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FUZZY REASONING UNDER NEW COMPOSITIONAL RULES OF INFERENCE

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This paper indicates that most of fuzzy translating rules for a fuzzy conditional proposition "If x is A then y is B" with A and B being fuzzy concepts can lead to very reasonable consequences which fit our intuition with respect to several criteria such as *modus ponens* and *modus tollens*. Moreover, it is shown that a syllogism holds for most of the methods under the new compositions, though they do not always satisfy the syllogism under the max-min composition.

1. INTRODUCTION

In our daily life we often infer that

$$\begin{array}{ll} \text{Ant 1:} & \text{If } x \text{ is } A \text{ then } y \text{ is } B \\ \text{Ant 2:} & x \text{ is } A' \\ \hline \text{Cons:} & y \text{ is } B' \end{array}$$

where A, A', B, B' are fuzzy concepts. In order to make such an inference with fuzzy concepts, Zadeh[1] suggested an inference rule called "compositional rule of inference" which infers B' of Cons from Ant 1 and Ant 2 by taking the max-min composition of fuzzy set A' and the fuzzy relation which is translated from the fuzzy conditional proposition "If x is A then y is B". In this connection, he,[1] Mamdani[2] and Mizumoto *et al.*[3,4] suggested several translating rules for translating the fuzzy proposition "If x is A then y is B" into a fuzzy relation.

We pointed out[3,4] that the consequences inferred by Zadeh's and Mamdani's translating rules do not always fit our intuition; we thus proposed some new translating rules which lead to the consequences coinciding with our intuition with respect to several criteria, such as *modus ponens* and *modus tollens*. Moreover, we suggested[4] new translating rules which are obtained by introducing implication rules of many-valued logic systems, but these translating rules were found not to lead to reasonable consequences.

We have also shown[5] that, although the translating rule by Zadeh, called "arithmetic rule", does not imply reasonable consequences in the compositional rule of inference which uses the max-min composition, the arithmetic rule can lead to very reasonable consequences when new compositions, termed "max-$\odot$ composition" and "max-$\mathbin{\wedge\!\!\!\!\cdot}$ composition", are used in the compositional rule of inference, where $\odot$ is the operation of a "bounded-product" which accompanies the "Bound-sum" introduced by Zadeh,[1] and $\mathbin{\wedge\!\!\!\!\cdot}$ is the operation of the "drastic product" $Tw(x, y)$ introduced by Dubois.[6]

As a continuation of our study,[5] this paper investigates the inference results by all the translating rules proposed until now under the max-$\odot$ composition and max-$\mathbin{\wedge\!\!\!\!\cdot}$ composition, and shows that the majority of the translating rules can lead to very reasonable consequences which fit our intuition.

2. TRANSLATING RULES

We shall first consider the following form of inference in which a fuzzy conditional proposition is contained.

$$\begin{array}{ll} \text{Ant 1:} & \text{If } x \text{ is } A \text{ then } y \text{ is } B \\ \text{Ant 2:} & x \text{ is } A' \\ \hline \text{Cons:} & y \text{ is } B' \end{array} \tag{1}$$

where x and y are the names of objects, and A, A', B and B' are fuzzy concepts represented by fuzzy sets in universes of discourse U, U, V and V, respectively. This form of inference may be viewed as *fuzzy modus ponens* which reduces to the *classical modus ponens* when $A' = A$ and $B' = B$.

Moreover, the following form of inference is possible which contains a fuzzy conditional proposition.

$$\begin{array}{ll} \text{Ant 1:} & \text{If } x \text{ is } A \text{ then } y \text{ is } B \\ \text{Ant 2:} & y \text{ is } B' \\ \hline \text{Cons:} & x \text{ is } A' \end{array} \tag{2}$$

This inference can be considered as *fuzzy modus tollens* which reduces to the *classical modus tollens* when $B' = \textit{not } B$ and $A' = \textit{not } A$.

The fuzzy proposition "If x is A then y is B" of (1) and (2) represents a certain relationship between A and B. From this point of view, a number of translating rules were proposed for translating the fuzzy proposition "If x is A then y is B" into a fuzzy relation in $U \times V$.

Let A and B be fuzzy sets in U and V, respectively, and let $\times$, $\cup$, $\cap$, $\neg$ and $\oplus$ be Cartesian product, union, intersection, complement and bounded-sum for fuzzy sets, respectively. Then the following fuzzy relations in $U \times V$ are obtained from the fuzzy proposition "If x is A then y is B". Rm (maximun rule) and Ra (arithmetic rule) were proposed by Zadeh,[1] Rc (min rule) by Mamdani,[2] and the others are by Mizumoto *et al.*[3,4] by introducing the implications of many-valued logic systems.

$$Rm = (A \times B) \cup (\neg A \times V) \Leftrightarrow [\mu_A(u) \wedge \mu_B(v)] \vee [1 - \mu_A(u)] \tag{3}$$

$$Ra = (\neg A \times V) \oplus (U \times B) \Leftrightarrow 1 \wedge [1 - \mu_A(u) + \mu_B(v)] \tag{4}$$

$$Rc = A \times B \Leftrightarrow \mu_A(u) \wedge \mu_B(v) \tag{5}$$

$$Rs = A \times V \underset{s}{\Rightarrow} U \times B \Leftrightarrow \begin{cases} 1 \ldots \mu_A(u) \leqq \mu_B(v), \\ 0 \ldots \mu_A(u) > \mu_B(v). \end{cases} \tag{6}$$

$$Rg = A \times V \underset{g}{\Rightarrow} U \times B \Leftrightarrow \begin{cases} 1 & \ldots \mu_A(u) \leqq \mu_B(v), \\ \mu_B(v) & \ldots \mu_A(u) > \mu_B(v). \end{cases} \tag{7}$$

$$Rsg = (A \times V \underset{s}{\Rightarrow} U \times B) \cap (\neg A \times V \underset{g}{\Rightarrow} U \times \neg B) \tag{8}$$

$$Rgg = (A \times V \underset{s}{\Rightarrow} U \times B) \cap (\neg A \times V \underset{g}{\Rightarrow} U \times \neg B) \tag{9}$$

$$Rgs = (A \times V \underset{g}{\Rightarrow} U \times B) \cap (\neg A \times V \underset{s}{\Rightarrow} U \times \neg B) \tag{10}$$

$$Rss = (A \times V \underset{s}{\Rightarrow} U \times B) \cap (\neg A \times V \underset{s}{\Rightarrow} U \times \neg B) \tag{11}$$

$$Rb = (\neg A \times V) \cup (U \times B) \Leftrightarrow [1 - \mu_A(u)] \vee \mu_B(v) \tag{12}$$

$$R_\triangle = A \times V \underset{\triangle}{\Rightarrow} U \times B \Leftrightarrow \begin{cases} 1 & \ldots \mu_A(u) \leqq \mu_B(v), \\ \dfrac{\mu_B(v)}{\mu_A(u)} & \ldots \mu_A(u) > \mu_B(v). \end{cases} \tag{13}$$

$$R_\blacktriangle = A \times V \underset{\blacktriangle}{\Rightarrow} U \times B \Leftrightarrow \begin{cases} 1 \wedge \dfrac{\mu_B(v)}{\mu_A(u)} \wedge \dfrac{1 - \mu_A(u)}{1 - \mu_B(v)} \ldots \\ \mu_A(u) > 0,\ 1 - \mu_B(v) > 0, \\ 1 \ldots \mu_A(u) = 0 \text{ or } 1 - \mu_B(v) = 0. \end{cases} \tag{14}$$

$$R_\bigstar = A \times V \underset{\bigstar}{\Rightarrow} U \times B \Leftrightarrow 1 - \mu_A(u) + \mu_A(u)\mu_B(v). \tag{15}$$

$$R_\# = A \times V \underset{\#}{\Rightarrow} U \times B \tag{16}$$

$$\Leftrightarrow [1 - \mu_A(u) \vee \mu_B(v)] \wedge [\mu_A(u) \vee 1 - \mu_A(u)] \wedge [\mu_B(v) \vee 1 - \mu_B(v)].$$

$$R_\square = A \times V \underset{\square}{\Rightarrow} U \times B \Leftrightarrow \begin{cases} 1 \ldots \mu_A(u) < 1 \text{ or } \mu_B(v) = 1 \\ 0 \ldots \mu_A(u) = 1,\ \mu_B(v) < 1 \end{cases} \tag{17}$$

In the *fuzzy modus ponens* of (1), the consequence B' can be deduced from Ant 1 and Ant 2 by taking the *max-min composition* "$\bigcirc$" of the fuzzy set A' and the fuzzy relation obtained above (the *compositional rule of inference*). For example, we have for the method of Rm of (3)

$$Bm' = A' \bigcirc Rm = A' \bigcirc [(A \times B) \cup (\neg A \times V)] \tag{18}$$

The membership function of the fuzzy set Bm' in V is given as

$$\mu_{Bm'}(v) = \bigvee_u \{\mu_{A'}(u) \wedge \mu_{Rm}(u, v)\} = \bigvee_u \{\mu_{A'}(u) \wedge [(\mu_A(u) \wedge \mu_B(v)) \vee (1 - \mu_A(u))]\} \tag{19}$$

Similarly, in the case of *fuzzy modus tollens* of (2), the consequence is given by A'

$$Am' = Rm \bigcirc B' \tag{20}$$

As simple examples, let $A' = A$ in (18) and $B' = not\ B$ in (20), then we can have such inference results [4] as

$$Bm' = A \bigcirc Rm \Leftrightarrow \mu_{Bm'}(v) = 0.5 \vee \mu_B(v)$$

$$Am' = Rm \bigcirc not\ B \Leftrightarrow \mu_{Am'}(u) = 0.5 \vee [1 - \mu_A(u)]$$

Similarly, for the arithmetic rule Ra of (4)

$$Ba' = A \bigcirc Ra \Leftrightarrow \mu_{Ba'}(v) = \frac{1 + \mu_B(v)}{2}$$

$$Aa' = Ra \bigcirc not\ B \Leftrightarrow \mu_{Aa'}(u) = \frac{1 - \mu_A(u)}{2}$$

These consequences B' and A' are found not to be equal to B and *not* A, respectively. In other words, these translating rules cannot satisfy the *modus ponens* and *modus tollens* which are quite reasonable demands in the fuzzy conditional inference. Therefore, it seems that these rules are not suitable

$$\frac{\begin{array}{l}\text{If } x \text{ is } A \text{ then } y \text{ is } B\\ x \text{ is } A\end{array}}{y \text{ is } B} \quad (\textit{modus ponens}) \qquad (21)$$

$$\frac{\begin{array}{l}\text{If } x \text{ is } A \text{ then } y \text{ is } B\\ y \text{ is } \textit{not } B\end{array}}{x \text{ is } \textit{not } B} \quad (\textit{modus tollens}) \qquad (22)$$

methods for the fuzzy conditional inference. In the next section, however, we shall show that not only these rules but also other translating rules in (5)–(16) can satisfy the *modus ponens* and *modus tollens* and infer the consequences which fit our intuition, if, instead of the max-min composition usually used in the compositional rule of inference, we use two kinds of new compositions called "max-$\odot$ composition" and "max-$\mathbin{\wedge\!\!\!\!\cdot}$ composition" in the compositional rule of inference.

3. FUZZY CONDITIONAL INFERENCE UNDER NEW COMPOSITIONS

We shall first give the operations of "bounded-product" $\odot$ and "drastic product" $\mathbin{\dot\wedge}$ in order to define new compositions of "max-$\odot$ composition" and "max-$\mathbin{\dot\wedge}$ composition" to be used in the compositional rule of inference. The more detailed properties of these operations are found in references 6–8.

For any $x, y \in [0, 1]$

$$\textit{Bounded-Product}: x \odot y = 0 \vee (x + y - 1) \qquad (23)$$

$$\textit{Drastic Product}: \quad x \mathbin{\dot\wedge} y = \begin{cases} x \ldots y = 1 \\ y \ldots x = 1 \\ 0 \ldots x, y < 1 \end{cases} \qquad (24)$$

Using these new operations we can easily define new compositions called max-$\odot$ composition "$\square$" and max-$\mathbin{\dot\wedge}$ composition "$\blacktriangle$" in the same way as the max-min composition of "$\bigcirc$" of (18) by replacing $\wedge$ with $\odot$ and $\mathbin{\dot\wedge}$ in (19). Therefore, it is possible to obtain consequences by using these compositions. For example, using the max-$\odot$ composition "$\square$", we can obtain the consequence in the same way as (18) and (20).

$$Bm' = A' \square Rm$$

$$\Leftrightarrow \mu_{Bm'}(v) = \bigvee_u \{\mu_{A'}(u) \odot \mu_{Rm}(u, v)\} \qquad (25)$$

$$Am' = Rm \square B' \qquad (26)$$

Similarly, under the max-$\mathbin{\dot\wedge}$ composition "$\blacktriangle$", we have

$$Bm' = A' \blacktriangle Rm$$

$$\Leftrightarrow \mu_{Bm'}(v) = \bigvee_u \{\mu_{A'}(u) \mathbin{\dot\wedge} \mu_{Rm}(u, v)\} \qquad (27)$$

$$Am' = Rm \blacktriangle B' \qquad (28)$$

The same ways are applicable to other translating rules $Ra, Rc, \ldots, R_{\square}$ of (4)–(17).

In the *fuzzy modus ponens*, we shall show what the consequences B' become under new compositions "$\square$" and "$\blacktriangle$" when A' is

$A' = A$

$A' = \textit{very } A = A^2$

$A' = \textit{more or less } A = A^{0.5}$

$A' = \textit{not } A = \neg A$

which are typical examples of A'.

Similarly, in the *fuzzy modus tollens* of (2), we show what the consequences A' is when B' is

$B' = \textit{not } B = \neg B$

$B' = \textit{not very } B = \neg B^2$

$B' = \textit{not more or less } B = \neg B^{0.5}$

$B' = B.$

We shall begin with the *fuzzy modus ponens* in (1). It is assumed in the discussion of the *fuzzy modus ponens* that $\mu_A(u)$ takes all values in [0, 1] according to u varying all over U, that is, μ_A is a function onto [0, 1]. Clearly, from this assumption, the fuzzy set A is a normally fuzzy set.

We shall first discuss Rm and obtain the consequence Bm' of (25) at $A' = A^\alpha$ which is a general case of A, *very* A and *more or less* A. From the above assumption that μ_A is a function onto [0, 1], (25) can be rewritten as

$$bm' = \bigvee_x \{x^\alpha \odot [(x \wedge b) \vee (1 - x)]\} \qquad (29)$$

and

$$f(x) = x^\alpha \odot [(x \wedge b) \vee (1 - x)] \qquad (30)$$

by letting

$$\mu_A(u) = x, \quad \mu_B(v) = b, \quad \mu_{Bm'}(v) = bm'. \qquad (31)$$

From the definition of bounded-product $\odot$ of (23), we have $f(x)$ of (30) as

$$f(x) = 0 \vee \{x^\alpha + [(x \wedge b) \vee (1 - x)] - 1\}$$
$$= 0 \vee [(x^\alpha + x - 1) \wedge (x^\alpha + b - 1)] \vee (x^\alpha - x) \qquad (32)$$

Case of $\alpha \geqq 1$: When $\alpha \geqq 1$, $x^\alpha - x \leqq 0$ is obtained. Thus, $f(x)$ reduces to

$$f(x) = 0 \vee [(x^\alpha + x - 1) \wedge (x^\alpha + b - 1)]$$

Figure 1(a) shows partly the expressions $x^\alpha + x - 1$

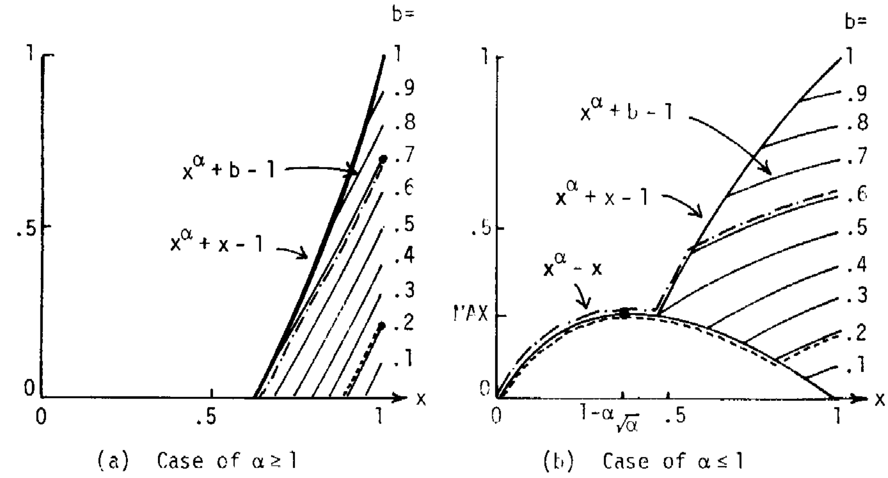

FIGURE 1 $f(x)$ of (32).

and $x^\alpha + b - 1$ by using a parameter b. When b is equal to, say, $0.2, f(x)$ is indicated by the broken line and thus $bm' = \bigvee_u [f(x)]$ of (29) at $b = 0.2$ becomes 0.2 by taking the maximum of this line. In the same way, at $b = 0.7$, $f(x)$ is shown by the line "–·–" whose maximum value is 0.7. Thus we have $bm' = 0.7$ at $b = 0.7$. In general, we can have $bm' = b$ for any b, that is, $bm' = b$ at $x' = x^\alpha (\alpha \geqq 1)$, which leads to $\mu_{Bm'}(v) = \mu_B(v)$ at $\mu_{A'}(u) = \mu_A(A)^\alpha$ $(\alpha \geqq 1)$ from (31). Namely, $Bm' = B$ at $A' = A^\alpha$ $(\alpha \geqq 1)$. Therefore,

$$A^\alpha \square Rm = B \ldots \alpha \geqq 1 \tag{33}$$

Case of $\alpha \leqq 1$: $f(x)$ is given by (32) and is drawn in Fig. 1(b). The expression $\chi^\alpha - x$ $(\alpha < 1)$ has the maximum value

$$\sqrt[1-\alpha]{\alpha}\left(\frac{1}{\alpha} - 1\right) \ (=\text{MAX}) \text{ at } x = \sqrt[1-\alpha]{\alpha}.$$

From Fig. 1(b) it follows that $bm' = \text{MAX}$ at $0 \leqq b \leqq \text{MAX}$, that is,

$$bm' = \bigvee_x f(x) = \sqrt[1-\alpha]{\alpha}\left(\frac{1}{\alpha} - 1\right)$$

On the other hand, when $\text{MAX} \leqq b \leqq 1$, we have

$$bm' = b.$$

Hence,

$$bm' = \sqrt[1-\alpha]{\alpha}\left(\frac{1}{\alpha} - 1\right) \vee b$$

Namely,

$$\mu_{Bm'}(v) = \sqrt[1-\alpha]{\alpha}\left(\frac{1}{\alpha} - 1\right) \vee \mu_B(v) \quad \text{at} \quad \alpha \leqq 1. \tag{34}$$

Therefore, the consequence $Bm = A^\alpha \square Rm$ under the max-$\odot$ composition "$\square$" is given by

$$\mu_{Bm'}(v) = \begin{cases} \sqrt[1-\alpha]{\alpha}\left(\frac{1}{\alpha} - 1\right) \vee \mu_B(v) \ldots \alpha \leqq 1 \\ \mu_B(v) \qquad\qquad \ldots \alpha \geqq 1 \end{cases} \tag{35}$$

From this result we can obtain the consequences Bm' at $A' = A$, *very* A $(= A^2)$ and *more or less* A $(= A^{0.5})$ by letting $\alpha = 1, 2, 0.5$, respectively.

$$Bm' = A \square Rm = B \tag{36}$$

$$Bm' = \textit{very } A \square Rm = B \tag{37}$$

$$Bm' = \textit{more or less } A \square Rm \Leftrightarrow \tfrac{1}{4} \vee \mu_B(v) \tag{38}$$

Equation (36) indicates that a *modus ponens* is satisfied by the method Rm under the max-$\odot$ composition "$\square$". It is noted that Rm does not satisfy *modus ponens* under the max-min composition "$\bigcirc$".

We shall next consider the inference result $Ba' = A^\alpha \blacktriangle Ra$ under the max-$\mathbin{\dot\wedge}$ composition "$\triangle$". Ba' is given by

$$\mu_{Ba'}(v) = \bigvee_u \{\mu_A\alpha(u) \mathbin{\dot\wedge} [1 \wedge (1 - \mu_A(u) + \mu_B(v))]\}$$

$$ba' = \bigvee_x \{x^\alpha \mathbin{\dot\wedge} [1 \wedge (1 - x + b)]\}$$

Let $g(x)$ be

$$g(x) = x^\alpha \mathbin{\dot\wedge} [1 \wedge (1 - x + b)] \tag{39}$$

then $g(x)$ is shown by the solid line and the black circle in Fig. 2. Namely,

$$g(x) = \begin{cases} x^\alpha \ldots 0 \leqq x \leqq b \\ b \ \ldots x = 1 \\ 0 \ \ldots \text{otherwise} \end{cases}$$

Thus,

$$ba' = \bigvee_x g(x) = \bigvee_{x[0,b]} x^\alpha \vee b = b^\alpha \vee b.$$

Therefore, we have $Ba' = A^\alpha \blacktriangle Ra$ as

$$Ba' = \begin{cases} B^\alpha \ldots \alpha \leqq 1 \\ B \ \ldots \alpha \geqq 1 \end{cases} \tag{40}$$

which gives the inference results Ba' at $A' = A$, *very* A and *more or less* A as follows.

$$\text{Ba}' = A \blacktriangle \text{Ra} = B \tag{41}$$

$$Ba' = \textit{very } A \blacktriangle Ra = B \tag{42}$$

$$Ba' = \textit{more or less } A \blacktriangle Ra = \textit{more or less } B \tag{43}$$

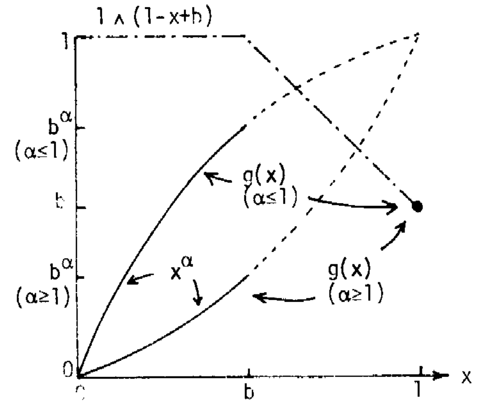

FIGURE 2 $g(x)$ of (39).

which also indicates the satisfaction of *modus ponens* under the max-$\wedge$ composition "▲".

In the same way, we can obtain the consequences by other methods Rc, Rs, . . . , $R_{\square}$. Tables 1–4 list the inference results by all the methods (3)–(17) under the max-$\odot$ composition and max-$\wedge$ composition.

In the form of fuzzy conditional inferences (1) and (2), it seems according to our intuition that criteria between A' in Ant 2 and B' in Cons of the *fuzzy modus ponens* (1) ought to be satisfied as shown in the left part of Table 5.[3,4] Similarly, criteria for the *fuzzy modus tollens* (2) are shown in this table. The right part of Table 5 indicates the satisfaction ($\otimes$) or failure ($\times$) of each criterion by each inference method by the use of the inference results in Tables 1–4. In order to compare the inference results under the max-$\odot$ composition "□" and max-$\wedge$ composition "▲" with those under the ordinal max-min composition "○", the satisfaction of each criterion under the max-min composition is listed in the table.[4]

From Tables 1–5 it follows that all the inference methods except $R_{\square}$ can satisfy so-called *modus ponens* (21) under the max-$\odot$ composition "□" and max-$\wedge$ composition "▲", but only the methods Rc, Rs, . . . , Rss can satisfy the *modus ponens* under the max-min composition "○". The almost same holds for the *modus tollens* of (22). Moreover, it is found that the majority of the methods can infer very reasonable consequences under the max-$\odot$ composition and max-$\wedge$ composition, though we can not always get reasonable consequences under the max-min composition as shown in Table 5.

4. SYLLOGISM UNDER NEW COMPOSITIONS

In this section we shall investigate a syllogism by each fuzzy inference method under new compositions of max-$\odot$ composition "□" and max-$\wedge$ composition "▲", and shows that the syllogism holds for many inference methods under the new compositions, though a few inference methods satisfy the syllogism under max-min composition "○".

Let P_1, P_2 and P_3 be fuzzy conditional propositions in (44), where A, B and C are fuzzy sets in U, V and W, respectively. If the proposition P_3 is deduced from the propositions P_1 and P_2, that is, the following holds, then it is said that a syllogism holds.

$$\begin{array}{ll} P_1\colon & \text{If } x \text{ is } A \text{ then } y \text{ is } B. \\ P_2\colon & \text{If } y \text{ is } B \text{ then } z \text{ is } C. \\ \hline P_3\colon & \text{If } x \text{ is } A \text{ then } z \text{ is } C. \end{array} \qquad (44)$$

TABLE 1
Inference results under max-$\odot$ composition (Case of *fuzzy modus ponens*)

	A	*very* A	*more or less* A	*not* A
Rm	B	B	$\frac{1}{4} \vee \mu_B$	*unknown*
Ra	B	B	$\begin{cases} \mu_B + \frac{1}{4} \cdots \mu_B \leqq \frac{1}{4} \\ \sqrt{\mu_B} \cdots \mu_B \geqq \frac{1}{4} \end{cases}$	*unknown*
Rc	B	B	B	ϕ
Rs	B	*very* B	*more or less* B	*unknown*
Rg	B	B	*more or less* B	*unknown*
Rsg	B	*very* B	*more or less* B	*not* B
Rgg	B	B	*more or less* B	*not* B
Rgs	B	B	*more or less* B	*not* B
Rss	B	*very* B	*more or less* B	*not* B
Rb	B	B	$\frac{1}{4} \vee \mu_B$	*unknown*
$R_{\triangle}$	B	B	*more or less* B	*unknown*
$R_{\blacktriangle}$	B	*very* B	*more or less* B	*unknown*
$R_{\star}$	B	B	$\begin{cases} \dfrac{1}{4(1-\mu_B)} \cdots \mu_B \leqq \frac{1}{2} \\ \mu_B \cdots \mu_B \geqq \frac{1}{2} \end{cases}$	*unknown*
$R_{\#}$	B	B	$\frac{1}{4} \vee \mu_B$	$B \cup$ *not* B
$R_{\square}$	*unknown*	*unknown*	*unknown*	*unknown*

TABLE 2
Inference results under max-⊙ composition (case of *fuzzy modus tollens*)

	not B	not very B	not more or less B	B
Rm	not A	$(1-\mu_A)\vee\frac{1}{4}$	not A	A ∪ not A
Ra	not A	$\begin{cases} 1-\mu_A^2 & \dots \mu_A \leqq \frac{1}{2} \\ \frac{1}{4}+(1-\mu_A) & \dots \mu_A \geqq \frac{1}{2} \end{cases}$	not A	unknown
Rc	ϕ	$\begin{cases} \mu_A-\mu_A^2 & \dots \mu_A \leqq \frac{1}{2} \\ \frac{1}{4} & \dots \mu_A \geqq \frac{1}{2} \end{cases}$	ϕ	A
Rs	not A	not very A	not more or less A	unknown
Rg	not A	$(1-\mu_A^2)\vee\frac{1}{4}$	not more or less A	unknown
Rsg	not A	not very A	not more or less A	A
Rgg	not A	$(1-\mu_A^2)\vee\frac{1}{4}$	not more or less A	A
Rgs	not A	$(1-\mu_A^2)\vee\frac{1}{4}$	not more or less A	A
Rss	not A	not very A	not more or less A	A
Rb	not A	$(1-\mu_A)\vee\frac{1}{4}$	not A	unknown
$R_\triangle$	not A	$\begin{cases} 1-\mu_A^2 & \dots \mu_A \leqq \sqrt{2}/2 \\ \frac{1}{4\mu_A^2} & \dots \mu_A \geqq \sqrt{2}/2 \end{cases}$	not more or less A	unknown
$R_\blacktriangle$	not A	not very A	not more or less A	unknown
$R_\bigstar$	not A	$\left(1-\frac{\mu_A}{2}\right)^2$	not A	unknown
$R_\#$	not A	$(1-\mu_A)\vee\frac{1}{4}$	not A	A ∪ not A
$R_\square$	$\begin{cases} 1 & \dots \mu_A < 1 \\ 0 & \dots \mu_A = 1 \end{cases}$	$\begin{cases} 1 & \dots \mu_A < 1 \\ 0 & \dots \mu_A = 1 \end{cases}$	$\begin{cases} 1 & \dots \mu_A < 1 \\ 0 & \dots \mu_A = 1 \end{cases}$	unknown

Let $R(A,B)$, $R(B,C)$ and $R(A,C)$ be fuzzy relations in $U\times V$, $V\times W$ and $U\times W$, respectively, which are obtained from the propositions P_1, P_2 and P_3. If the following equality holds, the syllogism holds under a composition $\bigstar$, where $\bigstar\in\{\square, \blacktriangle, \bigcirc\}$.

$$R(A,B)\bigstar R(B,C) = R(A,C) \tag{45}$$

That is to say,

P_1: If x is A then y is $B \to R(A,B)$
P_2: If y is B then z is $C \to R(B,C)$ (46)
P_3: If x is A then z is $C \leftarrow R(A,B)\bigstar R(B,C)$

The membership function of $R(A,B)\bigstar R(B,C)$ is given by the following:

When $\bigstar = \square$, we have

$$\mu_{R(A,B)\square R(B,C)}(u,w) = \bigvee_v \{\mu_{R(A,B)}(u,v) \odot \mu_{R(B,C)}(v,w)\} \tag{47}$$

Similarly, when $\bigstar = \blacktriangle$,

$$\mu_{R(A,B)\blacktriangle R(B,C)}(u,w) = \bigvee_v \{\mu_{R(A,B)}(u,v) \wedge \mu_{R(B,C)}(v,w)\} \tag{48}$$

TABLE 3
Inference results under max-∧ composition (case of *fuzzy modus ponens*)

	A	very A	more or less A	not A
Rm	B	B	B	unknown
Ra	B	B	more or less B	unknown
Rc	B	B	B	ϕ
Rs	B	very B	more or less B	unknown
Rg	B	B	more or less B	unknown
Rsg	B	very B	more or less B	not B
Rgg	B	B	more or less B	not B
Rgs	B	B	more or less B	not B
Rss	B	very B	more or less B	not B
Rb	B	B	B	unknown
$R_\triangle$	B	B	more or less B	unknown
$R_\blacktriangle$	B	very B	more or less B	unknown
$R_\bigstar$	B	B	B	unknown
$R_\#$	B	B	B	B ∪ not B
$R_\square$	unknown	unknown	unknown	unknown

TABLE 4
Inference results under max-$\wedge$ composition (case of *fuzzy modus tollens*)

	not B	*not very B*	*not more or less B*	*B*
Rm	*not A*	*not A*	*not A*	*A* ∪ *not A*
Ra	*not A*	*not very A*	*not A*	*unknown*
Rc	ϕ	ϕ	ϕ	*A*
Rs	*not A*	*not very A*	*not more or less A*	*unknown*
Rg	*not A*	*not very A*	*not more or less A*	*unknown*
Rsg	*not A*	*not very A*	*not more or less A*	*A*
Rgg	*not A*	*not very A*	*not more or less A*	*A*
Rgs	*not A*	*not very A*	*not more or less A*	*A*
Rss	*not A*	*not very A*	*not more or less A*	*A*
Rb	*not A*	*not A*	*not A*	*unknown*
$R_{\triangle}$	*not A*	*not very A*	*not more or less A*	*unknown*
$R_{\blacktriangle}$	*not A*	*not very A*	*not more or less A*	*unknown*
$R_{\bigstar}$	*not A*	*not A*	*not A*	*unknown*
$R_{\#}$	*not A*	*not A*	*not A*	*A* ∪ *not A*
$R_{\square}$	$\begin{cases}1 \ldots \mu_A < 1\\ 0 \ldots \mu_A = 1\end{cases}$	$\begin{cases}1 \ldots \mu_A < 1\\ 0 \ldots \mu_A = 1\end{cases}$	$\begin{cases}1 \ldots \mu_A < 1\\ 0 \ldots \mu_A = 1\end{cases}$	*unknown*

Now we shall obtain $R(A, B) \bigstar R(B, C)$ under each fuzzy inference method and show whether the syllogism holds or not under each of the compositons. It is assumed in the discussion of the syllogism that the membership function μ_B of the fuzzy set B is a function onto [0, 1].

We shall discuss only the case of Rb of (12) because of the limitation of space, and begin with the case of max-$\odot$ composition "$\square$". The fuzzy relations $Rb(A, B)$ and $Rb(B, C)$ are obtained from the propositions P_1 and P_2 by using (12).

$$Rb(A, B) = (7A \times V) \cup (U \times B)$$

$$Rb(B, C) = (7B \times W) \cup (V \times C)$$

The max-$\odot$ composition of $Rb(A, B)$ and $Rb(B, C)$ is

$$Rb(A, B) \square Rb(B, C)$$
$$= [(7A \times V) \cup (U \times B)] \square [(7B \times W) \cup (V \times C)]$$

and its membership function is as follows.

$$\mu_{Rb(A,B) \square Rb(B,C)(B,C)}(u, w) = \bigvee_v \{[(1 - \mu_A(u)) \vee \mu_B(v)] \odot [(1 - \mu_B(v)) \vee \mu_C(w)]\}. \quad (49)$$

This expression can be rewritten as

$$d = \bigvee_x \{[(1 - a) \vee x] \odot [(1 - x) \vee c]\} \quad (50)$$

$$f(x) = [(1 - a) \vee x] \odot [(1 - x) \vee c] \quad (51)$$

under the above assumption that μ_B is a function onto [0, 1], where

$$d = \mu_{Rb(A,B) \square Rb(B,C)}(u, w),$$

$$a = \mu_A(u), \quad x = \mu_B(v), \quad c = \mu_C(w) \quad (52)$$

From the definition of bounded-product $\odot$ of (23), we have $f(x)$ of (51) as

$$f(x) = 0 \vee \{[(1 - a) \vee x] + [(1 - x) \vee c] - 1\}$$
$$= 0 \vee (1 - a - x) \vee (x + c - 1) \vee (c - a) \quad (53)$$

When $c - a > 0$, the expression (53) is represented by the solid line as shown in Fig. 3(a) with parameters a and c. The maximum value of this line is $1 - a$ at $1 - a \geqq c$ and c at $1 - a \leqq c$. Thus, we have d of (50) as

$$d = \begin{cases} 1 - a \ldots 1 - a \geqq c \\ c \quad \ldots 1 - a \leqq c \end{cases}$$
$$= (1 - a) \vee c \quad \text{at } c - a > 0 \quad (54)$$

On the other hand, when $c - a \leqq 0$, $f(x)$ of (53) is shown by the solid line in Fig. 3(b) whose maximum value is $1 - a$ or c. Thus,

$$d = (1 - a) \vee c \quad \text{at } c - a \leqq 0 \quad (55)$$

From (54) and (55), d is given by

$$d = (1 - a) \vee c$$

for any a and c, which leads to

$$\mu_{Rb(A,B) \square Rb(B,C)}(u, w) = [1 - \mu_A(u)] \vee \mu_C(w) = \mu_{Rb(A,C)}(u, w) \quad (56)$$

from the notations in (52). Therefore, we have

$$Rb(A, B) \square Rb(B, C) = Rb(A, C) \quad (57)$$

which indicates the satisfaction of a syllogism of (45) under the max-$\odot$ composition "$\square$".

Next we shall discuss the case of max-$\wedge$ composition "$\blacktriangle$". The membership function of $Rb(A, B) \blacktriangle Rb(B, C)$ is given as

$$\mu_{Rb(A,B) \blacktriangle Rb(B,C)}(u, w) = \bigvee_v \{[(1 - \mu_A(u)) \mu_B(v)] \wedge [(1 - \mu_B(v)) \vee \mu_C(w)]\}$$

which can be rewritten as

$$d = \bigvee_x \{[(1 - a) \vee x] \wedge [(1 - x) \vee c]\} \quad (58)$$

$$g(x) = [(1 - a) \vee x] \wedge [(1 - x) \vee c] \quad (59)$$

by using the notations in (52). The expression $(1 - a) \vee x$ in (59) can be depicted in Fig. 4(a) by

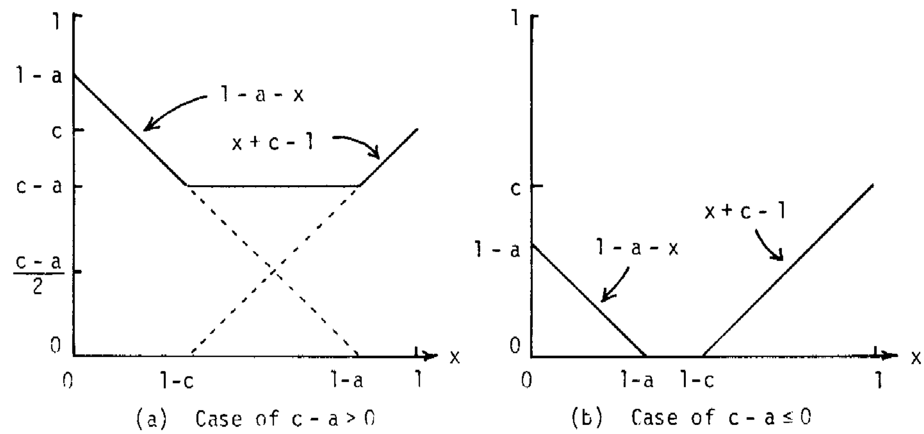

FIGURE 3 $f(x)$ of (51) (solid line).

using the parameter a, and the expression $(1-x) \vee c$ is shown by using a parameter c as in Fig. 4(b).

When $1-a \leqq c$, the function $g(x)$ of (59) is given by the black circles in Fig. 4(c), i.e.,

$$g(x) = \begin{cases} 1-a \ldots x = 0 \\ c \quad \ldots x = 1 \\ 0 \quad \ldots \text{otherwise} \end{cases}$$

Thus, d of (58) is obtained by

$$d = \bigvee_x g(x) = c \ldots \text{at } 1-a \leqq c \tag{60}$$

On the other hand, when $1-a \geqq c$, $g(x)$ is given by the black circles in Fig. 4(d). Thus,

$$d = 1-a \ldots \text{at } 1-a \geqq c \tag{61}$$

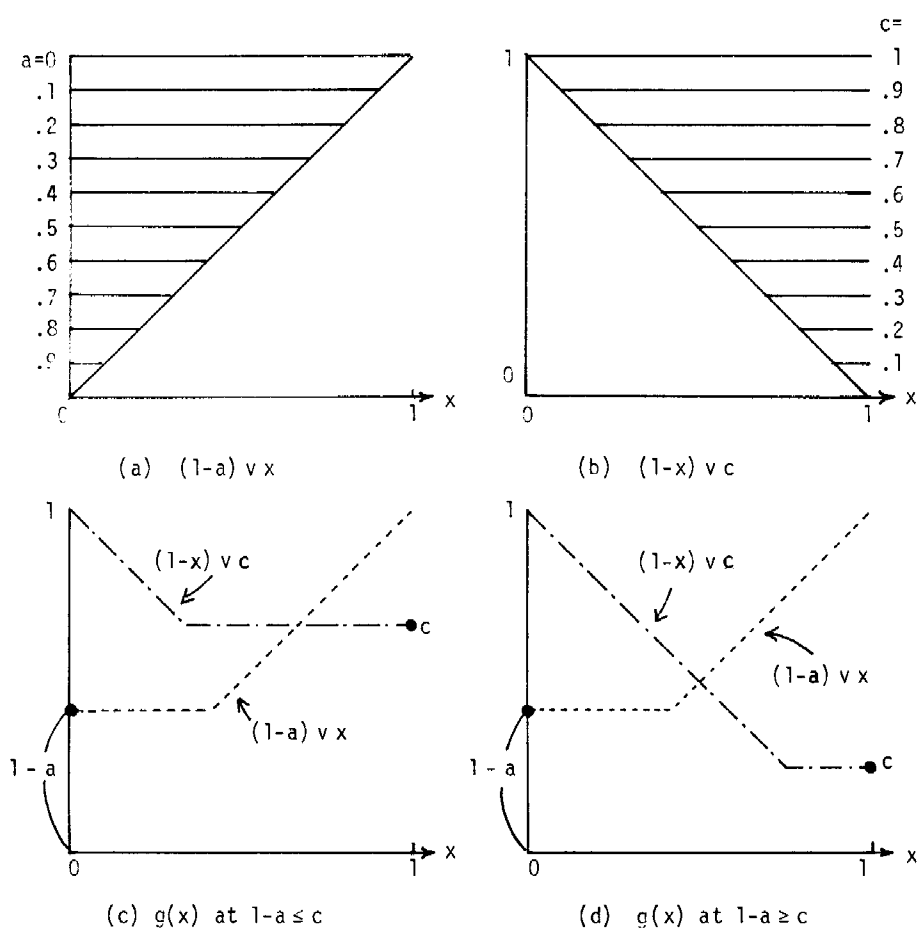

FIGURE 4 $g(x) = [(1-a) \vee x] \wedge [(1-x) \vee c]$ (black circles).

TABLE 5
Satisfaction (⊗) or failure (×) of criterion for each method under max-min composition "○", max-⊙ composition "□" and max-∧ composition "▲"

Criterion	Ant 2	Cons	*Rm* ○□▲	*Ra* ○□▲	*Rc* ○□▲	*Rs* ○□▲	*Rg* ○□▲	*Rsg* ○□▲	*Rgg* ○□▲	*Rgs* ○□▲	*Rss* ○□▲	*Rb* ○□▲	$R_\triangle$ ○□▲	$R_\blacktriangle$ ○□▲	$R_\bigstar$ ○□▲	$R_\#$ ○□▲	$R_\square$ ○□▲
I (*modus ponens*)	*A*	*B*	×⊗⊗	×⊗⊗	⊗⊗⊗	⊗⊗⊗	⊗⊗⊗	⊗⊗⊗	⊗⊗⊗	⊗⊗⊗	⊗⊗⊗	×⊗⊗	×⊗⊗	×⊗⊗	×⊗⊗	×⊗⊗	×××
II-1	*very A*	*very B*	×××	×××	×××	⊗⊗⊗	×××	⊗⊗⊗	×××	×××	⊗⊗⊗	×××	×××	×⊗⊗	×××	×××	×××
II-2	*very A*	*B*	×⊗⊗	×⊗⊗	⊗⊗⊗	×××	⊗⊗⊗	×××	⊗⊗⊗	⊗⊗⊗	×××	×⊗⊗	×⊗⊗	×××	×⊗⊗	×⊗⊗	×××
III-1	*more or less A*	*more or less B*	×××	××⊗	×××	⊗⊗⊗	⊗⊗⊗	⊗⊗⊗	⊗⊗⊗	⊗⊗⊗	⊗⊗⊗	×××	×⊗⊗	×⊗⊗	×××	×××	×××
III-2	*more or less A*	*B*	××⊗	×××	⊗⊗⊗	×××	×××	×××	×××	×××	×××	××⊗	×××	×××	××⊗	××⊗	×××
IV-1	*not A*	*unknown*	⊗⊗⊗	⊗⊗⊗	×××	⊗⊗⊗	⊗⊗⊗	×××	×××	×××	×××	⊗⊗⊗	⊗⊗⊗	⊗⊗⊗	⊗⊗⊗	×××	⊗⊗⊗
IV-2	*not A*	*not B*	×××	×××	×××	×××	×××	⊗⊗⊗	⊗⊗⊗	⊗⊗⊗	⊗⊗⊗	×××	×××	×××	×××	×××	×××
V (*modus tollens*)	*not B*	*not A*	×⊗⊗	×⊗⊗	×××	⊗⊗⊗	×⊗⊗	⊗⊗⊗	×⊗⊗	×⊗⊗	⊗⊗⊗	×⊗⊗	×⊗⊗	×⊗⊗	×⊗⊗	×⊗⊗	×××
VI-1	*not very B*	*not very B*	×××	××⊗	×××	⊗⊗⊗	××⊗	⊗⊗⊗	××⊗	××⊗	⊗⊗⊗	×××	××⊗	×⊗⊗	×××	×××	×××
VI-2	*not very B*	*not A*	××⊗	×××	×××	×××	×××	×××	×××	×××	×××	××⊗	×××	×××	××⊗	××⊗	×××
VII-1	*not more or less B*	*not more or less A*	×××	×××	×××	⊗⊗⊗	×⊗⊗	⊗⊗⊗	×⊗⊗	×⊗⊗	⊗⊗⊗	×××	×⊗⊗	×⊗⊗	×××	×××	×××
VII-2	*not more or less B*	*not A*	×⊗⊗	×⊗⊗	×××	×××	×××	×××	×××	×××	×××	×⊗⊗	×××	×××	×⊗⊗	×⊗⊗	×××
VIII-1	*B*	*unknown*	×××	⊗⊗⊗	×××	⊗⊗⊗	⊗⊗⊗	×××	×××	×××	×××	⊗⊗⊗	⊗⊗⊗	⊗⊗⊗	⊗⊗⊗	×××	⊗⊗⊗
VIII-2	*B*	*A*	×××	×××	⊗⊗⊗	×××	×××	×⊗⊗	×⊗⊗	⊗⊗⊗	⊗⊗⊗	×××	×××	×××	×××	×××	×××

TABLE 6
Syllogism under max-⊙ composition "□" and max-∧ composition "▲"

	$R(A,B)\square R(B,C)$	$R(A,B)\blacktriangle R(B,C)$
Rm	$[\mu_A(u)+\mu_C(w)-1]\vee[1-\mu_A(u)]$	$\begin{cases}\mu_C(w) & \ldots\mu_A(w)=1\\ \mu_A(u)\vee[1-\mu_A(u)] & \ldots\mu_C(w)=1\\ 0 & \ldots\text{otherwise}\end{cases}$
Ra	$Ra(A,C)$	$Ra(A,C)$
Rc	$0\vee[\mu_A(u)+\mu_C(w)-1]$	$\begin{cases}\mu_C(w)\ldots\mu_A(u)=1\\ \mu_A(u)\ldots\mu_C(w)=1\\ 0\quad\ldots\text{otherwise}\end{cases}$
Rs	$Rs(A,C)$	$Rs(A,C)$
Rg	$Rg(A,C)$	$Rg(A,C)$
Rsg	$Rsg(A,C)$	$Rsg(A,C)$
Rgg	$Rgg(A,C)$	$Rgg(A,C)$
Rgs	$Rgs(A,C)$	$Rgs(A,C)$
Rss	$Rss(A,C)$	$Rss(A,C)$
Rb	$Rb(A,C)$	$Rb(A,C)$
$R_\triangle$	$R_\triangle(A,C)$	$R_\triangle(A,C)$
$R_\blacktriangle$	$R_\blacktriangle(A,C)$	$R_\blacktriangle(A,C)$
$R_\bigstar$	$[1-\mu_A(A)]\vee\mu_C(w)$	$[1-\mu_A(u)]\vee\mu_C(w)$
$R_\#$	$[\mu_A(u)+\mu_C(w)-1]\vee(1-\mu_A(u)-\mu_C(w)]$ $\vee[\mu_C(w)-\mu_\wedge(u)]$	$\begin{cases}\mu_C(w)\vee[1-\mu_C(w)] & \ldots\mu_A(u)=0\\ \mu_C(w) & \ldots\mu_A(u)=1\\ 1-\mu_A(u) & \ldots\mu_C(w)=0\\ \mu_A(u)\vee[1-\mu_A(u)] & \ldots\mu_C(w)=1\\ 0 & \ldots\text{otherwise}\end{cases}$
$R_\square$	$R_\square(A,C)$	$R_\square(A,C)$

Therefore, from (60) and (61) we have

$$d=(1-a)\vee c$$

for any a and c, i.e.

$$\mu_{Rb(A,B)\blacktriangle Rb(B,C)}(u,w)=[1-\mu_A(u)]\vee\mu_C(w)\quad(62)$$

which indicates

$$Rb(A,B)\blacktriangle Rb(B,C)=Rb(A,C)\qquad(63)$$

Therefore, the syllogism holds for Rb under the max-∧ composition "▲".

It is found from the results of (57) and (63) that the inference method Rb satisfies the syllogism under the max-⊙ composition "□" and max-∧ composition "▲". It is noted[4] that Rb does not satisfy the syllogism under the max-min composition "○", i.e. we have

$$\mu_{Rb(A,B)\circ Rb(B,C)}(u,w)$$
$$=0.5\vee[1-\mu_A(u)]\vee\mu_C(w)\neq\mu_{Rb(A,C)}(u,w).$$

In the same way, we can obtain $R(A,B)\bigstar R(B,C)$ by the other fuzzy inference methods under the max-⊙ composition and max-∧ composition, and thus the results are listed in Table 6.

Using these results, the satisfaction (⊗) or failure (×) of the syllogism by each fuzzy inference method under the max-⊙ composition and max-∧ composition is listed in Table 7. This table also contains the results under the max-min composition.[4]

It follows from Table 7 that the methods Ra, Rb, $R_\triangle$ and $R_\blacktriangle$ can satisfy the syllogism under the max-⊙ composition and max-∧ composition, though they do not satisfy it under the max-min composition. But the converse holds for Rc.

5. CONCLUSION

We have shown that, when new compositions of max-⊙ composition and max-∧ composition are used in the compositional rule of inference, the majority of fuzzy inference methods can get very reasonable consequences, that coincide with our intuition with respect to several criteria such as *modus ponens*, *modus tollens* and syllogism.

It will be of interest to apply the new compositions to fuzzy inferences which are of the more complicated form such as

If x is A then y is B else y is C.
x is A'.

---

y is D.

If x is A_1 then y is B_1 else
if x is A_2 then y is B_2 else
.
.
.
if x is A_n then y is B_n.
x is A'.

---

y is B'.

TABLE 7
Satisfaction (⊗) or failure (×) of Syllogism under max-⊙ composition, max-∧ composition and max-min composition

	Rm	Ra	Rc	Rs	Rg	Rsg	Rgg	Rgs	Rss	Rb	$R_{\triangle}$	$R_{\blacktriangle}$	$R_{\bigstar}$	$R_{\#}$	$R_{\square}$
Max-⊙ composition	×	⊗	×	⊗	⊗	⊗	⊗	⊗	⊗	⊗	⊗	⊗	×	×	⊗
Max-∧ composition	×	⊗	×	⊗	⊗	⊗	⊗	⊗	⊗	⊗	⊗	⊗	×	×	⊗
Max-min composition	×	×	⊗	⊗	⊗	⊗	⊗	⊗	⊗	×	×	×	×	×	⊗

These results will be presented in subsequent papers.

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