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# PICTORIAL REPRESENTATIONS OF FUZZY CONNECTIVES, PART II: CASES OF COMPENSATORY OPERATORS AND SELF-DUAL OPERATORS

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*Abstract:* As the continuation of Part I in which pictorial representations of t-norms, t-conorms and averaging operators were made with the aid of a computer, this paper discusses compensatory operators and proposes generalized compensatory operators which can be obtained from averaging operators. Self-dual operators are discussed which can be defined by using t-norms, t-conorms and averaging operators. Finally, symmetric sums are reviewed which are also self-dual operators. The pictorial representations of these operators are also made.

*Keywords:* Fuzzy sets; fuzzy connectives; t-norms; t-conorms; quasi-t-norms; quasi-t-conorms; averaging operators; compensatory operators; generalized compensatory operators; self-dual operators; symmetric sums.

## 1. Introduction

In Part I [7] we showed a number of examples of existing and proposed t-norms and t-conorms and their pictorial representations were made with the aid of a computer. Moreover, averaging operators were summarized and their pictorial representations also made.

As the continuation of Part I, this paper proposes quasi-t-norms and quasi-t-conorms which are derived from t-(co)norms and do not necessarily satisfy the associativity. Compensatory operators are summarized and generalized compensatory operators are newly defined which can be obtained from averaging operators. Self-dual operators are discussed which can be obtained by using t-norms, t-conorms and averaging operators. Finally, symmetric sums are reviewed which are also self-dual operators.

## 2. Quasi-t-norms and quasi-t-conorms

In the definitions of t-norms [1–5, 7, 9], t-norms $T$ satisfy:

(i) $T(x, 1) = x,\ T(x, 0) = 0$;
(ii) $x' \leq x'',\ y' \leq y'' \Rightarrow T(x', y') \leq T(x'', y'')$;
(iii) $T(x, y) = T(y, x)$;
(iv) $T(x, T(y, z)) = T(T(x, y), z)$.

When $T$ does not necessarily satisfy the associativity (iv), we shall call it *quasi*-t-*norm*. For example, $Q(x, y) = xy(x + y - xy)$ does not satisfy (iv) but satisfies (i)–(iii). In this connection, we shall define the following operator $Q(x, y)$:

$$Q(x, y) = T'(T(x, y), S(x, y)). \tag{1}$$

where $T'$ and $T$ are t-norms and $S$ is a t-conorm which is not necessarily dual to $T$. $Q(x, y)$ satisfies the following:

(a) $Q(x, 1) = x$; $Q(x, 0) = 0$;
(b) $x' \leq x''$, $y' \leq y'' \Rightarrow Q(x', y') \leq Q(x'', y'')$;
(c) $Q(x, y) = Q(y, x)$;
(d) $Q(x, y) \leq T(x, y)$.

(a) is shown as follows:

$$Q(x, 1) = T'(T(x, 1), S(x, 1)) = T'(x, 1) = x,$$
$$Q(x, 0) = T'(T(x, 0), S(x, 0)) = T'(0, x) = 0.$$

The monotonicity (b) and commutativity (c) of $Q(x, y)$ are easily shown. Since $T'(x, y) \leq x \wedge y$ and $T(x, y) \leq S(x, y)$, we have $Q(x, y) = T'(T(x, y), S(x, y)) \leq T(x, y) \wedge S(x, y) = T(x, y)$ which shows (d).

De Morgan's like dual of a quasi-t-norm $Q(x, y) = T'(T(x, y), S(x, y))$ is called a *quasi*-t-*conorm* and is given by

$$Q^*(x, y) = S'(S''(x, y), T''(x, y)) \tag{2}$$

where $S'$ is a t-conorm dual to $T'$, and $S''$ and $T''$ are dual to $T$ and $S$, respectively. Namely, a quasi-t-conorm $Q^*(x, y)$ is derived from $Q(x, y)$ as follows:

$$\begin{aligned} Q^*(x, y) &= 1 - Q(1-x, 1-y) = 1 - T'(T(1-x, 1-y), S(1-x, 1-y)) \\ &= 1 - T'(1 - S''(x, y), 1 - T''(x, y)) = S'(S''(x, y), T''(x, y)). \end{aligned}$$

Clearly, if $T$ and $S$ are dual to each other in (1), $Q^*(x, y)$ is given as

$$Q^*(x, y) = S'(S(x, y), T(x, y)).$$

A quasi-t-conorm $Q^*(x, y)$ of (2) satisfies the following properties:

(a') $Q^*(x, 0) = x$, $Q^*(x, 1) = 1$;
(b') $x' \leq x''$, $y' \leq y'' \Rightarrow Q^*(x', y') \leq Q^*(x'', y'')$;
(c') $Q^*(x, y) = Q^*(y, x)$;
(d') $Q^*(x, y) \geq S''(x, y)$.

In the following, we shall derive some examples of quasi-t-norms $Q(x, t)$ of (1) and quasi-t-conorms $Q^*(x, y)$ of (2).

When $T' = \wedge$ (min), we have $Q(x, y) = T(x, y)$ since

$$Q(x, y) = T(x, y) \wedge S(x, y) = T(x, y).$$

Dually, when $S' = \vee$ (max), we have $Q^*(x, y) = S''(x, y)$.

In the case of $T'(a, b) = a \cdot b$ (algebraic product), $Q(x, y)$ is given as

$$Q(x, y) = T(x, y) \cdot S(x, y)$$

For example, let $T(x, y) = xy$ and $S(x, y) = x \vee y$; then we obtain

$$Q(x, y) = xy(x \vee y).$$

From $T(x, y) = xy$ and $S(x, y) = x + y - xy$, we have

$$Q(x, y) = xy(x + y - xy).$$

Moreover,

$$T(x, y) = \frac{xy}{x + y - xy} \quad \text{and} \quad S(x, y) = \frac{x + y - 2xy}{1 - xy}$$

(Hamacher product and Hamacher sum, respectively [5]) give the following $Q(x, y)$:

$$Q(x, y) = \frac{xy(x + y - 2xy)}{(x + y - xy)(1 - xy)}.$$

The three $Q(x, y)$ given above are quasi-t-norms but not t-norms.

Dually, from $S'(a, b) = a + b - ab$ (algebraic sum),

$$Q^*(x, y) = S''(x, y) + T''(x, y) - S''(x, y)T''(x, y)$$

is derived from (2). If $S''(x, y) = x + y - xy$ and $T''(x, y) = x \wedge y$, then $Q^*(x, y)$ is as follows:

$$Q^*(x, y) = x + y - xy + (x \wedge y) - (x + y - xy)(x \wedge y),$$

which is dual to $Q(x, y) = xy(x \vee y)$ given above. When $S''(x, y) = x + y - xy$ and $T''(x, y) = xy$, we have

$$Q^*(x, y) = x + y - xy(x + y - xy).$$

This is dual to $xy(x + y - xy)$. It is noted that this quasi-t-conorm $Q^*(x, y)$ and its dual quasi-t-norm $Q(x, y)$ satisfy

$$Q(x, y) + Q^*(x, y) = x + y.$$

Moreover, $S''(x, y) =$ Hamacher sum and $T''(x, y) =$ Hamacher product give the following $Q^*(x, y)$:

$$Q^*(x, y) = \frac{(x + y - 2xy)^2 + xy(1 - xy)}{(x + y - xy)(1 - xy)}.$$

Finally, when $T'(a, b) = 0 \vee (a + b - 1)$ (bounded product) in (1), $Q(x, y)$ is as follows:

$$Q(x, y) = 0 \vee (T(x, y) + S(x, y) - 1).$$

For example, for Hamacher product and sum we have

$$Q(x, y) = \frac{0 \vee (x + y - 2xy)(x + y - 1)}{(x + y - xy)(1 - xy)},$$

which is a quasi-t-norm but not a t-norm. For $T(x, y) = x \wedge y$, $xy$, $0 \vee (x + y - 1)$

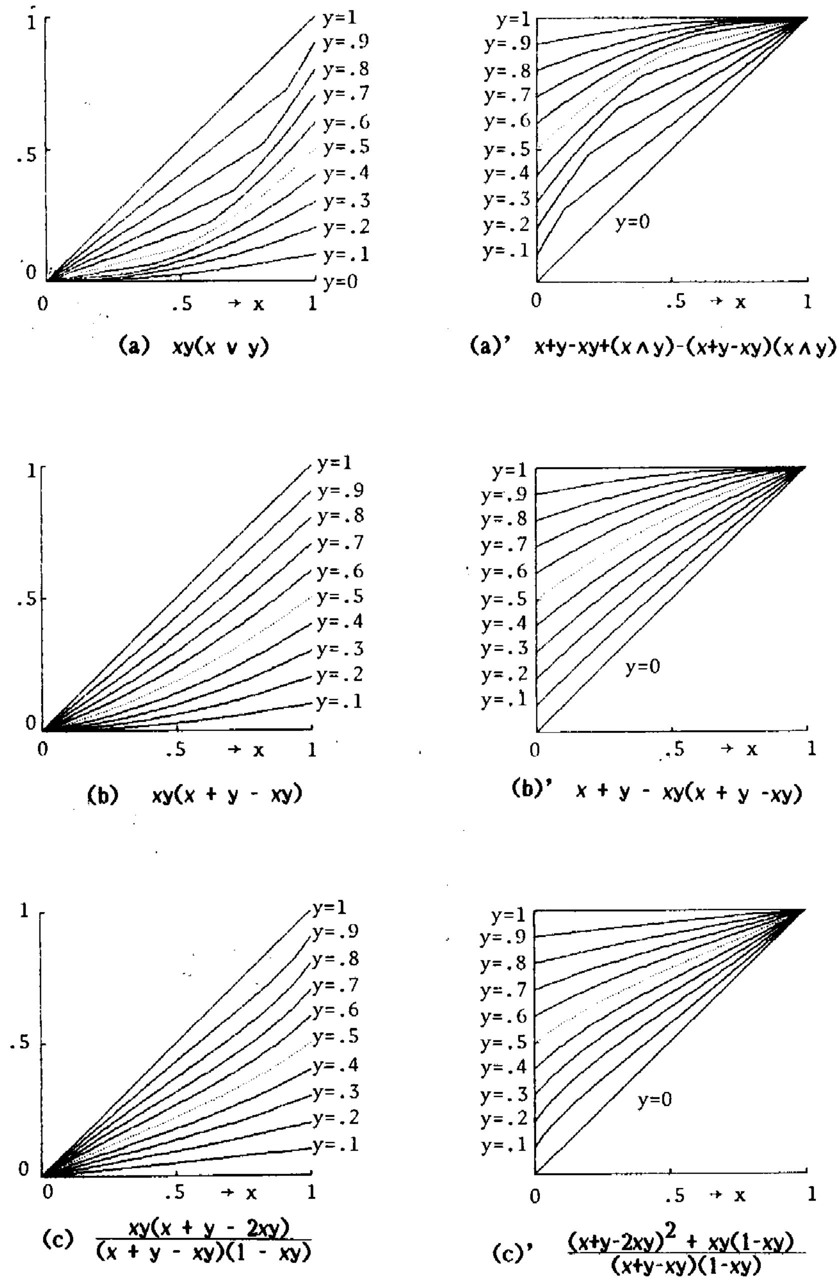
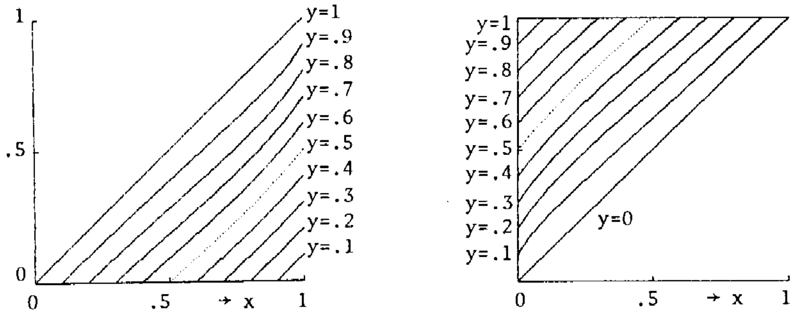



**Fig. 1.** Examples of quasi-t-norms $Q(x, y)$ (left) and quasi-t-conorms $Q^*(x, y)$ (right).




(d) $\dfrac{0 \vee (x+y-2xy)(x+y-1)}{(x+y-xy)(1-xy)}$ (d)' $1 \wedge \dfrac{(x+y)^2+xy[xy-3(x+y)-1]}{(x+y-xy)(1-xy)}$

Fig. 1 (continued).

and their duals $S(x, y)$, we have $Q(x, y) = 0 \vee (x + y - 1)$ which is bounded product.

Dually, when $S'(a, b) = 1 \wedge (a + b)$ (bounded sum), $S''$ = Hamacher sum and $T''$ = Hamacher product,

$$Q^*(x, y) = 1 \wedge (S''(x, y) + T''(x, y))$$
$$= 1 \wedge \frac{(x + y)^2 + xy[xy - 3(x + y) - 1]}{(x + y - xy)(1 - xy)}.$$

The above examples of $Q(x, y)$ are all quasi-t-norms but not t-norms, and $Q^*(x, y)$ are also quasi-t-conorms but not t-conorms, since they do not satisfy the associativity. These examples of quasi-t-norms $Q(x, y)$ and quasi-t-conorms $Q^*(x, y)$ are shown in Figure 1.

In a way similar to the cases of quasi-t-norms and quasi-t-conorms, we can define an operator

$$A(x, y) = T(T_1(x, y), T_2(x, y)) \tag{3}$$

and its dual

$$A^*(x, y) = S(S_1(x, y), S_2(x, y)) \tag{4}$$

where $S$, $S_1$ and $S_2$ are t-conorms dual to t-norms $T$, $T_1$, and $T_2$, respectively. $A(x, y)$ and its dual $A^*(x, y)$ satisfy the following properties:

(a) $A(x, 0) = A(0, x) = 0$, $A^*(x, 1) = A^*(1, x) = 1$,
$A(x, 1) = A(1, x) = T(x, x)$, $A^*(x, 0) = A^*(0, x) = S(x, x)$;

(b) $x' \leq x'', y' \leq y'' \Rightarrow A(x', y') \leq A(x'', y'')$
$\Rightarrow A^*(x', y') \leq A^*(x'', y'')$;

(c) $A(x, y) = A(y, x)$, $A^*(x, y) = A^*(y, x)$;

(d) $A(x, y) \leq T_1(x, y) \wedge T_2(x, y) \leq x \wedge y$,
$A^*(x, y) \geq S_1(x, y) \vee S_2(x, y) \geq x \vee y$;

(e) $A(x, y)$ and $A^*(x, y)$ are not necessarily associative.

For example, let $T(a, b) = ab$, $T_1(x, y) = xy$ and $T_2(x, y) = x \wedge y$ in (3); then

$$A(x, y) = xy(x \wedge y)$$

and its dual is

$$A^*(x, y) = x + y - xy + (x \vee y) - (x + y - xy)(x \vee y).$$

Furthermore, we can give an operator

$$B(x, y) = S(T_1(x, y), T_2(x, y)) \tag{5}$$

and its dual

$$B^*(x, y) = T(S_1(x, y), S_2(x, y)) \tag{6}$$

where $T$, $S_1$ and $S_2$ are dual to $S$, $T_1$ and $T_2$, respectively. They satisfy the following:

(a) $B(x, 0) = B(0, x) = 0$, $B^*(x, 1) = B^*(1, x) = 1$,
$B(x, 1) = B(1, x) = S(x, x)$, $B^*(x, 0) = B^*(0, x) = T(x, x)$,

(b) $B(x, y)$ and $B^*(x, y)$ are increasing and commutative;

(c) $B(x, y) \geqslant T_1(x, y) \vee T_2(x, y)$, $B^*(x, y) \leqslant S_1(x, y) \wedge S_2(x, y)$;

(d) $B(x, y)$ and $B^*(x, y)$ are not necessarily associative.

For example, if $S(a, b) = a + b - ab$, $T_1(x, y) = xy$ and $T_2(x, y) = x \wedge y$ in (5), we have

$$B(x, y) = xy + (x \wedge y) - xy(x \wedge y)$$

and its dual is

$$B^*(x, y) = (x + y - xy)(x \vee y).$$

Figure 2 shows examples of $A(x, y)$, $A^*(x, y)$, $B(x, y)$ and $B^*(x, y)$. These new operations will be found in the sequel to be useful in generating new compensatory operators, self-dual operators and symmetric sums.

## 3. Compensatory operators

Recent empirical works [12] indicate that min ($\wedge$) and algebraic product ($\cdot$) are not very appropriate to model the human use of the 'and'. The *compensatory operator*, which seems to be more adequate in human decision making, is defined by Zimmermann [12] as follows.

$$(xy)^{1-p} \cdot (x + y - xy)^p, \quad 0 \leqslant p \leqslant 1.$$

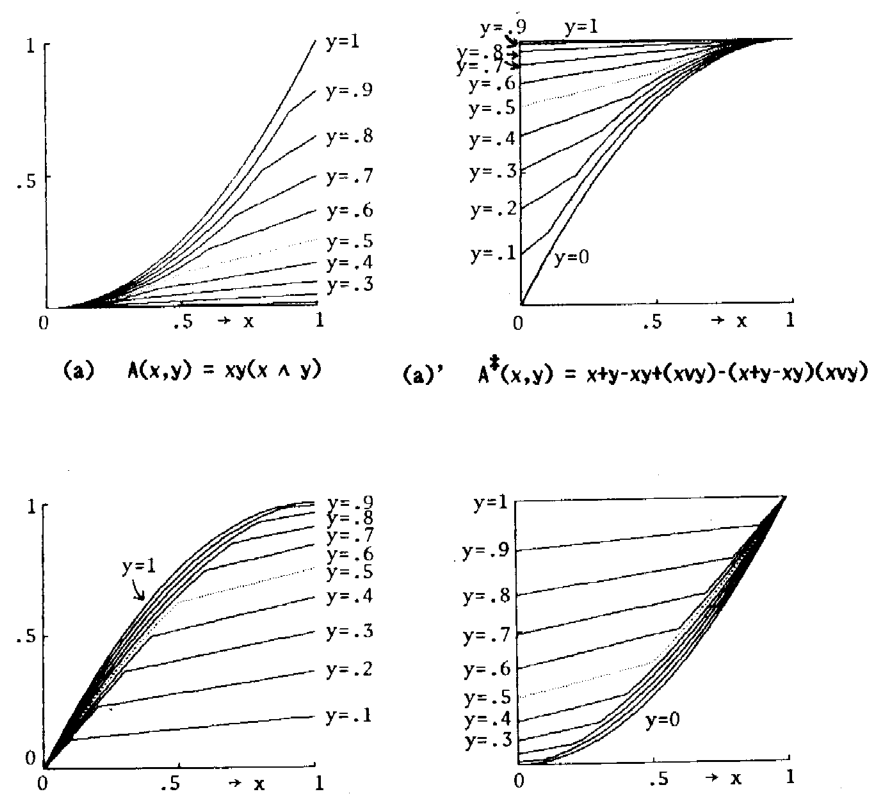
It is possible to define alternative compensatory operators by taking the convex combination of min ($\wedge$) and max ($\vee$) [13].

$$(x \wedge y)(1 - p) + (x \vee y)p, \quad 0 \leqslant p \leqslant 1.$$

In general, we can obtain many kinds of compensatory operators by using t-norms $T(x, y)$ and t-conorms $S(x, y)$ dual to $T(x, y)$. For $0 \leqslant p \leqslant 1$, we have

$$C(x, y) = T(x, y)^{1-p} \cdot S(x, y)^p, \tag{7}$$

$$C(x, y) = T(x, y)(1 - p) + S(x, y)p. \tag{8}$$




(a) A(x,y) = xy(x ∧ y) (a)' A*(x,y) = x+y-xy+(x∨y)-(x+y-xy)(x∨y)

(b) B(x,y) = xy+(x∧y)-xy(x∧y) (b)' B*(x,y) = (x+y-xy)(x∨y)

**Fig. 2.** Examples of $A(x, y)$, $A^*(x, y)$, $B(x, y)$ and $B^*(x, y)$ in (3)–(6).

Clearly we have

(i) $C(0, 0) = 0$, $C(1, 1) = 1$;

(v) $T(x, y)_{(p=0)} \leq C(x, y) \leq S(x, y)_{(p=1)}$;

and $C(x, y)$ is commutative, continuous and increasing.

It is found that compensatory operators $(x \wedge y)^{1-p} \cdot (x \vee y)^p$ and $(x \wedge y)(1-p) + (x \vee y)p$ are averaging operators since they are between $x \wedge y$ and $x \vee y$.

The De Morgan-like dual of compensatory operator (7) is given as

$$\begin{aligned} C^*(x, y) &= 1 - T(1-x, 1-y)^{1-p} \cdot S(1-x, 1-y)^p \\ &= 1 - (1 - S(x, y))^{1-p} \cdot (1 - T(x, y))^p. \end{aligned} \tag{9}$$

The dual of (8) is obtained as follows:

$$\begin{aligned} C^*(x, y) &= 1 - [T(1-x, 1-y)(1-p) + S(1-x, 1-y)p] \\ &= S(x, y)(1-p) + T(x, y)p, \end{aligned} \tag{10}$$

by using the equalities $S(x, y) = 1 - T(1-x, 1-y)$ and $T(x, y) = 1 - S(1-x, 1-y)$.

It is possible in (7) and (8) to use t-norms $T(x, y)$ and t-conorms $S'(x, y)$ which are not dual to $T(x, y)$. For $0 \leq p \leq 1$,

$$T(x, y)^{1-p} \cdot S'(x, y)^p, \tag{11}$$

$$T(x, y)(1-p) + S'(x, y)p. \tag{12}$$

For example, Luhandjula [6] gives as a special case of (12)

$$(x \wedge y)(1-p) + (1 \wedge (x+y))p, \quad 0 \leq p \leq 1.$$

Furthermore, we can define compensatory operators by combining two kinds of t-norms $T(x, y)$ and $T'(x, y)$, or t-cornorms $S(x, y)$ and $S'(x, y)$. For $0 \leq p \leq 1$,

$$T(x, y)^{1-p} \cdot T'(x, y)^p, \tag{13}$$

$$T(x, y)(1-p) + T'(x, y)p, \tag{14}$$

$$S(x, y)^{1-p} \cdot S'(x, y)^p, \tag{15}$$

$$S(x, y)(1-p) + S'(x, y)p. \tag{16}$$

For example, Sales [8] gives the following operators as special cases of (14) and (16):

$$[0 \vee (x+y-1)](1-p) + (x \wedge y)p,$$

$$[1 \wedge (x+y)](1-p) + (x \vee y)p.$$

He uses the former as 'and' and the latter as 'or'.

We can give other kinds of compensatory operators by combining t-norms $T(x, y)$ (or t-conorms $S(x, y)$) and averaging operators $M(x, y)$. For $0 \leq p \leq 1$,

$$T(x, y)^{1-p} \cdot M(x, y)^p, \tag{17}$$

$$S(x, y)^{1-p} \cdot M(x, y)^p, \tag{18}$$

$$T(x, y)(1-p) + M(x, y)p, \tag{19}$$

$$S(x, y)(1-p) + M(x, y)p. \tag{20}$$

For example, Werners [11] introduces compensatory operators by letting $T(x, y) = x \wedge y$, $S(x, y) = x \vee y$ and $M(x, y) = \frac{1}{2}(x+y)$ in (19) and (20):

$$(x \wedge y)(1-p) + \frac{x+y}{2}p, \qquad (x \vee y)(1-p) + \frac{x+y}{2}p.$$

She calls the former 'fuzzy and' and the latter 'fuzzy or'.

We can give the following compensatory operators by introducing two averaging operators $M(x, y)$ and $M'(x, y)$:

$$M(x, y)^{1-p} \cdot M'(x, y)^p, \tag{21}$$

$$M(x, y)(1-p) + M'(x, y)p. \tag{22}$$

Note that the compensatory operators are averaging operators.

In general, we can define compensatory operators by

$$F(x, y)^{1-p} \cdot G(x, y)^p, \tag{23}$$

$$F(x, y)(1-p) + G(x, y)p, \tag{24}$$

where quasi-t-norms, quasi-t-conorms, $A(x, y)$, $A^*(x, y)$, $B(x, y)$ and $B^*(x, y)$ of (3)–(6), compensatory operators, generalized compensatory operators, self-dual operators and symmetric sums to be discussed later can be used as $F(x, y)$ and $G(x, y)$.

A number of examples of compensatory operators are given in Table 1 and some of them at $p = 0.3$ are depicted in Figure 3.

In general, compensatory operators $C(x, y) = F(x, y)^{1-p} \cdot G(x, y)^p$ and $C(x, y) = F(x, y)(1-p) + G(x, y)p$ satisfy the following: For $0 \leqslant p \leqslant 1$,

$$\begin{aligned}
&\text{(i)} && C(0, 0) = 0, \quad C(1, 1) = 1; \\
&\text{(ii)} && C(x, y) = C(y, x); \\
&\text{(iii)} && x' \leqslant x'',\ y' \leqslant y'' \Rightarrow C(x', y') \leqslant C(x'', y''); \\
&\text{(iv)} && C(x, y) \text{ is continuous}; \\
&\text{(v)} && F(x, y)_{(p=0)} \leqslant C(x, y) \leqslant G(x, y)_{(p=1)},
\end{aligned} \tag{25}$$

where $F(x, y) \leqslant G(x, y)$ is assumed for $x, y \in [0, 1]$.

Table 1. Compensatory operators $C(x, y)$, $0 \leqslant p \leqslant 1$

| | | | |
|---|---|---|---|
| $C(x, y)$ of (7), (8) by t-norms and dual t-conorms | | | |
| ① | $(x \wedge y)^{1-p} \cdot (x \vee y)^p$ | ①′ | $(x \wedge y)(1-p) + (x \vee y)p$ (Zimmermann) |
| ② | $(xy)^{1-p} \cdot (x + y - xy)^p$ (Zimmermann) | ②′ | $xy(1-p) + (x + y - xy)p$ |
| ③ | $[0 \vee (x + y)]^{1-p} \cdot [1 \wedge (x + y)]^p$ | ③′ | $[0 \vee (x + y - 1)](1-p) + [1 \wedge (x + y)]p$ |
| $C(x, y)$ of (11), (12) by t-norms and t-conorms | | | |
| ④ | $(x \wedge y)^{1-p} \cdot (x + y - xy)^p$ | ④′ | $(x \wedge y)(1-p) + (x + y - xy)p$ |
| | | ↕ | |
| ⑤ | $(x \vee y)^{1-p} \cdot (xy)^p$ | ⑤′ | $(x \vee y)(1-p) + xyp$ |
| ⑥ | $(x \wedge y)^{1-p} \cdot [1 \wedge (x + y)]^p$ | ⑥′ | $(x \wedge y)(1-p) + [1 \wedge (x + y)]p$ (Luhandjula) |
| | | ↕ | |
| ⑦ | $(x \vee y)^{1-p} \cdot [0 \vee (x + y - 1)]^p$ | ⑦′ | $(x \vee y)(1-p) + [0 \vee (x + y - 1)]p$ |
| ⑧ | $(xy)^{1-p} \cdot (1 \wedge (x + y))^p$ | ⑧′ | $xy(1-p) + (1 \wedge (x + y))p$ |
| | | ↕ | |
| ⑨ | $(x + y - xy)^{1-p} \cdot [0 \vee (x + y - 1)]^p$ | ⑨′ | $(x + y - xy)(1-p) + [0 \vee (x + y - 1)]p$ |
| $C(x, y)$ of (13), (14), (15), (16) by t-norms or t-conorms | | | |
| ⑩ | $[0 \vee (x + y - 1)]^{1-p} \cdot (x \wedge y)^p$ | ⑩′ | $[0 \vee (x + y - 1)](1-p) + (x \wedge y)p$ (Sales) |
| | | ↕ | |
| ⑪ | $[1 \wedge (x + y)]^{1-p} \cdot (x \vee y)^p$ | ⑪′ | $[1 \wedge (x + y)](1-p) + (x \vee y)p$ (Sales) |
| ⑫ | $[0 \vee (x + y - 1)]^{1-p} \cdot (xy)^p$ | ⑫′ | $[0 \vee (x + y - 1)](1-p) + xyp$ |
| | | ↕ | |
| ⑬ | $[1 \wedge (x + y)]^{1-p} \cdot (x + y - xy)^p$ | ⑬′ | $[1 \wedge (x + y)](1-p) + (x + y - xy)p$ |
| ⑭ | $(xy)^{1-p} \cdot (x \wedge y)^p$ | ⑭′ | $xy(1-p) + (x \wedge y)p$ |
| | | ↕ | |
| ⑮ | $(x + y - xy)^{1-p} \cdot (x \vee y)^p$ | ⑮′ | $(x + y - xy)(1-p) + (x \vee y)p$ |

The symbol ↕ represents duality.

(*continued*).

Table 1 (continued).

| | |
|---|---|
| $C(x, y)$ of (17), (18), (19), (20) by t-norms (or t-conorms) and averaging operators | |
| (16) $(x \wedge y)^{1-p} \cdot \left(\frac{x+y}{2}\right)^p$ | (16)′ $(x \wedge y)(1-p) + \frac{x+y}{2}p$ (Werners) |
| | ↕ |
| (17) $(x \vee y)^{1-p} \cdot \left(\frac{x+y}{2}\right)^p$ | (17)′ $(x \vee y)(1-p) + \frac{x+y}{2}p$ (Werners) |
| (18) $(xy)^{1-p} \cdot \left(\frac{x+y}{2}\right)^p$ | (18)′ $xy(1-p) + \frac{x+y}{2}p$ |
| | ↕ |
| (19) $(x+y-xy)^{1-p} \cdot \left(\frac{x+y}{2}\right)^p$ | (18)′ $(x+y-xy)(1-p) + \frac{x+y}{2}p$ |
| (20) $[0 \vee (x+y-1)]^{1-p} \cdot \left(\frac{x+y}{2}\right)^p$ | (20)′ $[0 \wedge (x+y-1)](1-p) + \frac{x+y}{2}p$ |
| | ↕ |
| (21) $[1 \wedge (x+y)]^{1-p} \cdot \left(\frac{x+y}{2}\right)^p$ | (21)′ $[1 \wedge (x+y)](1-p) + \frac{x+y}{2}p$ |
| (22) $(x \wedge y)^{1-p} \cdot (\sqrt{xy})^p$ | (22)′ $(x \wedge y)(1-p) + (\sqrt{xy})p$ |
| | ↕ |
| (23) $(x \vee y)^{1-p} \cdot [1 - \sqrt{(1-x)(1-y)}]^p$ | (23)′ $(x \vee y)(1-p) + [1 - \sqrt{(1-x)(1-y)}]p$ |
| (24) $(x \wedge y)^{1-p} \cdot [1 - \sqrt{(1-x)(1-y)}]^p$ | (24)′ $(x \wedge y)(1-p) + [1 - \sqrt{(1-x)(1-y)}]p$ |
| | ↕ |
| (25) $(x \vee y)^{1-p} \cdot (\sqrt{xy})^p$ | (25)′ $(x \vee y)(1-p) + (\sqrt{xy})p$ |
| $C(x, y)$ of (21), (22) by averaging operators | |
| (26) $(\sqrt{xy})^{1-p} \cdot [1 - \sqrt{(1-x)(1-y)}]^p$ | (26)′ $\sqrt{xy}(1-p) + [1 - \sqrt{(1-x)(1-y)}]p$ |
| (27) $\left(\frac{2xy}{x+y}\right)^{1-p} \cdot \left(\frac{x+y-2xy}{2-x-y}\right)^p$ | (27)′ $\frac{2xy}{x+y}(1-p) + \frac{x+y-2xy}{2-x-y}p$ |
| (28) $\left(\frac{x+y}{2}\right)^{1-p} \cdot (\sqrt{xy})^p$ | (28)′ $\left(\frac{x+y}{2}\right)(1-p) + (\sqrt{xy})p$ |
| | ↕ |
| (29) $\left(\frac{x+y}{2}\right)^{1-p} \cdot [1 - \sqrt{(1-x)(1-y)}]^p$ | (29)′ $\left(\frac{x+y}{2}\right)(1-p) + [1 - \sqrt{(1-x)(1-y)}]p$ |
| (30) $\left(\frac{x+y}{2}\right)^{1-p} \cdot \left(\frac{2xy}{x+y}\right)^p$ | (30)′ $\left(\frac{x+y}{2}\right)(1-p) + \left(\frac{2xy}{x+y}\right)p$ |
| | ↕ |
| (31) $\left(\frac{x+y}{2}\right)^{1-p} \cdot \left(\frac{x+y-2xy}{2-x-y}\right)^p$ | (31)′ $\left(\frac{x+y}{2}\right)(1-p) + \left(\frac{x+y-2xy}{2-x-y}\right)^p$ |
| $C(x, y)$ of (23), (24) by any functions $F$ and $G$ | |
| (32) $[xy(x+y-xy)]^{1-p} \cdot [x+y-xy(x+y-xy)]^p$ | (32)′ $[xy(x+y-xy)](1-p) + [x+y-xy(x+y-xy)]p$ |
| (33) $[xy(x \wedge y)]^{1-p} \cdot [x+y-xy+x \vee y-(x+y-xy)(x \vee y)]^p$ | (33)′ $[xy(x \wedge y)](1-p) + [x+y-xy+x \vee y-(x+y-xy)(x \vee y)]p$ |
| (34) $[xy+(x \wedge y)-xy(x \wedge y)]^{1-p} \cdot [(x+y-xy)(x \vee y)]^p$ | (34)′ $[xy+(x \wedge y)-xy(x \wedge y)](1-p) + [(x+y-xy)(x \vee y)]p$ |

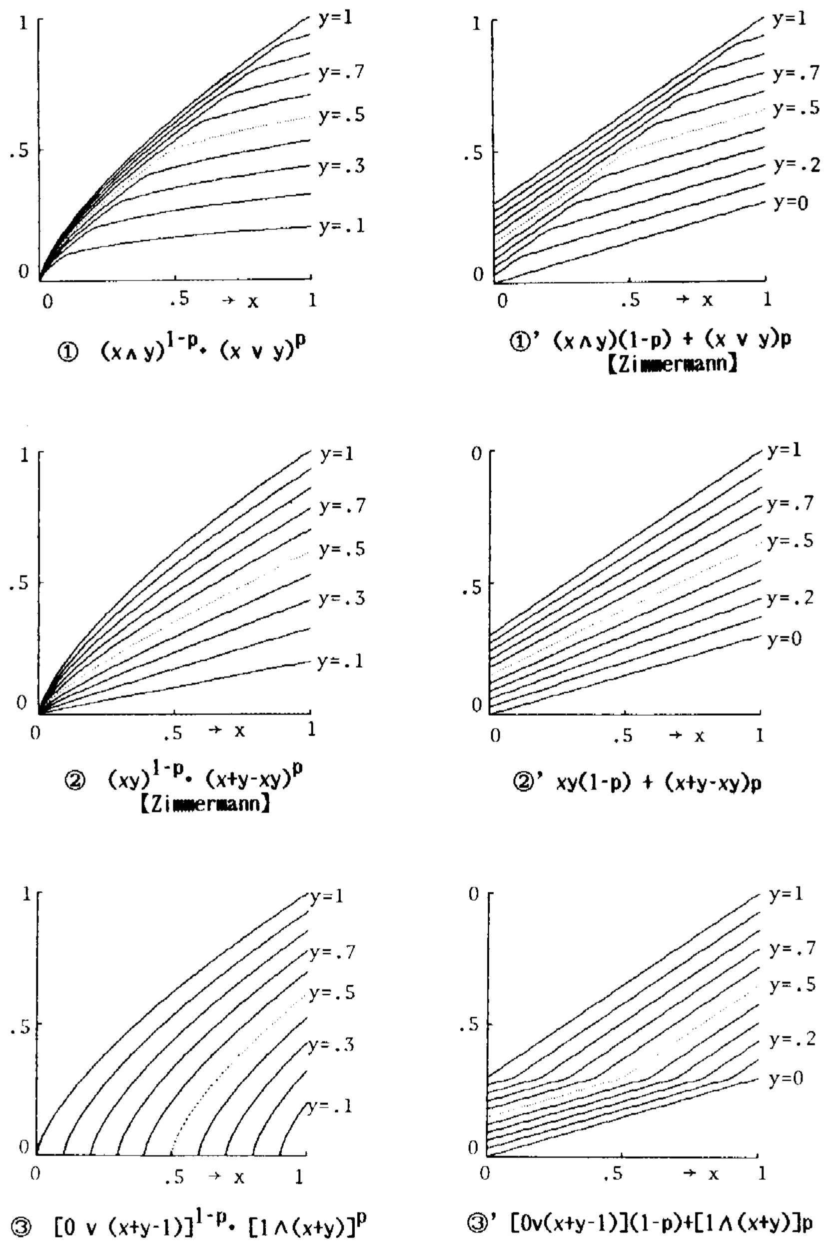



Fig. 3. Examples of compensatory operators in Table 1 ($p = 0.3$).

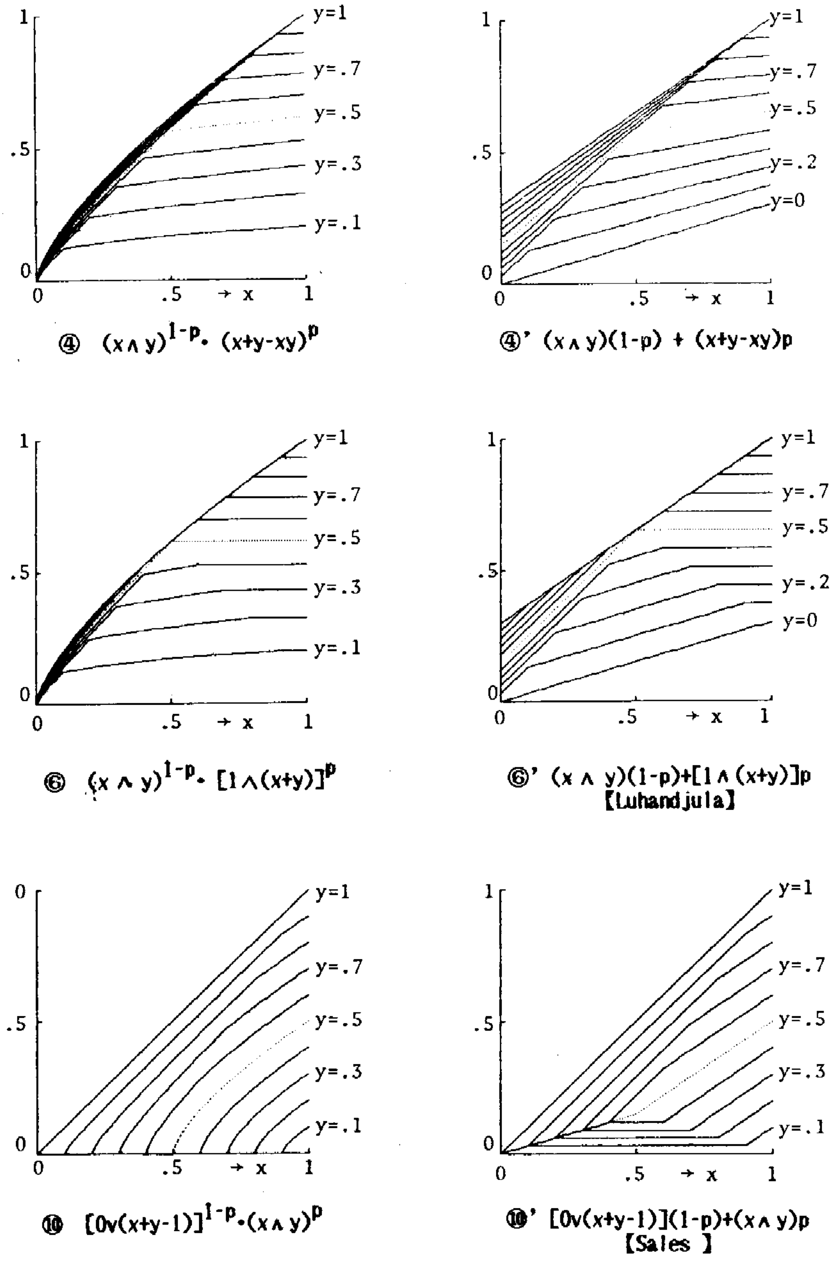
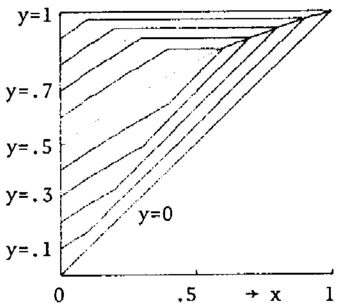
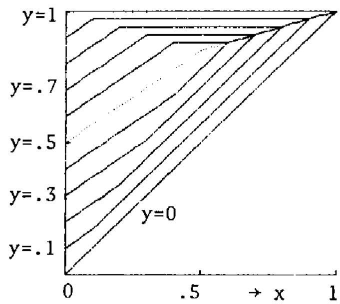


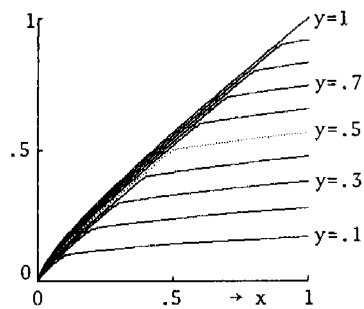

Fig. 3 (continued).



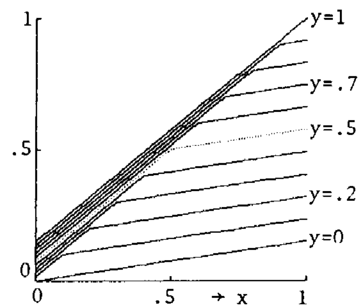

⑪ $[1 \wedge (x+y)]^{1-p} \cdot (x \vee y)^{p}$



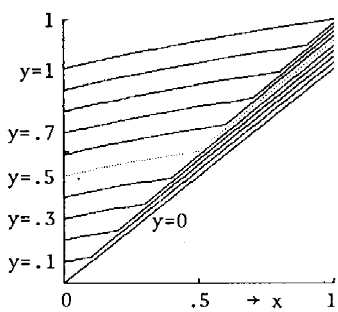

⑪' $[1 \wedge (x+y)](1-p)+(x \vee y)p$
[Sales]



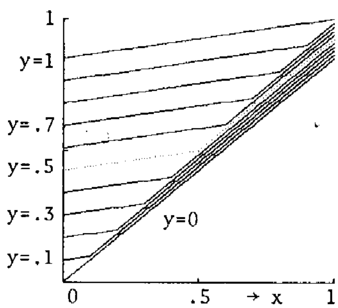

⑯ $(x \wedge y)^{1-p} \cdot [(x+y)/2]^{p}$




⑯' $(x \wedge y)(1-p)+[(x+y)/2]p$
[Werners]

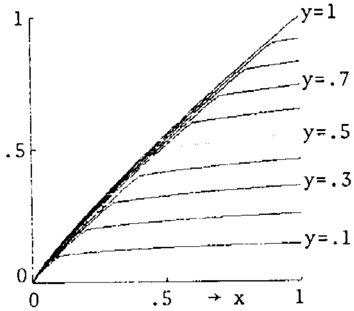



⑰ $(x \vee y)^{1-p} \cdot [(x+y)/2]^{p}$

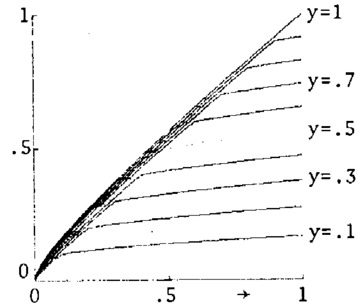



⑰' $(x \vee y)(1-p)+[(x+y)/2]p$
[Werners]

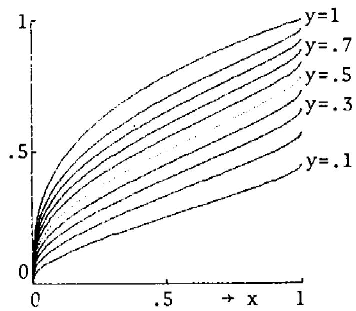
Fig. 3 (continued).



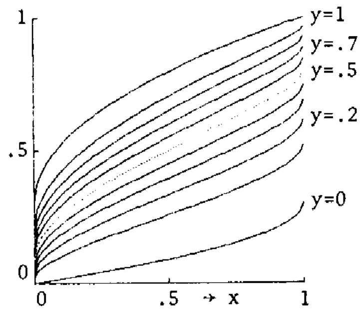

㉒ $(x \wedge y)^{1-p} \cdot (\sqrt{xy})^{p}$



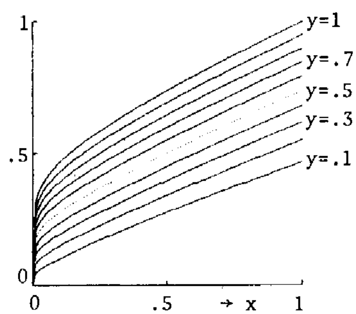

㉒' $(x \wedge y)(1-p)+(\sqrt{xy})p$



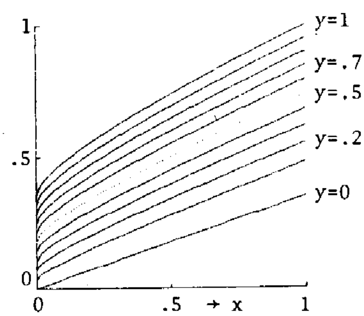

㉖ $(\sqrt{xy})^{1-p} \cdot [1-\sqrt{(1-x)(1-y)}]^{p}$




㉖' $\sqrt{xy}(1-p)+[1-\sqrt{(1-x)(1-y)}]p$

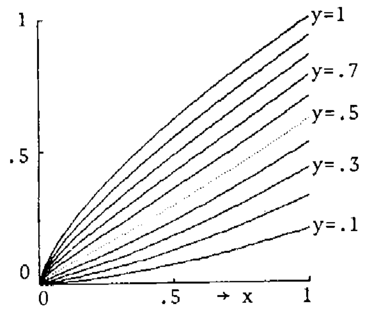



㉘ $[(x+y)/2]^{1-p} \cdot (\sqrt{xy})^{p}$

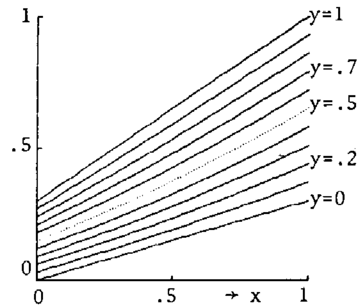



㉘' $[(x+y)/2](1-p)+(\sqrt{xy})p$

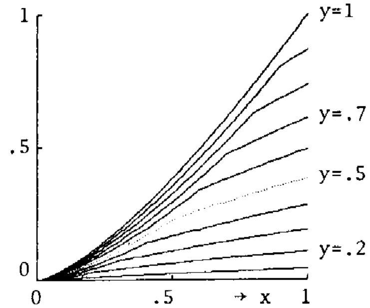
Fig. 3 (continued).



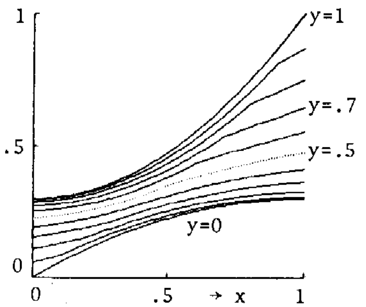

(32) $[xy(x+y-xy)]^{1-p} \cdot [x+y-xy(x+y-xy)]^{p}$



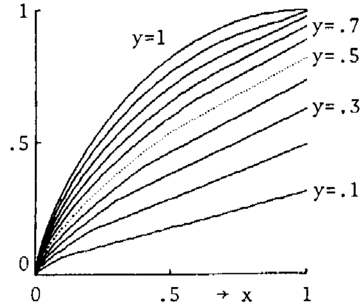

(32)' $[xy(x+y-xy)](1-p) + [x+y-xy(x+y-xy)]p$



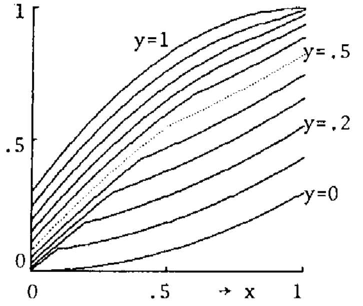

(33) $[xy(x \wedge y)]^{1-p} \cdot [x+y-xy+x \vee y-(x+y-xy)(x \vee y)]^{p}$




(33)' $[xy(x \wedge y)](1-p) + [x+y-xy+x \vee y-(x+y-xy)(x \vee y)]p$




(34) $[xy+(x \wedge y)-xy(x \wedge y)]^{1-p} \cdot [(x+y-xy)(x \vee y)]^{p}$




(34)' $[xy+(x \wedge y)-xy(x \wedge y)](1-p) + [(x+y-xy)(x \vee y)]p$

Fig. 3 (continued).

The duals of compensatory operators $C(x, y) = F(x, y)^{1-p} \cdot G(x, y)^p$ and $C(x, y) = F(x, y)(1-p) + G(x, y)p$ are given, respectively, as follows:

$$C^*(x, y) = 1 - (1 - F^*(x, y))^{1-p} \cdot (1 - G^*(x, y))^p, \tag{26}$$

$$C^*(x, y) = F^*(x, y)(1-p) + G^*(x, y)p, \tag{27}$$

where $F^*(x, y)$ and $G^*(x, y)$ are dual to $F(x, y)$ and $G(x, y)$, respectively.

For example, the dual of $C(x, y) = [0 \vee (x + y - 1)](1-p) + (x \wedge y)p$ by Sales given before is $C^*(x, y) = [1 \wedge (x + y)](1-p) + (x \vee y)p$. Similarly, for $C(x, y) = (x \wedge y)(1-p) + \frac{1}{2}(x + y)p$ by Werners, we have $C^*(x, y) = (x \vee y)(1-p) + \frac{1}{2}(x + y)p$.

Obviously, the duals of compensatory operators are also compensatory operators. It is noted that we have

$$G^*(x, y)_{(p=1)} \leq C^*(x, y) \leq F^*(x, y)_{(p=0)}$$

since $G^*(x, y) \leq F^*(x, y)$ owing to the assumption of $F(x, y) \leq G(x, y)$.

As a generalization of compensatory operators, we shall give an operator $\hat{C}$ as

$$\hat{C}(x, y) = M(F(x, y), G(x, y)) \tag{28}$$

where $M(x, y)$ is an averaging operator and $F$ and $G$ are t-norms, t-conorms or averaging operators. It is assumed that $F(x, y) \leq G(x, y)$ holds. Clearly we have the property such that $F(0, 0) = G(0, 0) = 0$, $F(1, 1) = G(1, 1) = 1$ and $F$ and $G$ are commutative and increasing. Any two-place functions satisfying the property can be used as $F$ and $G$. For example, quasi-t-norms, quasi-t-conorms, $A(x, y)$, $A^*(x, y)$, $B(x, y)$ and $B^*(x, y)$ in (3)–(6), self-dual operators, symmetric sums and compensatory operators may be adopted as $F$ and $G$.

It is shown that $\hat{C}(x, y)$ contains compensatory operators as special cases. Namely, let $M(x, y) = (x \wedge y)^{1-p} \cdot (x \vee y)^p$, which is a compensatory operator as well as an averaging operator. Then we have $x \wedge y \leq (x \wedge y)^{1-p} \cdot (x \vee y)^p \leq x \vee y$ for $0 \leq p \leq 1$ so that

$$\begin{aligned}\hat{C}(x, y) &= M(F(x, y), G(x, y)) = (F(x, y) \wedge G(x, y))^{1-p} \cdot (F(x, y) \vee G(x, y))^p \\ &= F(x, y)^{1-p} \cdot G(x, y)^p\end{aligned}$$

from the assumption of $F(x, y) \leq G(x, y)$. Moreover, we have $F(x, y) \wedge G(x, y) = F(x, y)$ and $F(x, y) \vee G(x, y) = G(x, y)$ which leads to

$$F(x, y) \leq F(x, y)^{1-p} \cdot G(x, y)^p \leq G(x, y).$$

In the same way, we can show that compensatory operator $F(x, y)(1-p) + G(x, y)p$ is a special case of $\hat{C}(x, y)$ by letting $M(x, y) = (x \wedge y)(1-p) + (x \vee y)p$.

In the following, we shall indicate that $\hat{C}(x, y)$, named as *generalized compensatory operator*, satisfies the properties (25) of compensatory operators $C(x, y)$:

$$\hat{C}(0, 0) = M(F(0, 0), G(0, 0)) = M(0, 0) = 0,$$

$$\hat{C}(1, 1) = M(F(1, 1), G(1, 1)) = M(1, 1) = 1.$$

The commutativity (ii) and increasingness (iv) of (25) are easily proved because $F$, $G$ and $M$ are commutative and increasing. Averaging operator $M(x, y)$ satisfies $x \wedge y \leq M(x, y) \leq x \vee y$. Thus we have

$$F(x, y) \wedge G(x, y) \leq M(F(x, y), G(x, y)) \leq F(x, y) \vee G(x, y)$$

so that

$$F(x, y) \leq M(F(x, y), G(x, y)) \leq G(x, y)$$

because of the assumption that $F(x, y) \leq G(x, y)$. Hence we obtain

$$F(x, y) \leq \hat{C}(x, y) \leq G(x, y)$$

In the sequel, $\hat{C}(x, y)$ satisfies the properties of (25).

It is noted that the dual of generalized compensatory operator $\hat{C}(x, y) = M(F(x, y), G(x, y))$ is given as

$$\hat{C}^*(x, y) = M^*(F^*(x, y), G^*(x, y)) \tag{29}$$

where $M^*$, $F^*$ and $G^*$ are dual to $M$, $F$ and $G$, respectively.

The dual $\hat{C}^*(x, y)$ is also a generalized compensatory operator and we have

$$G^*(x, y) \leq \hat{C}^*(x, y) \leq F^*(x, y).$$

We next show several examples of generalized compensatory operators $\hat{C}(x, y)$ in Table 2.

When $M(a, b)$ is the arithmetic mean $\frac{1}{2}(a + b)$, $\hat{C}(x, y)$ becomes

$$\hat{C}(x, y) = \tfrac{1}{2}(F(x, y) + G(x, y)). \tag{30}$$

Table 2. Generalized compensatory operators $\hat{C}(x, y)$ derived from (28)

| | | |
|---|---|---|
| ① | $\dfrac{xy + x \vee y}{2}$ | in $[xy, x \vee y]$ |
| ② | $\sqrt{xy(x + y - xy)}$ | in $[xy, x + y - xy]$ |
| ③ | $\dfrac{2(x \wedge y)(1 \wedge (x + y))}{x \wedge y + 1 \wedge (x + y)}$ | in $[x \wedge y, 1 \wedge (x + y)]$ |
| ④ | $\dfrac{2xy(x + y - xy)}{x + y}$ | in $[xy, x + y - xy]$ |
| ⑤ | $\dfrac{2xy(x + y)}{2xy + x + y}$ | in $[xy, \frac{1}{2}(x + y)]$ |
| ⑥ | $\dfrac{4xy}{x + y + 2}$ | in $[xy, 2xy/(x + y)]$ |
| ⑦ | $\dfrac{2xy(x + y - 2xy)}{(x + y)^2 + xy[xy - 3(x + y) + 1]}$ | in $\left[\dfrac{xy}{x + y - xy}, \dfrac{x + y - 2xy}{1 - xy}\right]$ |
| ⑧ | $\dfrac{2\sqrt{xy} + x + y}{4}$ | in $[\sqrt{xy}, \frac{1}{2}(x + y)]$ |
| ⑨ | $\sqrt[p]{\dfrac{(xy)^p + (x + y - xy)^p}{2}}$ | in $[xy, x + y - xy]$ |
| ⑩ | $\sqrt[p]{\dfrac{(xy)^p + (x \vee y)^p}{2}}$ | in $[xy, x \vee y]$ |

For example, let $F(x, y) = xy$ and $G(x, y) = x \vee y$, then we have $\hat{C}(x, y) = \frac{1}{2}(xy + x \vee y)$. Clearly, $\hat{C}(x, y)$ of (30) is a compensatory operator of the form $F(x, y)(1-p) + G(x, y)p$ for $p = 0.5$.

Similarly, when $M(a, b) = \sqrt{ab}$, we obtain $\hat{C}(x, y)$ as

$$\hat{C}(x, y) = \sqrt{F(x, y)G(x, y)}. \tag{31}$$

For example, if $F(x, y) = xy$ and $G(x, y) = x + y - xy$, then $\hat{C}(x, y)$ is

$$\hat{C}(x, y) = \sqrt{xy(x + y - xy)}.$$

$\hat{C}(x, y)$ of (31) corresponds to a compensatory operator $F(x, y)^{1-p} \cdot G(x, y)^p$ for $p = 0.5$.

Let $M(a, b)$ be the harmonic mean $2ab/(a + b)$; then $\hat{C}(x, y)$ is given as

$$\hat{C}(x, y) = \frac{2F(x, y)G(x, y)}{F(x, y) + G(x, y)}.$$

For example, when $F(x, y) = x \wedge y$ and $G(x, y) = 1 \wedge (x + y)$, we have

$$\hat{C}(x, y) = \frac{2(x \wedge y)(1 \wedge (x + y))}{x \wedge y + 1 \wedge (x + y)}.$$

Further, when $F(x, y) = xy$ and $G(x, y) = x + y - xy$, $\hat{C}(x, y)$ becomes

$$\hat{C}(x, y) = \frac{2xy(x + y - xy)}{x + y}.$$

In case $F(x, y) = xy$ and $G(x, y) = \frac{1}{2}(x + y)$, $\hat{C}(x, y)$ is

$$\hat{C}(x, y) = \frac{2xy(x + y)}{2xy + x + y}$$

and if $G(x, y) = 2xy/(x + y)$ then

$$\hat{C}(x, y) = \frac{4xy}{x + y + 2}.$$

From $F(x, y)$ = Hamacher product and $G(x, y)$ = Hamacher sum, we have

$$\hat{C}(x, y) = \frac{2xy(x + y - 2xy)}{(x + y)^2 + xy[xy - 3(x + y) + 1]}$$

Let $M(a, b) = \sqrt[p]{\frac{1}{2}(a^p + b^p)}$ be a quasi-linear averaging operator; then we can obtain the following generalized compensatory operator:

$$\hat{C}(x, y) = \sqrt[p]{\frac{F(x, y)^p + G(x, y)^p}{2}}, \quad -\infty \leq p \leq \infty, \tag{32}$$

where $F(x, y)_{(p=-\infty)} \leq \hat{C}(x, y) \leq G(x, y)_{(p=+\infty)}$. For example, if $F(x, y) = x \wedge y$ and $G(x, y) = x \vee y$, then

$$\hat{C}(x, y) = \sqrt[p]{\frac{(x \wedge y)^p + (x \vee y)^p}{2}} = \sqrt[p]{\frac{x^p + y^p}{2}}.$$

When $F(x, y) = xy$ and $G(x, y) = x + y - xy$, we have

$$\hat{C}(x, y) = \sqrt[p]{\frac{(xy)^p + (x + y - xy)^p}{2}}.$$

Examples of generalized compensatory operators $\hat{C}(x, y)$ are shown in Figure 4.

It should be noted that generalized compensatory operator $\hat{C}(x, y)$ becomes an averaging operator when $F(x, y)$ and $G(x, y)$ are averaging operators. Namely,

$$\hat{C}(x, y) = M(M'(x, y), M''(x, y)) \tag{33}$$

is an averaging operator, where $M$, $M'$ and $M''$ are averaging operators.

Since $M$, $M'$ and $M''$ are averaging operators, we have $x \wedge y \leqslant M(x, y)$, $M'(x, y)$, $M''(x, y) \leqslant x \vee y$. Thus,

$$\begin{aligned} x \wedge y &\leqslant M'(x, y) \wedge M''(x, y) \leqslant M(M'(x, y), M''(x, y)) \\ &\leqslant M'(x, y) \vee M''(x, y) \leqslant x \vee y \end{aligned}$$

so that

$$x \wedge y \leqslant \hat{C}(x, y) \leqslant x \vee y.$$

In the sequel, $\hat{C}(x, y)$ is an averaging operator.

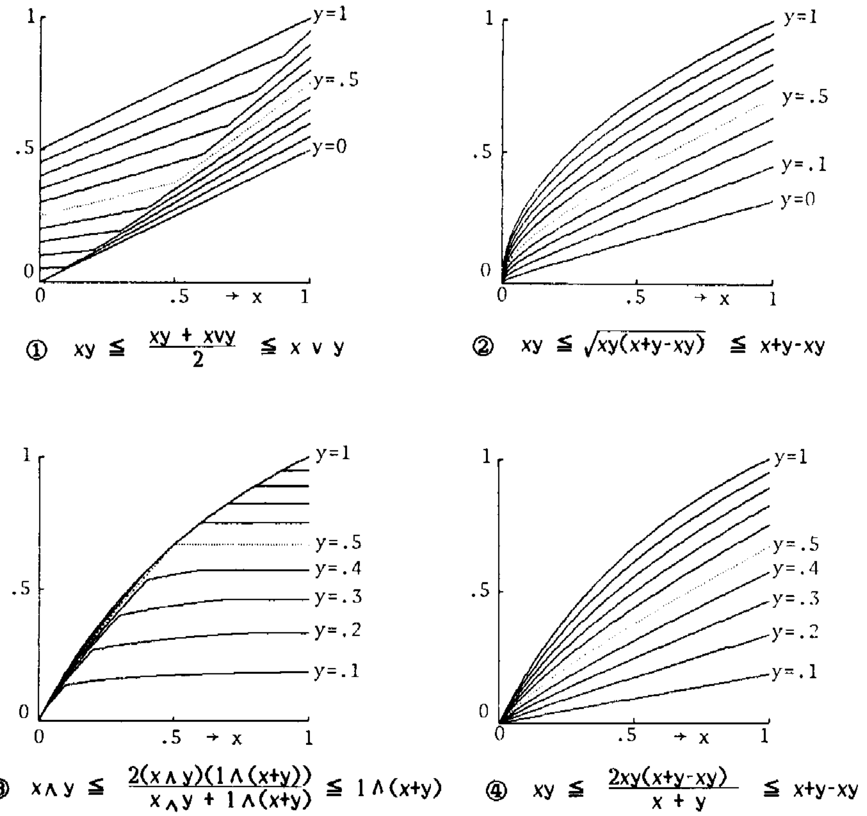
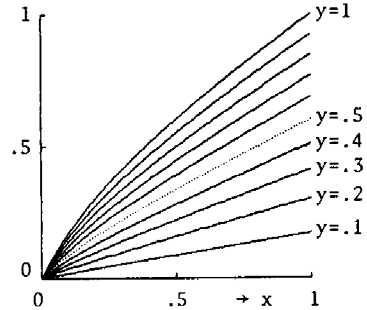
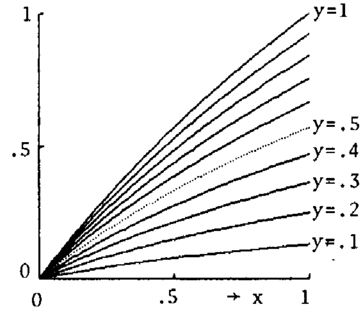


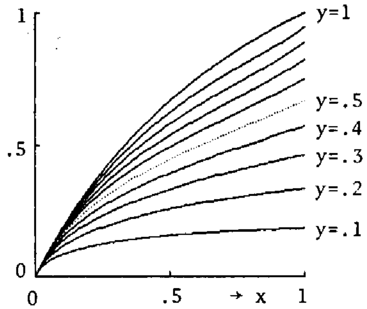

Fig. 4. Examples of generalized compensatory operators in Table 2 generated from (28).




⑤ $xy \leq \frac{2xy(x+y)}{2xy+x+y} \leq \frac{x+y}{2}$

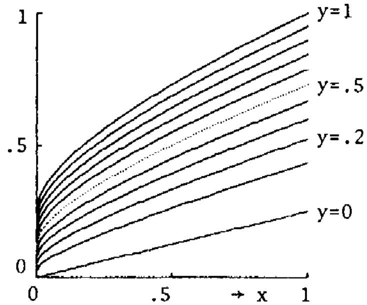



⑥ $xy \leq \frac{4xy}{x+y+2} \leq \frac{2xy}{x+y}$

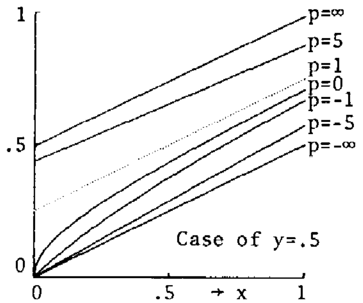



⑦ $\frac{2xy(x+y-2xy)}{(x+y)^2+xy[xy-3(x+y)+1]}$

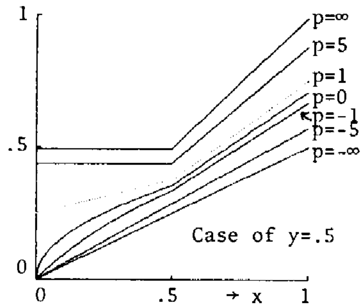
between $\frac{xy}{x+y-xy}$ and $\frac{x+y-2xy}{1-xy}$




⑧ $\sqrt{xy} \leq \frac{2\sqrt{xy}+x+y}{4} \leq \frac{x+y}{2}$




⑨ $xy \leq \sqrt[p]{\frac{(xy)^p+(x+y-xy)^p}{2}} \leq x+y-xy$




⑩ $xy \leq \sqrt[p]{\frac{(xy)^p+(x \vee y)^p}{2}} \leq x \vee y$

Fig. 4 (continued).

For example, let $M(a, b) = \frac{1}{2}(a+b)$, $M'(x, y) = \sqrt{xy}$ and $M''(x, y) = \frac{1}{2}(x+y)$, then

$$\hat{C}(x, y) = \frac{\sqrt{xy} + \frac{1}{2}(x+y)}{2} = \frac{2\sqrt{xy} + x + y}{4}$$

which is an averaging operator.

We shall next discuss self-dual operators which can be regarded as special cases of generalized compensatory operators in which $M(x, y) = \frac{1}{2}(x+y)$.

## 4. Self-dual operators

We shall next show self-dual operators which can be obtained by using t-norms, t-conorms and averaging operators. Moreover, we summarize self-dual operators named 'symmetric sums' by Silvert [10].

It is found from (8) and (10) that (8) is self-dual at $p = 0.5$. Namely, we have $C(x, y) = \frac{1}{2}(T(x, y) + S(x, y)) = 1 - C(1-x, 1-y)$. Thus,

$$D(x, y) = \frac{T(x, y) + S(x, y)}{2} \tag{34}$$

is a self-dual operator, where $T(x, y)$ is a t-norm and $S(x, y)$ is a t-conorm dual to $T(x, y)$. Therefore, the following are also self-dual operators:

$$D(x, y) = \frac{T(x, y) + 1 - T(1-x, 1-y)}{2}, \tag{35}$$

$$D(x, y) = \frac{S(x, y) + 1 - S(1-x, 1-y)}{2}. \tag{36}$$

Self-dual operators $D(x, y)$ satisfy the following:

$$\begin{aligned}
&\text{(i)} && D(0, 0) = 0, \quad D(1, 1) = 1, \\
&\text{(ii)} && D(x, y) = D(y, x), \\
&\text{(iii)} && D(x, y) \text{ is continuous and increasing,} \\
&\text{(iv)} && D(x, y) = 1 - D(1-x, 1-y), \\
&\text{(v)} && D(x, 1-x) = \tfrac{1}{2}.
\end{aligned} \tag{37}$$

For example, let $T(x, y) = x \wedge y$ and $S(x, y) = x \vee y$; then we have from (34),

$$D(x, y) = \frac{x \wedge y + x \vee y}{2} = \frac{x+y}{2},$$

which is self-dual. Similarly, we can have $\frac{1}{2}(x+y)$ from $T(x, y) = xy$ and $0 \vee (x+y-1)$. As for drastic product $x \mathbin{\dot\wedge} y$ and drastic sum $x \mathbin{\dot\vee} y$ (see Table 1 of

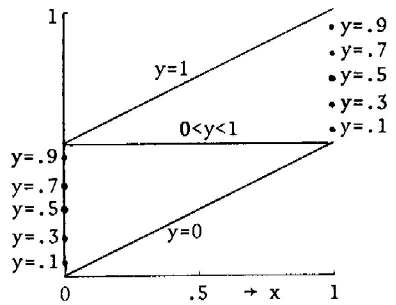



Fig. 5. Self-dual operator $D(x, y) = \frac{1}{2}(x \wedge y + x \vee y)$ of (38).

[7]), we have the following operator which can be regarded as a self-dual operator in a limit case (see Figure 5):

$$D(x, y) = \frac{x \wedge y + x \vee y}{2} = \begin{cases} \frac{1}{2}x, & y = 0, \\ \frac{1}{2}y, & x = 0, \\ \frac{1}{2}, & 0 < x, y < 1, \\ \frac{1}{2}(x+1), & y = 1, \\ \frac{1}{2}(y+1) & x = 1. \end{cases} \tag{38}$$

From the Hamacher product and sum, we can have a self-dual operator

$$D(x, y) = \frac{(x + y)^2 + xy[xy - 3(x + y) + 1]}{2(x + y - xy)(1 - xy)}.$$

As a general case for the Hamacher product and sum, the following self-dual operator with parameter $p$ is derived from Hamacher's parametrized t-norm and t-conorm

$$\frac{xy}{p + (1 - p)(x + y - xy)} \quad \text{and} \quad \frac{x + y + (p - 2)xy}{1 + (p - 1)xy}$$

[5], respectively:

$$D(x, y) = \frac{(1 - p)\{(x + y)^2 + xy[(1 - p)xy - (3 - p)(x + y) + 1 - p]\} + p(x + y)}{2[p + (1 - p)(x + y - xy)][1 + (p - 1)xy]}, \quad p \geq 0.$$

Note that we have $D(x, y) = \frac{1}{2}(x + y)$ at $p = 1$ (the case of $T(x, y) = xy$ and $S(x, y) = x + y - xy$). For Frank's t-norm and t-conorm [4], we have $T(x, y) + S(x, y) = x + y$ so that $D(x, y) = \frac{1}{2}(x + y)$ here.

We can obtain a number of self-dual operators by using t-norms and t-conorms. Some of them are listed in Table 3 and their pictorial representations are made in Figure 6, where self-dual operators are also drawn which are based on Yager's and Schweizer (3)'s parametrized t-(co)norms (see Table 3 of Part I [7]).

We can also obtain self-dual operators by introducing averaging operators. Let

Table 3. Self-dual operators $D(x, y)$ of (34), (39) and (40)

| | |
|---|---|
| **Self-dual operators of (34) by t-norms and t-conorms** | |
| ① $\dfrac{(x+y)^2+xy[xy-3(x+y)+1]}{2(x+y-xy)(1-xy)}$ | $T(x, y)$ = Hamacher product |
| ② $\dfrac{-(x+y)^2+xy(xy+x+y+1)+2(x+y)}{2(2-x-y+xy)(1+xy)}$ | $T(x, y)$ = Einstein product |
| ③ $\dfrac{(1-p)\{(x+y)^2+xy[(1-p)xy-(3-p)(x+y)+1-p]\}+p(x+y)}{2[p+(1-p)(x+y-xy)][1+(p-1)xy]}$ | $T(x, y)$ = Hamacher's t-norm ($p \geqslant 0$) |
| **Self-dual operators of (39) by averaging operators $M(x, y)$** | |
| ④ $\dfrac{\sqrt{xy}+1-\sqrt{(1-x)(1-y)}}{2}$ | $M(x, y)=\sqrt{xy}$ |
| ⑤ $\dfrac{(x+y)^2-4xy(x+y-1)}{2(x+y)(2-x-y)}$ | $M(x, y)=\dfrac{2xy}{x+y}$ |
| ⑥ $\dfrac{\sqrt[p]{\frac{1}{2}(x^p+y^p)}+1-\sqrt[p]{\frac{1}{2}((1-x)^p+(1-y)^p)}}{2}$ | $M(x, y)$ = Quasi-linear averaging operator |
| ⑦ $\dfrac{(x\wedge y)^{1-p}\cdot(x\vee y)^p+1-(1-x\wedge 1-y)^{1-p}\cdot(1-x\vee 1-y)^p}{2}$ | $M(x, y)=(x\wedge y)^{1-p}\cdot(x\vee y)^p$ |
| **Self-dual operators of (40) by any operator $F$ and its dual $F^*$** | |
| ⑧ $\dfrac{(x+y-1)(xy+1-x\wedge y)+1}{2}$ | $F(x, y)=xy(x\vee y)$ |
| ⑨ $\dfrac{(x+y-1)(xy+1-x\vee y)+1}{2}$ | $F(x, y)=xy(x\wedge y)$ |
| ⑩ $\dfrac{(x+y-1)(x\vee y+1-xy)+1}{2}$ | $F(x, y)=xy+(x\wedge y)-xy(x\wedge y)$ |
| ⑪ $\dfrac{(xy)^{1-p}\cdot(x+y-xy)^p+1-[(1-x)(1-y)]^{1-p}\cdot(1-xy)^p}{2}$ | $F(x, y)=(xy)^{1-p}\cdot(x+y-xy)^p$ |

$M(x, y)$ be an averaging operator and $M^*(x, y)$ be an averaging operator dual to $M(x, y)$. Then we can define a self-dual operator as follows:

$$D(x, y)=\frac{M(x, y)+M^*(x, y)}{2} \tag{39}$$

Note that this self-dual operator is an average operator (see (33)).

For example, let $M(x, y)=\frac{1}{2}(x+y)$ and $M^*(x, y)=\frac{1}{2}(x+y)$, we have $D(x, y)=\frac{1}{2}(x+y)$. For the harmonic mean $2xy/(x+y)$ and its dual $x+y-2xy/(2-x-y)$, the following self-dual operator can be obtained:

$$D(x, y)=\frac{(x+y)^2-4xy(x+y-1)}{2(x+y)(2-x-y)}.$$

From the compensatory operator $(x\wedge y)(1-p)+(x\vee y)p$ which is an averaging operator and its dual operator $(x\vee y)(1-p)+(x\wedge y)p$, we have

$$D(x, y)=\frac{(x\wedge y)(1-p)+(x\vee y)p+(x\vee y)(1-p)+(x\wedge y)p}{2}=\frac{x+y}{2}.$$

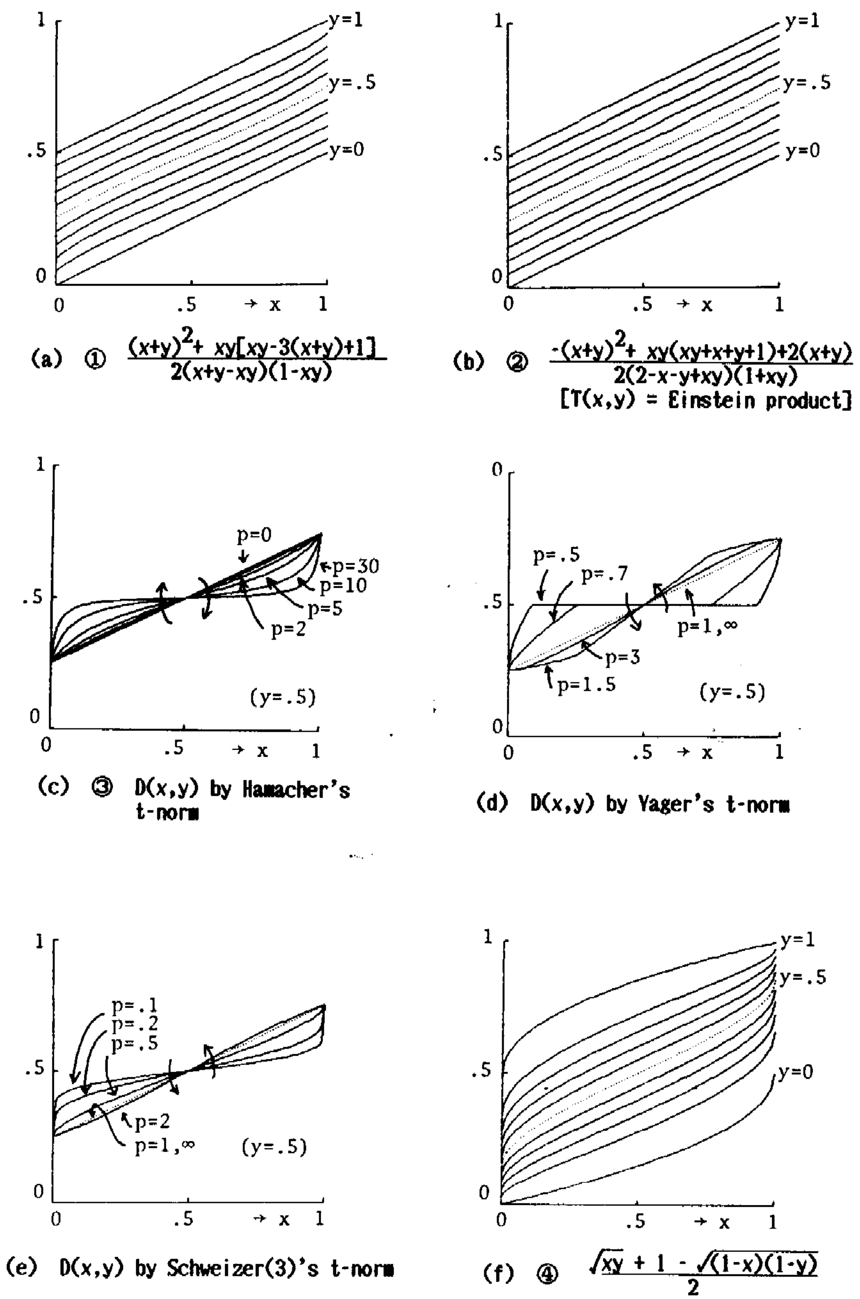
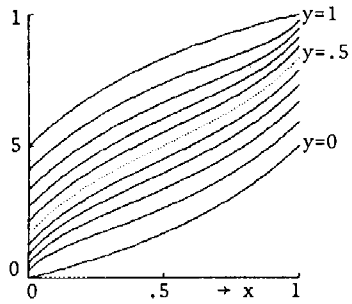
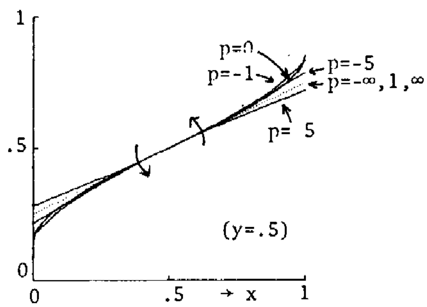


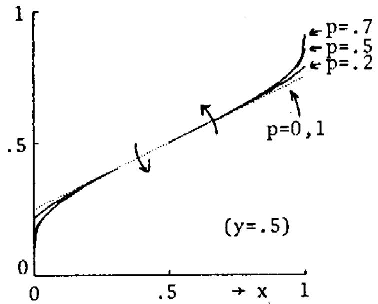

Fig. 6. Self-dual operators $D(x, y)$ in Table 3 derived from (34), (39) and (40).



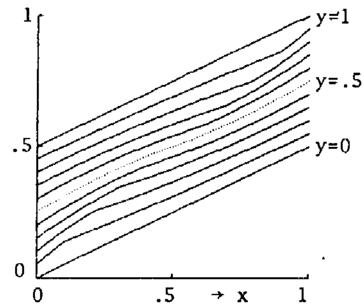

(g) ⑤ $\dfrac{(x+y)^2 - 4xy(x+y-1)}{2(x+y)(2-x-y)}$



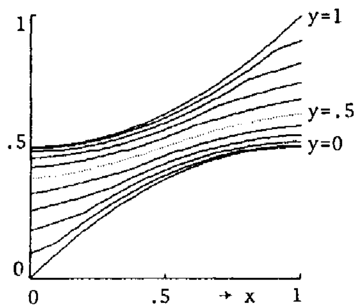

(h) ⑥ D(x,y) by quasi-linear averaging operator



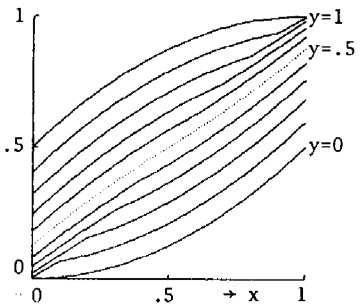

(i) ⑦ D(x,y) by $M(x,y)=(x\wedge y)^{1-p}\cdot(x\vee y)^p$




(j) ⑧ $\dfrac{(x+y-1)(xy+1-x\wedge y)+1}{2}$

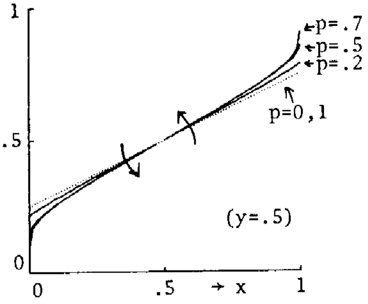



(k) ⑨ $\dfrac{(x+y-1)(xy+1-x\vee y)+1}{2}$




(l) ⑩ $\dfrac{(x+y-1)(x\vee y+1-xy)+1}{2}$

Fig. 6 (continued).




(m) ⑪ D(x,y) by $F(x,y)=(xy)^{1-p}\cdot(x+y-xy)^p$

Fig. 6 (continued).

On the other hand, from $(x \wedge y)^{1-p} \cdot (x \vee y)^p$ and $1-(1-x \wedge 1-y)^{1-p} \cdot (1-x \vee 1-y)^p$ which are dual to each other and averaging operators, we have a self-dual operator

$$D(x, y)=\frac{(x \wedge y)^{1-p} \cdot (x \vee y)^p+1-(1-x \wedge 1-y)^{1-p} \cdot (1-x \vee 1-y)^p}{2},$$

which is an averaging operator.

In the same way, we can obtain a number of self-dual operators by using averaging operators. Some of them are listed in Table 3 and their figures are shown in Figure 6, in which all of them are averaging operators.

As general cases of (34) and (39) which use t-norms, t-conorms and averaging operators, we can define a self-dual operator

$$D(x, y)=\frac{F(x, y)+F^*(x, y)}{2} \tag{40}$$

by introducing any functions $F(x, y)$ and $F^*(x, y)$ which are dual to each other. As candidates of $F(x, y)$ and $F^*(x, y)$, we can adopt quasi-t-norms (1) and quasi-t-conorms (2); $A(x, y)$ and $A^*(x, y)$ of (3)–(4); $B(x, y)$ and $B^*(x, y)$ of (5)–(6); (generalized) compensatory operators and their duals.

For example, let $F(x, y)=xy(x \vee y)$ be a quasi-t-norm and $F^*(x, y)=x+y-xy+(x \wedge y)-(x+y-xy)(x \wedge y)$ be a quasi-t-conorm dual to $xy(x \vee y)$. Then we have a self-dual operator

$$D(x, y)=\frac{(x+y-1)(xy+1-x \wedge y)+1}{2}.$$

Moreover, if $F(x, y)=xy(x \wedge y)$ and $F^*(x, y)=x+y-xy+(x \vee y)-(x+y-xy)(x \vee y)$, which are dual and derived from $A(x, y)$ and $A^*(x, y)$ of (3) and (4), then

$$D(x, y)=\frac{(x+y-1)(xy+1-x \vee y)+1}{2}.$$

Similarly, when $F(x, y) = xy(x \wedge y) - xy(x \wedge y)$ and $F^*(x, y) = (x + y - xy)(x \vee y)$, which are dual and obtained from $B(x, y)$ and $B^*(x, y)$ of (5) and (6), we have

$$D(x, y) = \frac{(x + y - 1)(x \vee y + 1 - xy) + 1}{2}.$$

We can also use a compensatory operator and its dual as $F(x, y)$ and $F^*(x, y)$. For example, let $F(x, y) = (xy)^{1-p} \cdot (x + y - xy)^p$ and its dual $F^*(x, y) = 1 - [(1 - x)(1 - y)]^{1-p} \cdot (1 - xy)^p$, then the following self-dual operator is obtained: For $0 \leqslant p \leqslant 1$,

$$D(x, y) = \frac{(xy)^{1-p} \cdot (x + y - xy)^p + 1 - [(1 - x)(1 - y)]^{1-p} \cdot (1 - xy)^p}{2}.$$

Similarly, from a compensatory operator $F(x, y) = xy(1 - p) + (x + y - xy)p$ and its dual $F^*(x, y) = (x + y - xy)(1 - p) + xyp$, we have

$$D(x, y) = \frac{x + y}{2}.$$

Generally, from the compensatory operator $F(x, y)(1 - p) + G(x, y)p$ of (24) and its dual $F^*(x, y)(1 - p) + G^*(x, y)p$ of (27), we obtain

$$D(x, y) = \frac{[F(x, y) + F^*(x, y)](1 - p) + [G(x, y) + G^*(x, y)]p}{2} \tag{41}$$

where $F^*(x, y)$ and $G^*(x, y)$ are dual to $F(x, y)$ and $G(x, y)$, respectively.

Self-dual operators in the sense of De Morgan's laws have been extensively studied by Silvert [10] under the name *symmetric sums*. He has shown that any symmetric sum is of the form

$$S(x, y) = \frac{g(x, y)}{g(x, y) + g(1 - x, 1 - y)} \tag{42}$$

where $g(x, y)$ is any non-negative, increasing, commutative, continuous real mapping such that $g(0, 0) = 0$. For example, t-(co)norms, quasi-t-(co)norms, $A(x, y)$, $A^*(x, y)$, $B(x, y)$ and $B^*(x, y)$ of (3)–(6), averaging operators, compensatory operators, self-dual operators, symmetric sums and so on can be used as $g(x, y)$.

The main properties of symmetric sums are:

(a) $S(x, y)$ satisfies (37).

(b) $S(x, 1 - x) = \frac{1}{2}$ for $0 < x < 1$.

(c) If $g(0, x) = 0$ for any $x$, then $S(0, 1)$ is not defined and thus $S$ is not continuous at the points $(0, 1)$ and $(1, 0)$; otherwise $S(0, 1) = S(1, 0) = \frac{1}{2}$.

Examples of symmetric sums are listed in Table 4. Their figures are in Figure 7 in which symmetric sums are also depicted whose generators $g(x, y)$ are Schweizer (3)'s, Frank's t-norms and t-conorms in Table 3 of [7] and the quasi-linear averaging operator. In this table, ①, ①′, ⑥, ⑦, ⑦′, ⑧, ⑧′ are averaging operators and the others are not averaging operators. It is found that

Table 4. Examples of symmetric sums $S(x, y)$ of (42)

| | $S(x, y)$ | $g(x, y)$ |
|---|---|---|
| ① | $\dfrac{x \wedge y}{1 - \lvert x - y \rvert}$ | $x \wedge y$ |
| ①′ | $\dfrac{x \vee y}{1 + \lvert x - y \rvert}$ | $x \vee y$ |
| ② | $\dfrac{xy}{1 - x - y + 2xy}$ | $xy$ |
| ②′ | $\dfrac{x + y - xy}{1 + x + y - 2xy}$ | $x + y - xy$ |
| ③ | $\dfrac{xy(x + y - xy)}{1 - x - y + 2xy(x + y - xy)}$ | $xy(x + y - xy)$ |
| ③′ | $\dfrac{x + y - xy(x + y - xy)}{1 + x + y - 2xy(x + y - xy)}$ | $x + y - xy(x + y - xy)$ |
| ④ | $\dfrac{0 \vee (x + y - 1)}{0 \vee (x + y - 1) + 0 \vee (1 - x - y)}$ | $0 \vee (x + y - 1)$ |
| ④′ | $\dfrac{1 \wedge (x + y)}{1 \wedge (x + y) + 1 \wedge (2 - x - y)}$ | $1 \wedge (x + y)$ |
| ⑤ | $\dfrac{x \dot{\vee} y}{x \dot{\vee} y + (1 - x) \dot{\vee} (1 - y)}$ | $x \dot{\vee} y = \begin{cases} x, & y = 0, \\ y, & x = 0, \\ 1, & x, y > 0, \end{cases}$ |
| ⑥ | $\dfrac{x + y}{2}$ | $x + y$ |
| ⑦ | $\dfrac{\sqrt{xy}}{\sqrt{xy} + \sqrt{(1 - x)(1 - y)}}$ | $\sqrt{xy}$ |
| ⑦′ | $\dfrac{1 - \sqrt{(1 - x)(1 - y)}}{2 - \sqrt{(1 - x)(1 - y)} - \sqrt{xy}}$ | $1 - \sqrt{(1 - x)(1 - y)}$ |
| ⑧ | $\dfrac{xy(2 - x - y)}{x(1 - x) + y(1 - y)}$ | $\dfrac{2xy}{x + y}$ |
| ⑧′ | $\dfrac{x + y}{2}$ | $\dfrac{x + y - 2xy}{2 - x - y}$ |

these symmetric sums are ordered as follows when $x + y \leqslant 1$:

$$
\begin{aligned}
&\frac{0 \vee (x + y - 1)}{0 \vee (x + y - 1) + 0 \vee (1 - x + y)} \leqslant \frac{xy(x + y - xy)}{1 - x - y + 2xy(x + y - xy)} \\
&\leqslant \frac{xy}{1 - x - y + 2xy} \leqslant x \wedge y \leqslant \frac{x \wedge y}{1 - |x - y|} \leqslant \frac{xy(2 - x - y)}{x(1 - x) + y(1 - y)} \\
&\leqslant \frac{\sqrt{xy}}{\sqrt{xy} + \sqrt{(1 - x)(1 - y)}} \leqslant \frac{1 - \sqrt{(1 - x)(1 - y)}}{2 - \sqrt{(1 - x)(1 - y)} - \sqrt{xy}} \leqslant \frac{x + y}{2} \\
&\leqslant \frac{x \vee y}{1 + |x - y|} \leqslant x \vee y \leqslant \frac{x + y - xy}{1 + x + y - 2xy} \leqslant \frac{x + y - xy(x + y - xy)}{1 - x - y + 2xy(x + y - xy)} \\
&\leqslant \frac{1 \wedge (x + y)}{1 \wedge (x + y) + 1 \wedge (2 - x - y)} \leqslant \frac{x \dot{\vee} y}{x \dot{\vee} y + (1 - x) \dot{\vee} (1 - y)}.
\end{aligned}
$$

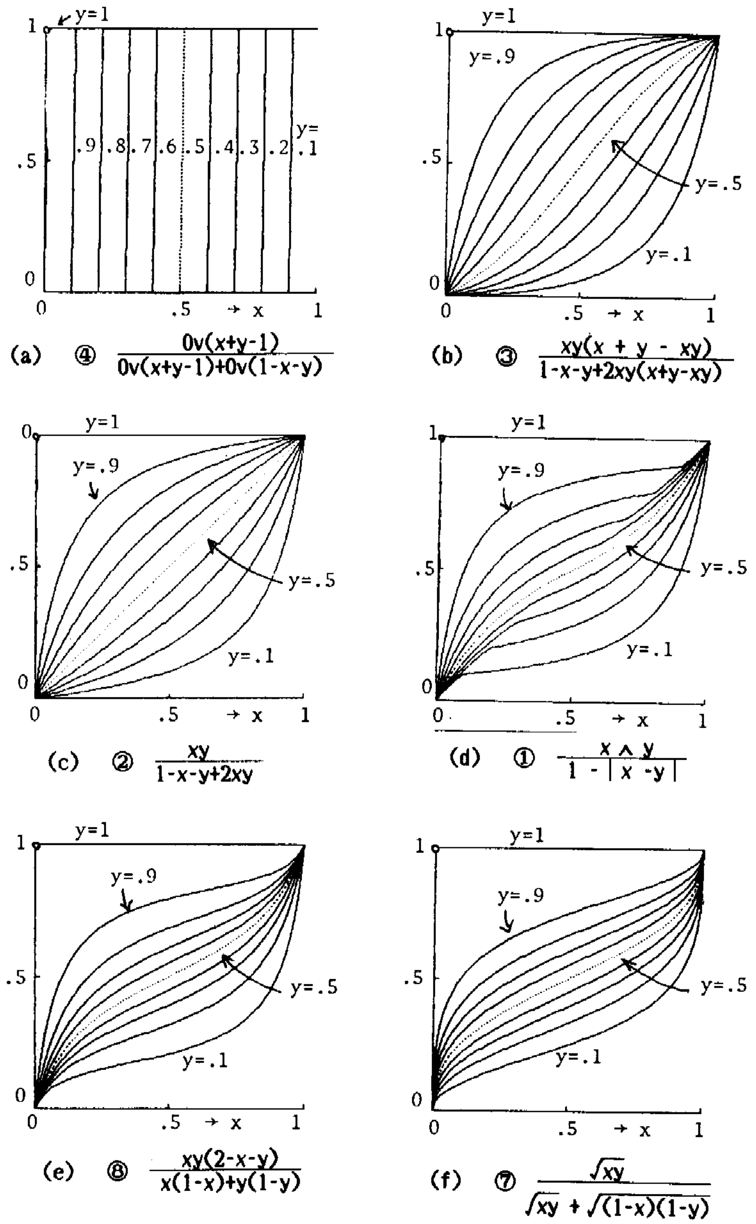
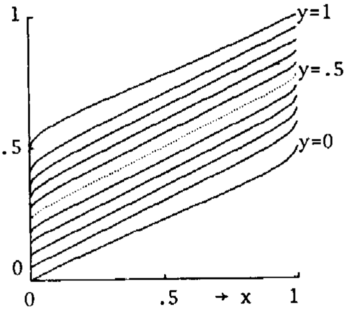
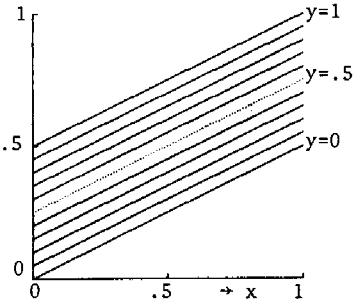


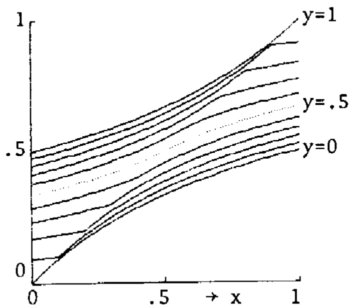

Fig. 7. Symmetric sums $S(x, y)$ in Table 4.



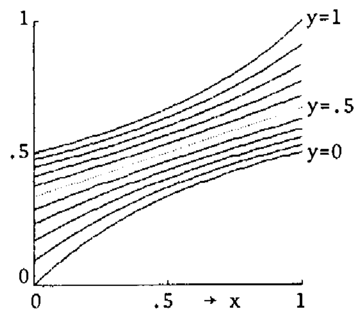

(g) ⑦' $\frac{1-\sqrt{(1-x)(1-y)}}{2-\sqrt{(1-x)(1-y)}-\sqrt{xy}}$



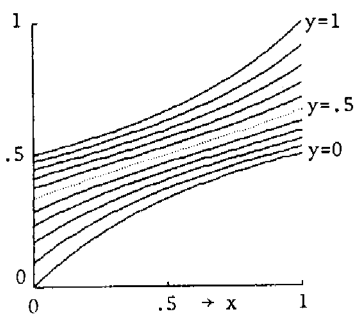

(h) ⑥ $\frac{x+y}{2}$



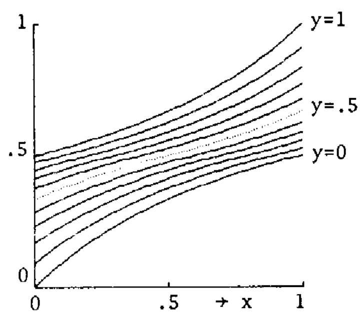

(i) ①' $\frac{x \vee y}{1+|x-y|}$




(j) ②' $\frac{x+y-xy}{1+x+y-2xy}$

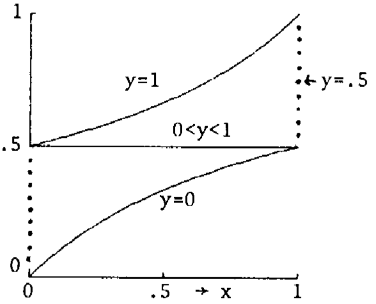



(k) ③' $\frac{x+y-xy(x+y-xy)}{1+x+y-2xy(x+y-xy)}$

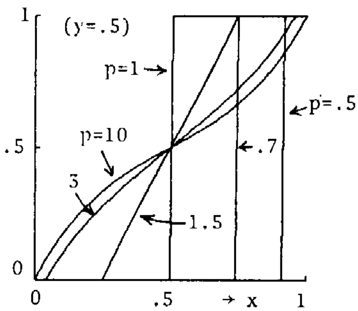



(l) ④ $\frac{1\wedge(x+y)}{1\wedge(x+y)+1\wedge(2-x-y)}$

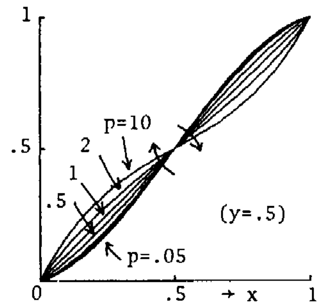
Fig. 7 (continued).



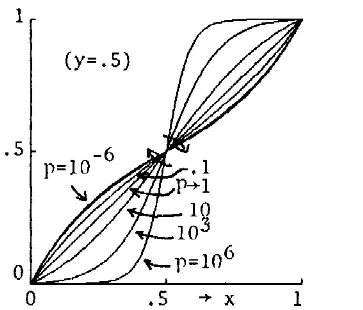

(m) ⑤ $\dfrac{x \dot{\vee} y}{x\dot{\vee}y+(1-x)\dot{\vee}(1-y)}$



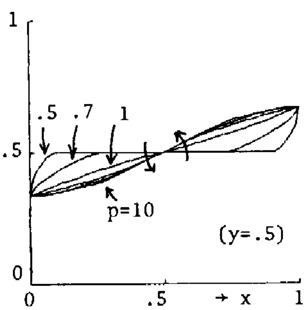

(n) S(x,y) by Yager's t-norm



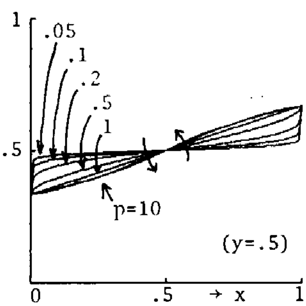

(o) S(x,y) by Schweizer(3)'s t-norm




(p) S(x,y) by Frank's t-norm

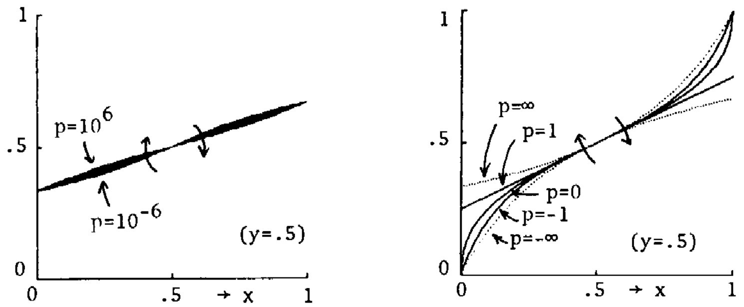



(q) S(x,y) by Yager's t-conorm




(r) S(x,y) by Schweizer(3)'s t-conorm

Fig. 7 (continued).




(s) S(x,y) by Frank's t-conorm (t) S(x,y) by quasi-linear averaging operator

Fig. 7 (continued).

If $x+y \geqslant 1$ we have the converse inequalities. Particularly, if $x+y=1$, all these expressions are equal to 0.5.

Let $D(a, b)$ be a self-dual operator; then we have $D(a, b)=1-D(1-a, 1-b)$ and $D(a, b)=D(b, a)$. From the self-dual operator $D(a, b)$, we can obtain other self-dual operators $D'$ as

$$D'(x, y)=D(T(x, y), S(x, y)), \tag{43}$$

$$D'(x, y)=D(M(x, y), M^*(x, y)), \tag{44}$$

where $T(x, y)$ is a t-norm and $S(x, y)$ is a t-conorm dual to $T(x, y)$, and $M(x, y)$ is an averaging operator and $M^*(x, y)$ is dual to $M(x, y)$.

More generally, we can define self-dual operators by introducing any $F(x, y)$ and $F^*(x, y)$ which are dual to each other:

$$D'(x, y)=D(F(x, y), F^*(x, y)), \tag{45}$$

where $F(x, y)$ and $F^*(x, y)$ are operators as in (40).

The self-duality of $D'(x, y)$ of (43) is shown as follows.

$$\begin{aligned} D'(x, y) &= D(T(x, y), S(x, y)) = D(S(x, y), T(x, y)) \\ &= 1-D(1-S(x, y), 1-T(x, y)) \\ &= 1-D(T(1-x, 1-y), S(1-x, 1-y)) \\ &= 1-D'(1-x, 1-y). \end{aligned}$$

Thus $D'$ is a self-dual operator. In a similar way, the self-duality of (44) and (45) is proved.

For example, since the symmetric sum $D(a, b)=(a \vee b)/(1+|a-b|)$ in Table 4 is a self-dual operator, we can have the following self-dual operator by letting $T(x, y)=xy$ and $S(x, y)=x+y-xy$ in (43):

$$D'(x, y)=\frac{xy \vee (x+y-xy)}{1+|xy-(x+y-xy)|}=\frac{x+y-xy}{1+x+y-2xy},$$

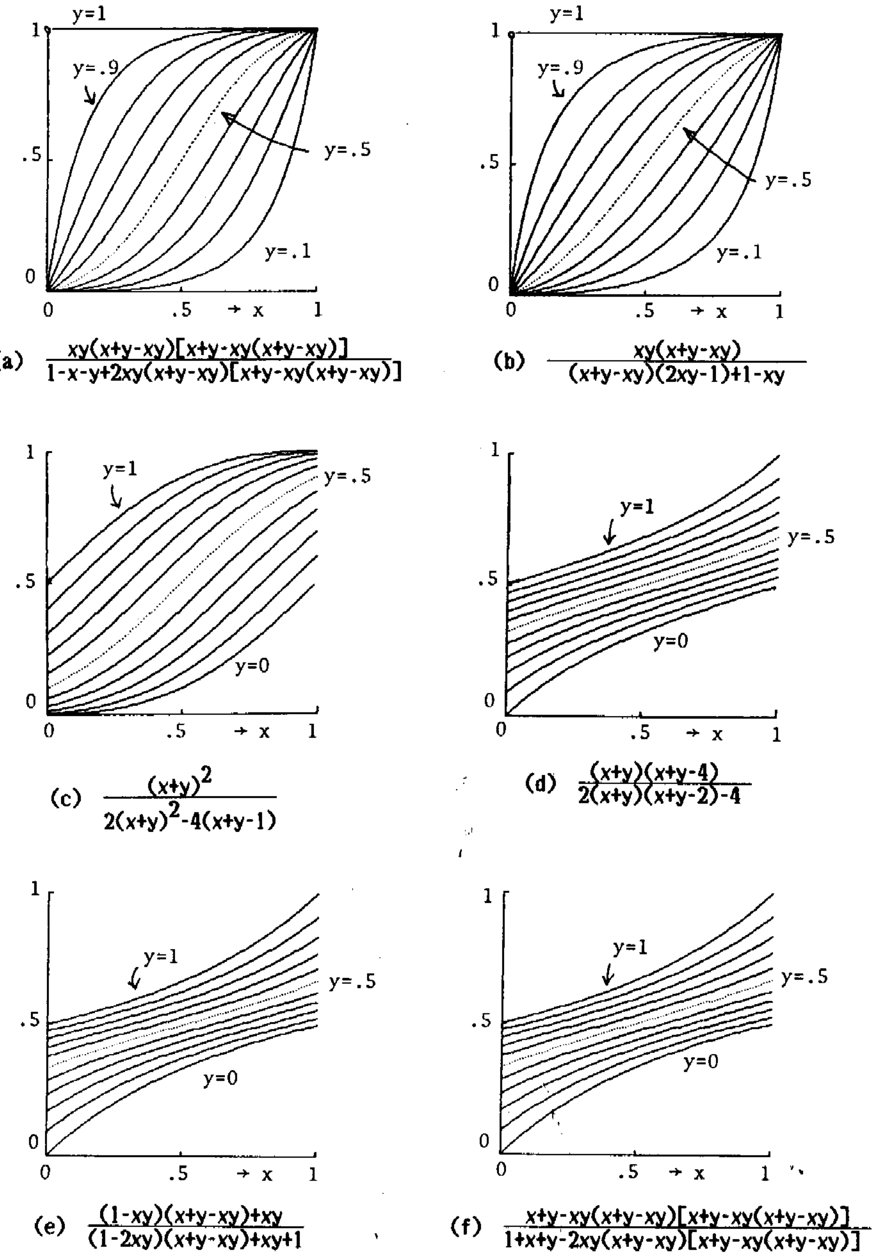



Fig. 8. Self-dual operators $D'(x, y)$ derived from (43)–(45).

which is the same as ②′ in Table 4. Similarly, from $S(x, y) = x \vee y$, $1 \wedge (x + y)$ and $x \dot{\vee} y$ we have the same symmetric sums ①′, ④′ and ⑤ in Table 4, respectively.

Let $D(a, b) = ab/(1 - a - b + 2ab)$ which is symmetric sum ② in Table 4, and let $T(x, y) = xy$ and $S(x, y) = x + y - xy$. Then we have

$$D'(x, y) = \frac{xy(x + y - xy)}{(x + y - xy)(2xy - 1) + 1 - xy}.$$

When $T(x, y) = x \wedge y$ and $S(x, y) = x \vee y$, we have $D'(x, y) = D(x, y)$ since the properties $(x \wedge y)(x \vee y) = xy$ and $(x \wedge y) + (x \vee y) = x + y$ hold. In the case where $M(x, y) = M'(x, y) = \frac{1}{2}(x + y)$, we can obtain $D'(x, y)$ from (44) as follows:

$$D'(x, y) = \frac{(x + y)^2}{2(x + y)^2 - 4(x + y - 1)}.$$

From a quasi-t-norm $F(x, y) = xy(x + y - xy)$ and its dual $F^*(x, y) = x + y - xy(x + y - xy)$, the following self-dual operator is derived by using (45):

$$D'(x, y) = \frac{xy(x + y - xy)[x + y - xy(x + y - xy)]}{1 - x - y + 2xy(x + y - xy)[x + y - xy(x + y - xy)]}.$$

As for $D(a, b) = (a + b - ab)/(1 + a + b - 2ab)$ in Table 4, the following self-dual operator is obtained from $T(x, y) = xy$ and $S(x, y) = x + y - xy$:

$$D'(x, y) = \frac{(1 - xy)(x + y - xy) + xy}{(1 - 2xy)(x + y - xy) + xy + 1}.$$

From $M(x, y) = M'(x, y) = \frac{1}{2}(x + y)$, we have

$$D'(x, y) = \frac{(x + y)(x + y - 4)}{2(x + y)(x + y - 2) - 4}.$$

$F(x, y) = xy(x + y - xy)$ and $F^*(x, y) = x + y - xy(x + y - xy)$ give

$$D'(x, y) = \frac{x + y - xy(x + y - xy)[x + y - xy(x + y - xy)]}{1 + x + y - 2xy(x + y - xy)[x + y - xy(x + y - xy)]}.$$

These self-dual operators $D(x, y)$ which are derived from (43)–(45) are depicted in Figure 8.

## 5. Conclusion

We have summarized fuzzy connectives of quasi-t-norms, quasi-t-conorms, compensatory operators, generalized compensatory operators, self-dual operators and symmetric sums. The pictorial representations of these fuzzy connectives have been made with the aid of a computer. These figures of fuzzy connectives, especially of fuzzy connectives with parameter, will be useful when we use parameterized connectives in various applications.

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