomato is very red.

Consequence: This tomato is very ripe.

However, difficulty arises in the following reasoning:

Antecedent 1: If a tomato is red then the tomato is ripe.
Antecedent 2: If a tomato is green then the tomato is unripe.
Antecedent 3: This tomato is yellow.

Consequence: ???

What would be the consequence? Intuitively, one would have the consequence that the tomato is half ripe when it is yellow. But the consequence would be nothing according to the conventional fuzzy inference methods. This problem can be represented as in Fig. 1.

Kóczy and Hirota have looked into this problem and proposed a method named a linear interpolative reasoning, which is stated briefly as follows [3,4].

Firstly, we have the following definitions.

Definition 1. Denote the set of all normal and convex fuzzy sets of the universe X_i by $P(X_i)$. Then for $A_1, A_2 \in P(X_i)$, if $\forall \alpha \in (0,1]$ the following conditions hold:

$$\inf\{A_{1\alpha}\} < \inf\{A_{2\alpha}\}, \qquad \sup\{A_{1\alpha}\} < \sup\{A_{2\alpha}\},$$

then A_1 is said to be less than A_2, that is, $A_1 < A_2$, where $A_{1\alpha}$ and $A_{2\alpha}$ are α-cut sets of A_1 and A_2, respectively, and $\inf\{A_{i\alpha}\}$ is the infimum of $A_{i\alpha}$ and $\sup\{A_{i\alpha}\}$ is the supremum of $A_{i\alpha}$ $(i = 1, 2)$.

Definition 2. Given a fuzzy relation $R_< = \{(A_1, A_2) \mid A_1, A_2 \in P(X), A_1 < A_2\}$, if fuzzy sets A_1 and A_2 satisfy $R_<$, the lower and the upper fuzzy distances between A_1 and A_2 are defined as follows, by using the resolution

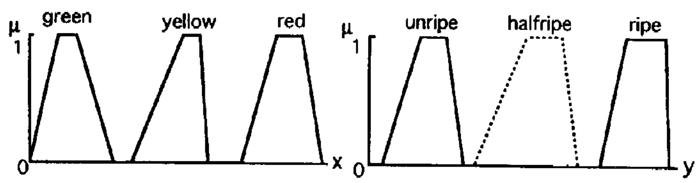

Fig. 1. Fuzzy reasoning assumption of the Tomato Problem.

principle [3]:

$$d_L(A_1, A_2): R_< \to P([0,1]),$$

$$\mu_{d_L(A_1,A_2)}(\delta) = \sum_{\alpha \in [0,1]} \alpha / d(\inf\{A_{1\alpha}\}, \inf\{A_{2\alpha}\});$$

$$d_U(A_1, A_2): R_< \to P([0,1]),$$

$$\mu_{d_U(A_1,A_2)}(\delta) = \sum_{\alpha \in [0,1]} \alpha / d(\sup\{A_{1\alpha}\}, \sup\{A_{2\alpha}\}),$$

where $\delta \in [0,1]$ [3,4], and d is the Euclidean distance, or, more generally, Minkowski distance.

Definition 3. Let $A_1 \Rightarrow B_1$, $A_2 \Rightarrow B_2$ be disjoint fuzzy rules on the universe of discourse $X \times Y$, and A_1, A_2 and B_1, B_2, be fuzzy sets on X and Y, respectively. Assume that A^* is the observation of the input universe X. If $A_1 < A^* < A_2$ then the linear fuzzy rule interpolation between R_1 and R_2 is defined as

$$d(A^*, A_1): d(A^*, A_2) = d(B^*, B_1): d(B^*, B_2). \tag{1}$$

Definition 4. Let A_1 and A_2 be fuzzy sets on the universe of discourse X with $|X| < \infty$, then the lower and the upper distances between α-cut sets $A_{1\alpha}$ and $A_{2\alpha}$ are defined as

$$d_L(A_{1\alpha}, A_{2\alpha}) = d(\inf\{A_{1\alpha}\}, \inf\{A_{2\alpha}\}), \tag{2}$$

$$d_U(A_{1\alpha}, A_{2\alpha}) = d(\sup\{A_{1\alpha}\}, \sup\{A_{2\alpha}\}). \tag{3}$$

From Definitions 2 and 4 and resolution principle of fuzzy sets, the fuzzy rule interpolation given in (1) can be redefined as

$$d_L(A^*_\alpha, A_{1\alpha}): d_L(A^*_\alpha, A_{2\alpha}) = d_L(B^*_\alpha, B_{1\alpha}): d_L(B^*_\alpha, B_{2\alpha}),$$

$$d_U(A^*_\alpha, A_{1\alpha}): d_U(A^*_\alpha, A_{2\alpha}) = d_U(B^*_\alpha, B_{1\alpha}): d_U(B^*_\alpha, B_{2\alpha}),$$

which can be rewritten as

$$\inf\{B^*_\alpha\} = \frac{d_L(A^*_\alpha, A_{1\alpha})\inf\{B_{2\alpha}\} + d_L(A^*_\alpha, A_{2\alpha})\inf\{B_{1\alpha}\}}{d_L(A^*_\alpha, A_{1\alpha}) + d_L(A^*_\alpha, A_{2\alpha})}, \tag{4}$$

$$\sup\{B^*_\alpha\} = \frac{d_U(A^*_\alpha, A_{1\alpha})\sup\{B_{2\alpha}\} + d_U(A^*_\alpha, A_{2\alpha})\sup\{B_{1\alpha}\}}{d_U(A^*_\alpha, A_{1\alpha}) + d_U(A^*_\alpha, A_{2\alpha})}. \tag{5}$$

The authors of [3,4] claim that the following statement holds by this linear interpolation method.

If fuzzy rules $A_1 \Rightarrow B_1$ and $A_2 \Rightarrow B_2$, and the observation A^* are defined by triangular membership functions, the interpolated conclusion B^* is also the triangular type.

We have, however, found some different conclusions as [3,4]. The detail is described in the following sections.

3. Discussion on the fuzzy interpolation method

In Eq. (4), let

$$M = \frac{d_L(A_\alpha^*, A_{1\alpha})}{d_L(A_\alpha^*, A_{1\alpha}) + d_L(A_\alpha^*, A_{2\alpha})}, \quad (6)$$

$$N = \frac{d_L(A_\alpha^*, A_{2\alpha})}{d_L(A_\alpha^*, A_{1\alpha}) + d_L(A_\alpha^*, A_{2\alpha})}, \quad (7)$$

then Eq. (4) can be rewritten as

$$\inf\{B_\alpha^*\} = M \inf\{B_{2\alpha}\} + N \inf\{B_{1\alpha}\}. \quad (8)$$

From the equation, $\inf\{B_\alpha^*\}$ seems to be the linear combination of $\inf\{B_{1\alpha}\}$ and $\inf\{B_{2\alpha}\}$. But one finds that the coefficients M and N are not constants but variables with α as $\inf\{B_{1\alpha}\}$ and $\inf\{B_{2\alpha}\}$. This implies that $\inf\{B_\alpha^*\}$ is not always linear with respect to $\inf\{B_{1\alpha}\}$ and $\inf\{B_{2\alpha}\}$.

We shall get the interpolated conclusion by the method of Kóczy and Hirota more precisely. Fig. 2 shows the example of fuzzy interpolation when rules $A_1 \Rightarrow B_1$ and $A_2 \Rightarrow B_2$, and the observation A^* are all defined by triangular-type membership functions.

The left sides of triangular membership functions of fuzzy sets A_1, A_2, A^*, B_1 and B_2 can be given as

$$\mu = 1/k_1(x - a_1), \qquad \mu = 1/k_2(x - a_2),$$

$$\mu = 1/k(x - a), \qquad \mu = 1/h_1(y - b_1), \qquad \mu = 1/h_2(y - b_2),$$

where

$$a_1 = \inf\{A_{1\alpha}\}, \qquad a_2 = \inf\{A_{2\alpha}\}, \qquad a = \inf\{A_\alpha^*\},$$

$$b_1 = \inf\{B_{1\alpha}\}, \qquad b_2 = \inf\{B_{2\alpha}\}$$

at $\alpha = 0$. $1/k_1$, $1/k_2$, $1/k$, $1/h_1$ and $1/h_2$ represent the slopes of left sides of the membership functions.

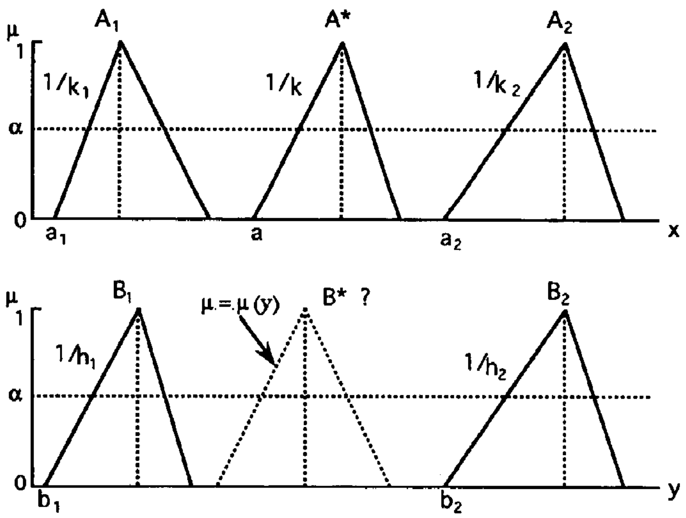

Fig. 2. Interpolation with triangular membership functions.

Thus, we have the following:

$$x = k_1\mu + a_1, \tag{9}$$

$$x = k_2\mu + a_2, \tag{10}$$

$$x = k\mu + a, \tag{11}$$

$$y = h_1\mu + b_1, \tag{12}$$

$$y = h_2\mu + b_2 \tag{13}$$

where $\mu \in [0, 1]$.

Assume that the left side of the membership function of the consequence fuzzy set B^* is represented as

$$y = y(\mu) \tag{14}$$

in the form of a reverse function. Then the function (14) is deduced by using the interpolation method in the previous section as follows.

For all $\alpha \in [0, 1]$, we have

$$\inf\{A_{1\alpha}\} = k_1\alpha + a_1, \tag{15}$$

$$\inf\{A_{2\alpha}\} = k_2\alpha + a_2, \tag{16}$$

$$\inf\{A_\alpha^*\} = k\alpha + a, \tag{17}$$

$$\inf\{B_{1\alpha}\} = h_1\alpha + b_1, \tag{18}$$

$$\inf\{B_{2\alpha}\} = h_2\alpha + b_2. \tag{19}$$

Substituting Eqs. (15)–(19) into (4), we get

$$\begin{aligned} y(\alpha) = \inf\{B_\alpha^*\} &= \frac{d_L(A_\alpha^*, A_{1\alpha})\inf\{B_{2\alpha}\} + d_L(A_\alpha^*, A_{2\alpha})\inf\{B_{1\alpha}\}}{d_L(A_\alpha^*, A_{1\alpha}) + d_L(A_\alpha^*, A_{2\alpha})} \\ &= \frac{[(k - k_1)\alpha + (a - a_1)](h_2\alpha + b_2) + [(k_2 - k)\alpha + (a_2 - a)](h_1\alpha + b_1)}{(k_2 - k_1)\alpha + (a_2 - a_1)} \\ &= \frac{\{[h_1(k_2 - k) + h_2(k - k_1)]\alpha^2 + [b_1(k_2 - k) + b_2(k - k_1) + h_1(a_2 - a) + h_2(a - a_1)]\alpha + [b_1(a_2 - a) + b_2(a - a_1)]\}}{(k_2 - k_1)\alpha + (a_2 - a_1)}. \end{aligned} \tag{20}$$

As seen from Eq. (20), $y = y(\alpha)$ is a non-linear relation of α in general. Since $\alpha(\alpha \in [0, 1])$ is arbitrary, $y = y(\mu)$ is a non-linear function of μ. That is, if fuzzy rules $A_1 \Rightarrow B_1$ and $A_2 \Rightarrow B_2$ are defined by triangular membership functions and the observation is defined so, the membership function of the consequence fuzzy set B^* derived by the above interpolation method is not always of triangular type.

As a special case, assume $k_1 = k_2 \neq k$, and $h_1 \neq h_2$, then Eq. (20) becomes

$$y(\alpha) = \frac{[(h_2 - h_1)(k - k_1)]\alpha^2 + [(b_2 - b_1)(k - k_1) + h_1(a_2 - a) + h_2(a - a_1)]\alpha + [b_1(a_2 - a) + b_2(a - a_1)]}{(a_2 - a_1)} \tag{21}$$

which indicates that $y(\alpha)$ is a second-order function of α.

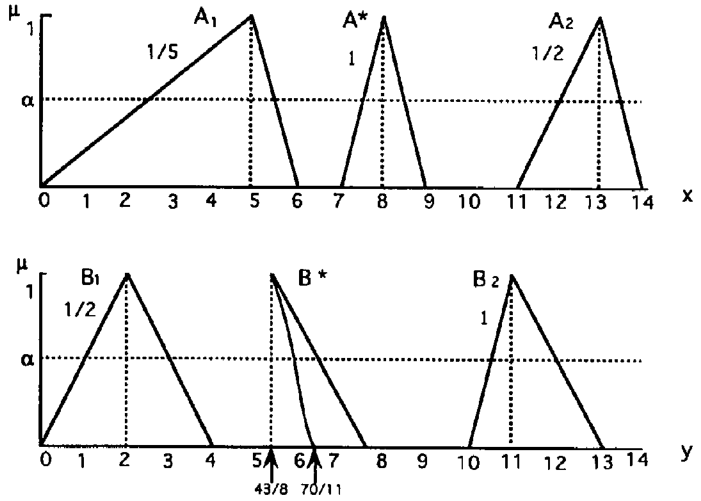

Fig. 3. Example of fuzzy interpolation.

We shall next prove that the fuzzy interpolation reasoning method cannot always guarantee B^* to be convex fuzzy set.

As a simple example, let $a_1 = 0, a_2 = 11, a = 7, b_1 = 0, b_2 = 10, k_1 = 5, k_2 = 2, k = 1, h_1 = 2$ and $h_2 = 1$ as shown in Fig. 3, then we have from (20),

$$y(\alpha) = (2\alpha^2 + 25\alpha - 70)/(3\alpha - 11). \tag{22}$$

When α is 0 and 1, we obtain

$$y(0) = \tfrac{70}{11}, \qquad y(1) = \tfrac{43}{8}; \tag{23}$$

obviously, $y(1) < y(0)$, i.e., $\inf\{B_1^*\} < \inf\{B_0^*\}$, which indicates that the membership function of B^* is an abnormal fuzzy set, as shown in Fig. 3.

In some cases, the membership function of consequence B^* can be of triangular type. In the next section, we shall obtain two reasoning conditions for the fuzzy interpolation under which the membership function of consequence B^* can be guaranteed to be triangular type. The following are sufficient conditions to guarantee the linearity of $\inf\{B_\alpha^*\}$.

4. Conditions of the interpolation reasoning method

Condition I. In Eqs. (6) and (7), if $\inf\{A_\alpha^*\}$ is such that the coefficients M and N become constants, then the consequence $\inf\{B_\alpha^*\}$ of (8) will be a linear combination of $\inf\{B_{1\alpha}\}$ and $\inf\{B_{2\alpha}\}$.

Let

$$M = \frac{d_L(A_\alpha^*, A_{1\alpha})}{d_L(A_\alpha^*, A_{1\alpha}) + d_L(A_\alpha^*, A_{2\alpha})} = \beta, \quad \beta \in [0, 1],$$

be a constant; then substituting (15)–(17), the above equation becomes

$$M = \frac{(k - k_1)\mu + (a - a_1)}{(k_2 - k_1)\mu + (a_2 - a_1)} = \beta, \tag{24}$$

which requires $(k - k_1) = \beta(k_2 - k_1)$, and $(a - a_1) = \beta(a_2 - a_1)$, that is, k and a of A^* are obtained as

$$k = \beta k_2 + (1 - \beta)k_1, \tag{25}$$

$$a = \beta a_2 + (1 - \beta)a_1. \tag{26}$$

In this case, $y(\alpha)$ of (20) becomes a linear function such as

$$\begin{aligned} y(\alpha) &= [\beta h_2 + (1 - \beta)h_1]\alpha + [\beta b_2 + (1 - \beta)b_1] \\ &= \{[h_1(a_2 - a) + h_2(a - a_1)]/(a_2 - a_1)\}\alpha + [b_1(a_2 - a) + b_2(a - a_1)]/(a_2 - a_1) \end{aligned} \tag{27}$$

and

$$y'(\alpha) = \beta h_2 + (1 - \beta)h_1 > 0. \tag{28}$$

It is noted that a similar discussion is possible for the right sides of the membership functions. Therefore, conditions similar to (25) and (26) are applied for the right sides. Thus, the membership function of the interpolated conclusion B^* will be triangular type.

As an illustration, let $\beta = \frac{1}{2}$. When the parameters k and a of $\inf\{A_\alpha^*\}$ satisfy Eqs. (25) and (26), that is

$$k = (\tfrac{1}{2})k_2 + (1 - \tfrac{1}{2})k_1 = (k_1 + k_2)/2,$$

$$a = (\tfrac{1}{2})a_2 + (1 - \tfrac{1}{2})a_1 = (a_1 + a_2)/2,$$

we have

$$\inf\{B_\alpha^*\} = y(\alpha) = [(h_1 + h_2)\alpha + (b_1 + b_2)]/2.$$

Fig. 4 shows an interpolation result under Reasoning Condition I when $k_1 = 3$, $k_2 = 1$, $a_1 = 0$, $a_2 = 11$; $h_1 = 2$, $h_2 = 1$, $b_1 = 0$, $b_2 = 10$.

Condition II. Rewrite Eq. (4) as follows:

$$\inf\{B_\alpha^*\} = \frac{d_L(B_{2\alpha}, B_{1\alpha})}{d_L(A_{2\alpha}, A_{1\alpha})}\, d_L(A_\alpha^*, A_{1\alpha}) + \inf\{B_{1\alpha}\}. \tag{29}$$

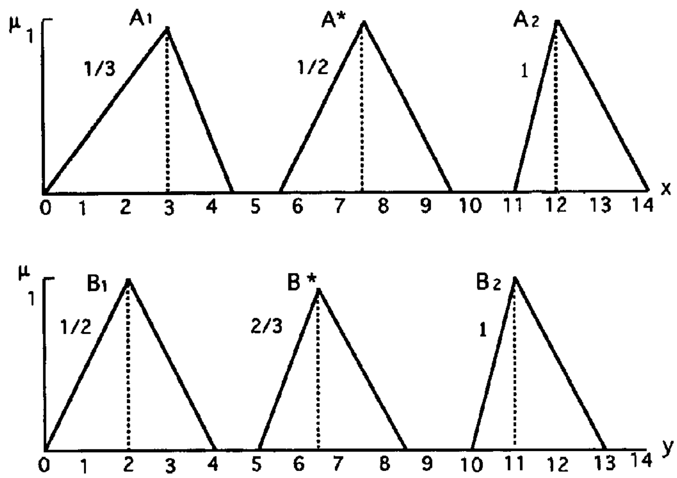

Fig. 4. An interpolation result under the Reasoning Condition I.

If fuzzy rules $A_1 \Rightarrow B_1$, $A_2 \Rightarrow B_2$ and the observation A^* are defined in a triangular type, that is, $\inf\{A_{1\alpha}\}$, $\inf\{B_{1\alpha}\}$, $\inf\{A_{2\alpha}\}$, $\inf\{B_{2\alpha}\}$ and $\inf\{A^*\}$ are all linear with respect to α, it requires that

$$\frac{d_L(B_{2\alpha}, B_{1\alpha})}{d_L(A_{2\alpha}, A_{1\alpha})} = \gamma \quad (\gamma > 0)$$

is a constant in order to make $\inf\{B^*\}$ also linear. Substituting (15), (16), (18) and (19), the above equation becomes

$$\frac{(h_2 - h_1)\alpha + (b_2 - b_1)}{(k_2 - k_1)\alpha + (a_2 - a_1)} = \gamma$$

and this requires

$$(b_2 - b_1) = \gamma(a_2 - a_1), \tag{30}$$

$$(h_2 - h_1) = \gamma(k_2 - k_1). \tag{31}$$

In this case, $y(\alpha)$ becomes a linear function like

$$\begin{aligned} y(\alpha) &= [h_1 + \gamma(k - k_1)]\alpha + [b_1 + \gamma(a - a_1)] \\ &= \{[(a_2 - a_1)h_1 + (b_2 - b_1)(k - k_1)]/(a_2 - a_1)\}\alpha + [b_1(a_2 - a) + b_2(a - a_1)]/(a_2 - a_1) \end{aligned} \tag{32}$$

and

$$y'(\alpha) = h_1 + \gamma(k - k_1). \tag{33}$$

In order to guarantee the monotonicity of $\inf\{B^*\}$ (monotone increasing), it is required, for $y'(\alpha) \geqslant 0$, i.e., $h_1 + \gamma(k - k_1) \geqslant 0$, because $\gamma = (b_2 - b_1)/(a_2 - a_1) > 0$, that

$$k \geqslant k_1 - h_1/\gamma \tag{34}$$

holds.

For example, when $\gamma = 1$, let $k_1 = 3$, $k_2 = 4$, $a_1 = 0$, $a_2 = 9$; $h_1 = 2$, $h_2 = 3$, $b_1 = 1$, $b_2 = 10$ and $k = 2$, $a = 5$, then the conclusion of the interpolation under the above condition is

$$\inf\{B^*_\alpha\} = y(\alpha) = [h_1 + \gamma(k - k_1)]\alpha + [b_1 + \gamma(a - a_1)] = [2 + (2 - 3)]\alpha + [1 + (5 - 0)\} = \alpha + 6,$$

where $\alpha \in [0, 1]$. This is shown in Fig. 5.

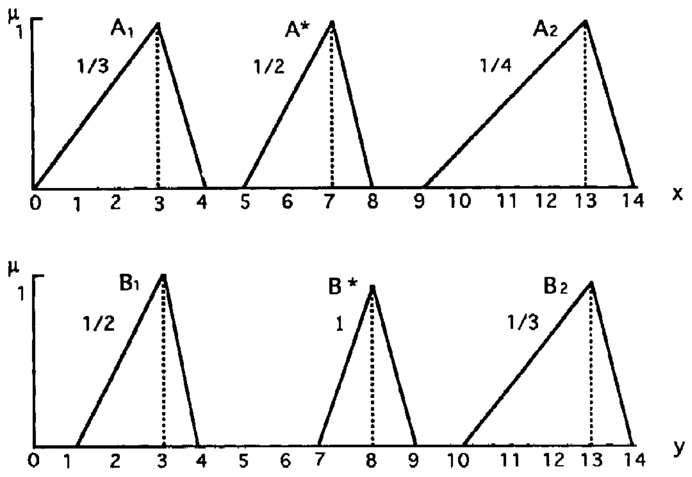

Fig. 5. An interpolation result under the Reasoning Condition II.

A similar discussion is possible for the case of the right sides of the membership functions. If conditions similar to (30), (31) and (34) are applied for the right sides, the membership function of the interpolated conclusion B^* will be triangular type.

5. Conclusion

We have illustrated that if fuzzy rules $A_1 \Rightarrow B_1$, $A_2 \Rightarrow B_2$ and an observation A^* are defined by triangular membership functions, the interpolation method described in (4) and (5) cannot always guarantee the membership function of the interpolated conclusion B^* to be triangular type.

Furthermore, we have given reasoning conditions under which the membership function of the interpolated conclusion can be triangular type.

Although there are such sufficient conditions, it is still limited in practical applications of the interpolative reasoning method. A new interpolative reasoning method will be needed which can guarantee that the interpolated conclusion will also be triangular type for a triangular-type observation. Further, it will be interesting to search a new fuzzy reasoning method where if fuzzy rules $A_1 \Rightarrow B_1$, $A_2 \Rightarrow B_2$ and the observation A^* are defined by normal "convex" fuzzy sets, the interpolated conclusion B^* will also be a normal "convex" fuzzy set.

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