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Blin (1974) considers a reflexive fuzzy relation satisfying Eqn. (5) and gives an algorithm for deriving a linear ordering out of it. Dubois and Prade (1980) relax Eqn. (5) into $\min(\mu_R(x,y),\ \mu_R(y,x)) \leq 0.5$ and suggest a ranking algorithm which searches for inclusion relationships between fuzzy dominated classes.

Saaty (1978) provides a ranking procedure which acts on fuzzy relations such that

$$\mu_R(x,y)\cdot\mu_R(y,x) = 1 \text{ (antisymmetry)}$$

$$\mu_R(x,y)\cdot\mu_R(y,z) = \mu_R(x,y) \text{ consistency}$$

Here the unit interval is replaced by the positive real line. Saaty's method consists in finding an eigenvector of the matrix μ_R whose rank is always 1. Other techniques for the rank ordering of fuzzily related objects are devised in Shimura (1973) and Navarrete *et al.* (1979). Various fuzzy ordering structures in a mathematical setting are discussed in Roubens and Vincke (1985).

NB: It is patent that the scalar pairwise ranking indices described in the first part of this article equip a set of fuzzy numbers with fuzzy ordering relations which can be processed by the cited methods.

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D. Dubois and H. Prade

Fuzzy Reasoning Methods

In much of human reasoning, the form of reasoning is approximate rather than exact as in the statement:

If a tomato is red then the tomato is ripe

This tomato is very red

This tomato is very ripe

Such reasoning cannot be made sufficient by the inference rules of classical two-valued logic and many-valued logic.

To make such reasoning with fuzzy concepts, Zadeh (1975a,b) first suggested an inference rule called "the compositional rule of inference" and proposed translation rules for translating a fuzzy conditional proposition "If x is A then y is B" into a fuzzy relation. Since then several alternative approaches to that given by Zadeh for fuzzy reasoning have been presented by several researchers such as Baldwin (1979a,b), Tsukamoto (1979a,b) and Yager (1980) by introducing fuzzy logic with fuzzy truth values.

In the following we shall introduce their methods for fuzzy reasoning in which a fuzzy conditional proposition "If . . . then . . ." is contained.

Ant 1: If x is A then y is B

Ant 2: x is A' (1)

Cons: y is B'

where x and y are the names of objects, and A, A', B and B' are fuzzy concepts represented by fuzzy sets in universes of discourse U, U, V and V, respectively. This form of fuzzy reasoning can be considered as a fuzzy modus ponens which reduces to the classical modus ponens when $A' = A$ and $B' = B$.

1. Zadeh's Fuzzy Reasoning Method

The Ant 1 of the form "If x is A then y is B" in (1) may represent a certain relationship between A and B. From this point of view, Zadeh (1975a) proposed a translation rule called an "arithmetic rule" for translating "If x is

A then y is B" into a fuzzy relation in $U \times V$. Let A and B be fuzzy sets in U and V, respectively, then the arithmetic rule is given as

$$Ra = (\neg A \times V) \oplus (U \times B)$$
$$= \int_{U \times V} 1 \wedge (1 - \mu_A(u) + \mu_B(v)) / (u, v) \quad (2)$$

It is noted that the arithmetic rule is based on the implication rule of Lukasiewicz's logic, that is,

$$a \rightarrow b = 1 \wedge (1 - a + b) \quad a, b \in [0, 1] \quad (3)$$

Therefore, it is possible to introduce other implication rules of many-valued logic systems to a translation rule for the fuzzy conditional proposition (Mizumoto and Zimmermann 1982).

$$a \rightarrow b = (a \wedge b) \vee (1 - a) \quad (4)$$
$$a \rightarrow b = a \wedge b \quad (5)$$
$$a \rightarrow b = \begin{cases} 1 & \text{if } a \leq b \\ 0 & \text{otherwise} \end{cases} \quad (6)$$
$$a \rightarrow b = \begin{cases} 1 & \text{if } a \leq b \\ b & \text{otherwise} \end{cases} \quad (7)$$
$$a \rightarrow b = (1 - a) \vee b \quad (8)$$
$$a \rightarrow b = 1 \wedge b/a \quad (9)$$
$$a \rightarrow b = 1 - a + ab \quad (10)$$
$$a \rightarrow b = \begin{cases} 1 & \text{if } a \neq 1 \text{ or } b = 1 \\ 0 & \text{otherwise} \end{cases} \quad (11)$$

The consequence B' of (1) can be deduced from Ant 1 and Ant 2 by taking the max–min composition ($\circ$) of the fuzzy set A' and the fuzzy relation Ra (the compositional rule of inference).

$$B' = A' \circ Ra = A' \circ [(\neg A \times V) \oplus (U \times B)] \quad (12)$$
$$\mu_{B'}(v) = \bigvee_u \{\mu_{A'}(u) \wedge [1 \wedge (1 - \mu_A(u) + \mu_B(v))]\} \quad (13)$$

For example, when $A' = A$ the arithmetic rule infers such a consequence as

$$B' = \int_V \frac{1 + \mu_B(v)}{2} \Big/ v \neq B \quad (14)$$

This inference result indicates that the arithmetic rule does not satisfy the modus ponens which is quite a reasonable demand in fuzzy reasoning.

If x is A then y is B
x is A (modus ponens) (15)
———————
y is B

However, if new compositions "max-$\odot$ composition" and "max-$\wedge\!\!\!\cdot$ composition" are used in the compositional rule of inference, then the arithmetic rule satisfies the modus ponens of (15). The operations $\odot$ and $\wedge\!\!\!\cdot$ are defined as: For $x, y \in [0, 1]$,

$$\textit{Bounded product: } x \odot y = 0 \vee (x + y - 1) \quad (16)$$
$$\textit{Drastic product: } x \wedge\!\!\!\cdot\, y = \begin{cases} x \ldots y = 1 \\ y \ldots x = 1 \\ 0 \ldots \text{otherwise} \end{cases} \quad (17)$$

Max-$\odot$ composition ($\square$) and max-$\wedge\!\!\!\cdot$ composition ($\blacktriangle$) are obtained from (13) by replacing $\wedge$ by $\odot$ and $\wedge\!\!\!\cdot$, respectively. Using these new compositions, we have at $A' = A$

$$B' = A \,\square\, Ra = B$$
$$B' = A \blacktriangle Ra = B$$

which shows the satisfaction of the modus ponens of (15).

Moreover, when A' are fuzzy sets such as *very A*, *more or less A*, *not A*, the inference results under these new compositions are much better than those under the max–min composition (Mizumoto 1981). Such a tendency can be observed in the case where the implication rules of (4)–(11) are used in the translation rules (Mizumoto 1982, 1985).

2. Baldwin's Fuzzy Reasoning Method

Baldwin (1979a,b) gives an alternative approach for fuzzy reasoning using fuzzy logic with fuzzy truth values.

A fuzzy truth value τ is a fuzzy set in the truth value space [0, 1] and is defined by a membership function μ_τ as

$$\mu_\tau : [0, 1] \rightarrow [0, 1] \quad (18)$$

Some of the fuzzy truth values are given by (for $t \in [0, 1]$):

$$\mu_t(t) = t \quad (19)$$
$$\mu_f(t) = 1 - t \quad (20)$$
$$\mu_{vt}(t) = \mu_t(t)^2 = t^2 \quad (21)$$
$$\mu_{mlt}(t) = \mu_t(t)^{0.5} = t^{0.5} \quad (22)$$
$$\mu_{at}(t) = \begin{cases} 1 & \text{if } t = 1 \\ 0 & \text{otherwise} \end{cases} \quad (23)$$
$$\mu_{af}(t) = \begin{cases} 1 & \text{if } t = 0 \\ 0 & \text{otherwise} \end{cases} \quad (24)$$

(t, true; f, false; vt, very true; mlt, more or less true; at, absolutely true; af, absolutely false).

We shall consider a fuzzy proposition "x is F" with a fuzzy truth value τ, that is,

$$(x \text{ is } F) \text{ is } \tau$$

and reduce this proposition to a fuzzy proposition

$$x \text{ is } G$$

where fuzzy sets F and G are in the same universe of discourse U. Then the fuzzy set G is given as follows (truth functional modification):

$$\mu_G(u) = \mu_\tau(\mu_F(u)) \tag{25}$$

For example, if τ is true, very true, false, we have

$$(x \text{ is } F) \text{ is true} \leftrightarrow x \text{ is } F \tag{26}$$

$$(x \text{ is } F) \text{ is very true} \leftrightarrow x \text{ is very } F \tag{27}$$

$$(x \text{ is } F) \text{ is false} \leftrightarrow x \text{ is not } F$$

Conversely, we derive a fuzzy truth value for a fuzzy proposition "x is F" when given a (reference) proposition "x is G." The fuzzy truth value τ of "x is F" relative to "x is G" is defined by the following (inverse truth functional modification).

$$\tau = \text{truth } (x \text{ is } F/x \text{ is } G)$$

$$= \int_{[0,1]} \mu_G(u)/\mu_F(u) \tag{28}$$

A more explicit expression for τ is

$$\mu_\tau(t) = \bigvee_u \mu_G(u), \qquad t \in [0,1]$$

$$\mu_F(u) = t$$

As a simple example, we obtain

$$\text{truth } (x \text{ is } F/x \text{ is } F) = \text{true} \tag{29}$$

$$\text{truth } (x \text{ is } F/x \text{ is not } F) = \text{false}$$

$$\text{truth } (x \text{ is } F/x \text{ is very } F) = \text{very true} \tag{30}$$

In what follows we shall explain Baldwin's method for the fuzzy reasoning of (1). The consequence B' of (1) is obtained by the following.

(a) Using the inverse truth functional modification of (28), we obtain the fuzzy truth value τ_A of "x is A" relative to "x is A'." That is to say,

$$\tau_A = \text{truth}(x \text{ is } A/x \text{ is } A')$$

(b) The fuzzy truth values of "x is A" and "y is B" of (1) are considered as true from (26). Thus, the fuzzy conditional proposition "If true then true" can be translated into a fuzzy relation in $[0,1] \times [0,1]$ which is obtained by using, say, the arithmetic rule of (2).

$$\text{true} \Rightarrow \text{true}$$

$$= (\neg \text{true} \times [0,1]) \oplus ([0,1] \times \text{true})$$

$$= \int_{[0,1]^2} 1 \wedge (1 - \mu_{\text{true}}(t) + \mu_{\text{true}}(s))/(t,s)$$

$$= \int_{[0,1]^2} 1 \wedge (1 - t + s)/(t,s)$$

The truth value of "y is B" given "y is A" is obtained by taking the max–min composition "$\circ$" of τ_A and (true $\Rightarrow$ true), that is,

$$\tau_B = \tau_A \circ (\text{true} \Rightarrow \text{true})$$

$$\mu_{\tau_B}(s) = \bigvee_t \{\mu_{\tau_A}(t) \wedge [1 \wedge (1 - t + s)]\} \tag{31}$$

(c) Using the truth functional modification of (25), the consequence "y is B'" is obtained as

$$y \text{ is } B' \leftrightarrow (y \text{ is } B) \text{ is } \tau_B$$

where

$$\mu_{B'}(v) = \mu_{\tau_B}(\mu_B(v))$$

As an example, we shall consider the case of $A' = A$ and $A' = \text{very } A$ $(= A^2)$. When $A' = A$, we derive

$$\tau_A = \text{truth } (x \text{ is } A/x \text{ is } A) = \text{true} \ldots \text{from (29)}$$

$$\tau_B = \text{true} \circ (\text{true} \Rightarrow \text{true})$$

$$= \int_{[0,1]} \frac{1+s}{2}/s$$

$$\mu_{B'}(v) = \mu_{\tau_B}(\mu_B(v)) = \frac{1 + \mu_B(v)}{2}$$

Thus,

$$B' = \int_V \frac{1 + \mu_B(v)}{2}/v$$

This inference result B' at $A' = A$ is equal to (14) and thus this method does not satisfy the modus ponens of (15). However, if new compositions "max-$\odot$ composition" and "max-$\wedge$ composition" discussed in Zadeh's method are used in (31), Baldwin's method can also get the consequence which satisfies the modus ponens.

When $A' = \text{very } A$ $(= A^2)$, the consequence B' is obtained as follows.

$$\tau_A = \text{truth } (x \text{ is } A/x \text{ is very } A)$$

$$= \text{very true} \ldots \text{from (30)}$$

$$\tau_B = \text{very true} \circ (\text{true} \Rightarrow \text{true})$$

$$= \int_{[0,1]} \frac{3 + 2s - \sqrt{5 + 4s}}{2}/s$$

Thus,

$$\mu_{B'}(v) = \mu_{\tau_B}(\mu_B(v)) = \frac{3 + 2\mu_B(v) - \sqrt{5 + 4}\mu_B(v)}{2}$$

which is also equal to the result by Zadeh's method at $A' = \text{very } A$ (Mizumoto 1981). As was indicated in the above example, the inference results by Baldwin's method are the same as those by Zadeh's method. In fact, Tong and Festathiou (1982) show that if μ_A^{-1} is a subjective mapping, Baldwin's method is equivalent to Zadeh's method and so Baldwin's method which is based on fuzzy truth values is inherently redundant and computationally inefficient.

As a generalization of the fuzzy modus ponens of (1), Baldwin (1979b) also investigated fuzzy reasoning whose antecedents have fuzzy truth values τ_1 and τ_2.

$$\begin{array}{l}\text{Ant 1: (If } x \text{ is } A \text{ then } y \text{ is } B) \text{ is } \tau_1\\ \text{Ant 2: } (x \text{ is } A') \text{ is } \tau_2\\ \hline \text{Cons: } y \text{ is } B'\end{array}$$

For this form of fuzzy reasoning, the consequence B' is inferred by the following.

(a) Using the truth functional modification of (25), we obtain new proposition "x is $A\dagger$" which is equivalent to Ant 2.

$$x \text{ is } A\dagger \leftrightarrow (x \text{ is } A') \text{ is } \tau_2$$

where

$$\mu_{A\dagger}(u) = \mu_{\tau_2}(\mu_{A'}(u))$$

(b) Using (28), the fuzzy truth value τ_A of "x is A" relative to "x is $A\dagger$" is obtained as

$$\tau_A = \text{truth } (x \text{ is } A/x \text{ is } A\dagger)$$

(c) The truth value τ_B of "y is B" given "x is A" is given by

$$\tau_B = \tau_A \circ \tau_1(\text{true} \Rightarrow \text{true})$$

$$\mu_{\tau_B}(s) = \bigvee_t \{\mu_{\tau_A}(t) \wedge \mu_{\tau_1}(1 \wedge (1 - t + s))\}$$

(d) The consequence B' is as follows:

$$\mu_{B'}(v) = \mu_{\tau_B}(\mu_B(v))$$

3. *Tsukamoto's Fuzzy Reasoning Method*

Tsukamoto introduced a different fuzzy reasoning method which is also based on a fuzzy logic with fuzzy truth values.

Let τ_A and τ_B be fuzzy truth values of fuzzy propositions "x is A" and "y is B," respectively, and let $a \to b$ be the implication rule of Lukasiewicz logic, that is,

$$a \to b = 1 \wedge (1 - a + b) \qquad a, b \in [0, 1]$$

Then the fuzzy truth value of "If x is A then y is B" ($A \Rightarrow B$, for short) is given by

$$\tau_{A \Rightarrow B} = 1 \wedge (1 - \tau_A + \tau_B) \qquad (32)$$

where the operations $\wedge$, $-$ and $+$ are fuzzified ones which are defined by using the extension principle (Zadeh 1975b).

Introducing α-level sets of these fuzzy truth values, Eqn. (32) is rewritten as

$$(\tau_{A \Rightarrow B})^\alpha = 1 \wedge (1 - \tau_A^\alpha + \tau_B^\alpha) \qquad (33)$$

where

$$\tau_A^\alpha = \{t | \mu_{\tau_A}(t) \geqslant \alpha\}$$

When τ_A and $\tau_{A\Rightarrow B}$ are given, τ_B is obtained by solving Eqn. (33). Let τ_A^α and $(\tau_{A\Rightarrow B})^\alpha$ be given as intervals in [0, 1], say,

$$\tau_A^\alpha = [a_1, a_2], \qquad (\tau_{A\Rightarrow B})^\alpha = [c_1, c_2]$$

Then we can obtain by (33)

$$\tau_B^\alpha = \begin{cases} [a_1 + c_1 - 1 \vee 0, 1] & c_2 = 1 \\ [a_1 + c_1 - 1 \vee 0, a_2 + c_2 - 1] & a_2 + c_2 \geqslant 1, c_2 \neq 1 \\ \varnothing & 0 \leqslant a_2 + c_2 < 1 \end{cases} \qquad (34)$$

As a simple case, let us assume that τ_A is normal and convex, and $\tau_{A\Rightarrow B}$ is normal and its membership function is nondecreasing in its domain [0, 1]. Then we have $\tau_A^\alpha = [a_1, a_2]$ and $(\tau_{A\Rightarrow B})^\alpha = [c, 1]$. From this we obtain readily τ_B^α as follows, by setting $c_2 = 1$ and $c_1 = c$ in Eqn. (34):

$$\tau_B^\alpha = [a_1 + c - 1 \vee 0, 1] \qquad (35)$$

Now, we shall apply Tsukamoto's method to the fuzzy reasoning of the form:

$$\begin{array}{l}\text{Ant 1: (If } x \text{ is } A \text{ then } y \text{ is } B) \text{ is } \tau\\ \text{Ant 2: } x \text{ is } A'\\ \hline \text{Cons: } y \text{ is } B'\end{array} \qquad (36)$$

which is a general case of (1).

The consequence B' is deduced by the following.

(a) Using (28), the fuzzy truth value of "x is A" relative to "x is A'" is given as

$$\tau_A = \text{truth } (x \text{ is } A/x \text{ is } A')$$

(b) From τ_A and τ ($= \tau_{A\Rightarrow B}$ of (36)), we calculate τ_B by using (34) or (35).

(c) The consequence B' is obtained as

$$y \text{ is } B' \leftrightarrow (y \text{ is } B) \text{ is } \tau_B$$

where

$$\mu_{B'}(v) = \mu_{\tau_B}(\mu_B(v))$$

As a simple example, we shall consider the case of τ = true. When τ = true, (36) reduces to the fuzzy modus ponens of (1). When $A' = A$, we have

$$\tau_A = \text{truth } (x \text{ is } A/x \text{ is } A) = \text{true}$$

α-level sets of τ_A (= true) and τ ($= \tau_{A\Rightarrow B}$) = true are given as

$$\tau_A^\alpha = [\alpha, 1], \qquad \tau^\alpha = [\alpha, 1]$$

From (35), τ_B^α becomes

$$\tau_B^\alpha = [\alpha + \alpha - 1 \vee 0, 1] = [2\alpha - 1 \vee 0, 1]$$

Thus,

$$\tau_B = \bigcup_\alpha \tau_B^\alpha = \int_{[0,1]} \frac{1+s}{2}\Big/ s$$

Hence

$$B' = \int_V \frac{1 + \mu_B(v)}{2} / v$$

The consequence B' is equal to (14) but not equal to B, and hence this method does not satisfy the modus ponens of (15).

When A' = very A, the consequence B' is as follows.

$$\tau_A = \text{truth } (x \text{ is } A/x \text{ is very } A) = \text{very true}$$

$$\tau_A^\alpha = \text{very true}^\alpha = [\sqrt{\alpha}, 1],\ \tau^\alpha = \text{true}^\alpha = [\alpha, 1]$$

$$\tau_B^\alpha = [\sqrt{\alpha} + \alpha - 1 \vee 0, 1]$$

$$\tau_B = \int_{[0,1]} \frac{3 + 2s - \sqrt{5 + 4s}}{2} / s$$

$$B' = \int_V \frac{3 + 2\mu_B(v) - \sqrt{5 + 4\mu_B(v)}}{2} / v$$

which is equal to the inference result by Zadeh's and Baldwin's methods (see Mizumoto and Zimmermann 1982).

4. *Yager's Fuzzy Reasoning Method*

This method is based on a similarity measure and a new implication rule. We shall be concerned with the fuzzy modus ponens of (1). The consequence B' is obtained as follows.

(a) Let S be a similarity measure of the fuzzy sets A and A' in (1), which indicates the degree to which "x is A" can be derived from "x is A'." In Yager's method two similarity measures are introduced:

$$S_1 = \frac{\bigvee_u \{\mu_A(u) \wedge \mu_{A'}(u)\}}{\bigvee_u \mu_{A'}(u)} \quad (37)$$

$$S_2 = \int_{[0,1]} \mu_{A'}(u) / \mu_A(u) \quad (38)$$

The similarity measure S_1 has crisp numerical values in [0, 1]. The second measure S_2 is given as a fuzzy set in [0, 1] and is based on the compatibility of A and A' (Zadeh 1975a) which has the same definition as (28).

(b) In the fuzzy modus ponens of (1), it is natural to expect $B' \approx B$ at $A' \approx A$. Thus, the more similar A' is to A, the more similar B' is to B. Therefore, the larger the similarity is, the more important it is that "y is B" is satisfied. From this fact, Yager suggested that the consequence B' is given as

$$B' = B^S \quad (39)$$

which is based on a new implication rule proposed by him, that is,

$$a \to b = b^a \quad (40)$$

When S is a numerical value, B^S is defined as

$$B^S = \int_V \mu_B(v)^S / v \quad (41)$$

When S is a fuzzy set in [0, 1], B^S is raised to a fuzzy set of type 2 (Zadeh 1975b) whose grades are fuzzy sets in [0, 1]. Thus, the grade $\mu_{B^S}(v)$ is a fuzzy set in [0, 1] and defined by

$$\mu_{B^S}(v) = \int \mu_S(s) / \mu_B(v)^s \qquad s \in [0, 1] \quad (42)$$

As an example of the above, firstly we shall consider the numerical similarity measure S_1 given in (37). When $A' = A$, we have $S_1 = 1$ by (37). Thus, from (41) the consequence B' is obtained as

$$B' = B^1 = B \ldots \text{at } A' = A$$

which indicates the satisfaction of the modus ponens of (15). Similarly, at A' = very A, we have $S_1 = 1$. Hence,

$$B' = B^1 = B \ldots \text{at } A' = \text{very } A$$

Next, we shall consider the fuzzy similarity measure S_2 of (38). When $A' = A$, S_2 is given from (38) as

$$S_2 = \int_{[0,1]} s/s$$

Thus, the consequence B' is inferred as

$$B' = B^{S_2}$$

$$\mu_{B'}(v) = \int \mu_{S_2}(s) / \mu_B(v)^s \qquad s \in [0, 1] \qquad \text{from (42)}$$

$$= \int s / \mu_B(v)^s$$

$$= \int \log_{\mu_B(v)} z / z \qquad z \in [\mu_B(v), 1]$$

Similarly, when A' = very A

$$S_2 = \int_{[0,1]} s^2 / s$$

$$\mu_{B'}(v) = \int s^2 / \mu_B(v)^s \qquad s \in [0, 1]$$

$$= \int [\log_{\mu_B(v)} z]^2 / z \qquad z \in [\mu_B(v), 1]$$

It is possible in this method to use other implication rules in (40). For example, if we use the Lukasiewicz implication rule of (3), the consequence B' is given as

$$\mu_{B'}(v) = 1 \wedge (1 - S + \mu_B(v)) \quad (43)$$

For example, when $A' = A$, we have $S_1 = 1$ and $S_2 =$

$\int_{[0,1]} s/s$. Therefore,

$$\mu_{B'}(v) = 1 \wedge (1 - 1 + \mu_B(v))$$
$$= \mu_B(v) \ldots \text{at } S_1 = 1$$
$$\mu_{B'}(v) = 1 \wedge (1 - S_2 + \mu_B(v))$$
$$= \int -z + \mu_B(v) + 1/z$$
$$z \in [\mu_B(v), 1] \ldots \text{at } S_2$$

5. *Mizumoto's Fuzzy Reasoning Method*

The fuzzy truth values of "x is A" and "y is B" of (1) are considered to be true from (26). Thus, the fuzzy truth value of the implication true → true is obtained as follows by the use of the extension principle (Zadeh 1975b).

$$\text{true} \to \text{true} = \int \mu_{\text{true}}(t) \wedge \mu_{\text{true}}(s)/(t \to s) \quad (44)$$

Thus, the fuzzy truth value τ_B for "x is B" given "x is A'" is obtained as

$$\tau_B = \tau_A \,\hat{\wedge}\, (\text{true} \to \text{true})$$

where $\hat{\wedge}$ stands for a fuzzified "min" and τ_A = truth (x is A/x is A').

For example, if the implication rule of (3) is used in (44), the truth value true → true is obtained as

$$\text{true} \to \text{true} = \int \mu_{\text{true}}(t) \wedge \mu_{\text{true}}(s)/(1 \wedge (1 - t + s))$$
$$= \int t \wedge s/(1 \wedge (1 - t + s))$$
$$= 1 \wedge (1 - \text{true} + \text{true})$$
$$= \text{true}$$

When $A' = A$, τ_A becomes true from (26). Hence we have τ_B as

$$\tau_B = \text{true} \,\hat{\wedge}\, (\text{true} \to \text{true}) = \text{true} \,\hat{\wedge}\, \text{true} = \text{true}$$

Therefore, we obtain $B' = B$ by using (25), which shows the satisfaction of the modus ponens. In the same way, when A' = very A, more or less A, not A, the fuzzy truth values τ_B become as follows. Hence, the consequence B' is: $B' = B$ at A' = very A; B' = more or less B at A' = more or less A; B' = not B at A' = not A.

$$\tau_B = \text{very true} \,\hat{\wedge}\, (\text{true} \to \text{true})$$
$$= \text{very true} \,\hat{\wedge}\, \text{true} = \text{true at } A' = \text{very } A$$
$$\tau_B = \text{more or less true} \,\hat{\wedge}\, \text{true}$$
$$= \text{more or less true at } A' = \text{more or less } A$$
$$\tau_B = \text{not true} \,\hat{\wedge}\, \text{true}$$
$$= \text{not true } (= \text{false}) \text{ at } A' = \text{not } A$$

See also: Modus Ponens: Generalization

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Fuzzy Relational Equations: Max–Min Fuzzy Relational Equations

Let

$$\boldsymbol{A} \subset X, \quad \mu_A(x): X \to [0, 1]$$
$$\boldsymbol{B} \subset Y, \quad \mu_B(y): Y \to [0, 1]$$

be two fuzzy sets, and $\boldsymbol{R}(X \times Y)$ be a fuzzy relation.

Given the composition of $\boldsymbol{A}$ with $\boldsymbol{R}$ denoted by $\boldsymbol{A} \circ \boldsymbol{R}$ (where $\circ$ is the max–min operator), then the problem of finding all the $\boldsymbol{A}$ such that $\boldsymbol{A} \circ \boldsymbol{R} = \boldsymbol{B}$, is called a fuzzy max–min relational equation.

There are different definitions of the composite operators. More generally, let

$$\boldsymbol{P}(X \times Y), \qquad \boldsymbol{Q}(Z), \qquad \boldsymbol{R}(X \times Y \times Z)$$

be the three fuzzy relations, then the problem of finding all the $\boldsymbol{P}$ such that $\boldsymbol{P} * \boldsymbol{R} = \boldsymbol{Q}$ is called the fuzzy relational