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FUZZY INFERENCE USING MAX-Δ COMPOSITION IN THE COMPOSITIONAL RULE OF INFERENCE

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This paper shows that most of fuzzy inference methods for fuzzy conditional "If x is A then y is B" with A and B being fuzzy concepts can infer quite reasonable consequences which fit our intuition with respect to several criteria such as modus ponens and modus tollens, if a new composition called "max-Δ composition" is used in the compositional rule of inference, though, as was pointed out before, reasonable consequences can not always be obtained when using the max-min composition which is used usually in the compositional rule of inference. Moreover, it is shown that a syllogism holds for most of the methods under the max-Δ composition, though they do not always satisfy the syllogism under the max-min composition.

Keywords: Fuzzy conditional inference, Compositional rule of inference, Fuzzy modus ponens, Fuzzy modus tollens, Max-Δ composition, Syllogism

1. INTRODUCTION

In our daily life we often make such an inference of the form:

Ant 1:	If x is A then y is B
Ant 2:	x is A'
Cons:	y is B'

where A, A', B and B' are fuzzy concepts. In order to make such an inference with fuzzy concepts, Zadeh [1] suggested an inference rule called "compositional rule of inference" which infers B' of Cons from Ant 1 and Ant 2 by taking the max-min composition of A' and the fuzzy relation which is translated from a fuzzy conditional proposition "If x is A then y is B." In this connection, he [1], Mamdani [2] and Mizumoto et al. [3-6] suggested several translating rules for translating the fuzzy proposition "If x is A then y is B" into a fuzzy relation.

In [4-6] we pointed out that the consequences inferred by Zadeh's and Mamdani's methods do not always fit our intuition, and suggested some new methods which get the consequences coinciding with our intuition with respect to several criteria such as modus ponens and modus tollens. Moreover, we have proposed in [7] a number of translating rules which are obtained by introducing implication rules of many-valued logic systems, but they were found not to infer reasonable consequences.

In this paper, on the contrary, we show that almost all the methods proposed before can infer quite reasonable consequences if, instead of the max-min composition usually used in the compositional rule of inference, we use a new composition called "max-Δ composition" in the compositional rule of inference, where Δ is the operation of "drastic product" introduced by Dubois [8]. Moreover, it is shown that the syllogism holds for most of these methods when using the max-Δ composition, though they do not satisfy the syllogism under the max-min composition [7].

2. FUZZY INFERENCE METHODS

We shall first consider the following form of inference in which a fuzzy conditional proposition is contained.

Ant 1:	If x is A then y is B	
Ant 2:	x is A'	(1)
Cons:	y is B'	

where x and y are the names of objects, and A, A', B and B' are fuzzy concepts represented by fuzzy sets in universes of discourse U, U, V and V, respectively. This form of inference may be viewed as a fuzzy modus ponens which reduces to the classical modus ponens when A' = A and B' = B.

Moreover, the following form of inference is also possible which also contains a fuzzy conditional proposition.

Ant 1:	If x is A then y is B	
Ant 2:	y is B'	(2)
Cons:	x is A'	

This inference can be considered as a fuzzy modus tollens which reduce to modus tollens when B' = not B and A' = not A.

The Ant 1 of the form "If x is A then y is B" in (1) and (2) may represent a certain relationship between A and B. From this point of view, several methods were proposed for the form of fuzzy conditional proposition "If x is A then y is B."

Let A and B be fuzzy sets in U and V, respectively, which are represented as

$$A = \int_U \mu_A(u)/u \; ; \quad B = \int_V \mu_B(v)/v$$

and let $\times$, $\cup$, $\cap$, $\neg$ and $\oplus$ be cartesian product, union, intersection, complement and bounded-sum for fuzzy sets, respectively. Then the following fuzzy relations in U x V can be derived from the proposition "If x is A then y is B." The fuzzy relation Rm and Ra were proposed by Zadeh [1], Rc by Mamdani [2], and the others are by Mizumoto et al. [3-7] by introducing the implications of many-valued logic systems [9-11].

$$Rm = (A \times B) \cup (\neg A \times V) \tag{3}$$
$$= \int_{U\times V} (\mu_A(u) \wedge \mu_B(v)) \vee (1-\mu_A(u)) \;/\; (u,v).$$

$$Ra = (\neg A \times V) \oplus (U \times B) \tag{4}$$
$$= \int_{U\times V} 1 \wedge (1-\mu_A(u)+\mu_B(v)) \;/\; (u,v).$$

$$Rc = A \times B \tag{5}$$
$$= \int_{U\times V} \mu_A(u) \wedge \mu_B(v) \;/\; (u,v).$$

$$Rs = A \times V \underset{s}{\Longrightarrow} U \times B \tag{6}$$
$$= \int_{U\times V} [\mu_A(u) \underset{s}{\to} \mu_B(v)] \;/\; (u,v),$$

where

$$\mu_A(u) \underset{s}{\to} \mu_B(v) = \begin{cases} 1 & \ldots \; \mu_A(u) \le \mu_B(v), \\ 0 & \ldots \; \mu_A(u) > \mu_B(v). \end{cases}$$

$$Rg = A \times V \underset{g}{\Longrightarrow} U \times B \tag{7}$$
$$= \int_{U\times V} [\mu_A(u) \underset{g}{\to} \mu_B(v)] \;/\; (u,v),$$

where

$$\mu_A(u) \underset{g}{\to} \mu_B(v) = \begin{cases} 1 & \ldots \; \mu_A(u) \le \mu_B(v), \\ \mu_B(v) & \ldots \; \mu_A(u) > \mu_B(v). \end{cases}$$

$$Rsg = (A \times V \underset{s}{\Longrightarrow} U \times B) \cap (\neg A \times V \underset{g}{\Longrightarrow} U \times \neg B) \tag{8}$$
$$= \int_{U\times V} [\mu_A(u) \underset{s}{\to} \mu_B(v)] \wedge [1-\mu_A(u) \underset{g}{\to} 1-\mu_B(v)] / (u,v).$$

$$Rgg = (A \times V \underset{g}{\Longrightarrow} U \times B) \cap (\neg A \times V \underset{g}{\Longrightarrow} U \times \neg B) \tag{9}$$
$$= \int_{U\times V} [\mu_A(u) \underset{g}{\to} \mu_B(v)] \wedge [1-\mu_A(u) \underset{g}{\to} 1-\mu_B(v)] / (u,v).$$

$$Rgs = (A \times V \underset{g}{\Longrightarrow} U \times B) \cap (\neg A \times V \underset{s}{\Longrightarrow} U \times \neg B) \tag{10}$$
$$= \int_{U\times V} [\mu_A(u) \underset{g}{\to} \mu_B(v)] \wedge [1-\mu_A(u) \underset{s}{\to} 1-\mu_B(v)] / (u,v).$$

$$Rss = (A \times V \underset{s}{\Longrightarrow} U \times B) \cap (\neg A \times V \underset{s}{\Longrightarrow} U \times \neg B) \tag{11}$$
$$= \int_{U\times V} [\mu_A(u) \underset{s}{\to} \mu_B(v)] \wedge [1-\mu_A(u) \underset{s}{\to} 1-\mu_B(v)] / (u,v).$$

$$Rb = (\neg A \times V) \cup (U \times B) \tag{12}$$
$$= \int_{U\times V} (1-\mu_A(u)) \vee \mu_B(v) \;/\; (u,v).$$

$$R_\Delta = A \times V \underset{\Delta}{\Longrightarrow} U \times B \tag{13}$$
$$= \int_{U\times V} [\mu_A(u) \underset{\Delta}{\to} \mu_B(v)] \;/\; (u,v),$$

where

$$\mu_A(u) \underset{\Delta}{\to} \mu_B(v) = \begin{cases} 1 & \ldots \; \mu_A(u) \le \mu_B(v), \\ \dfrac{\mu_B(v)}{\mu_A(u)} & \ldots \; \mu_A(u) > \mu_B(v). \end{cases}$$

$$R_\blacktriangle = A \times V \underset{\blacktriangle}{\Longrightarrow} U \times B \tag{14}$$
$$= \int_{U\times V} [\mu_A(u) \underset{\blacktriangle}{\to} \mu_B(v)] \;/\; (u,v),$$

$$\mu_A(u) \underset{\blacktriangle}{\to} \mu_B(v)$$
$$= [\mu_A(u) \underset{\Delta}{\to} \mu_B(v)] \wedge [1-\mu_B(v) \underset{\Delta}{\to} 1-\mu_A(u)]$$
$$= \begin{cases} 1 \wedge \dfrac{\mu_B(v)}{\mu_A(u)} \wedge \dfrac{1-\mu_A(u)}{1-\mu_B(v)} & \ldots \; \mu_A(u)>0,\ 1-\mu_B(v)>0, \\ 1 & \ldots \; \mu_A(u)=0 \text{ or } 1-\mu_B(v)=0. \end{cases}$$

$$R_* = A \times V \underset{*}{\Longrightarrow} U \times B \tag{15}$$
$$= \int_{U\times V} [\mu_A(u) \underset{*}{\to} \mu_B(v)] \;/\; (u,v),$$
$$\mu_A(u) \underset{*}{\to} \mu_B(v) = 1 - \mu_A(u) + \mu_A(u)\mu_B(v).$$

$$R_\# = A \times V \underset{\#}{\Longrightarrow} U \times B \tag{16}$$
$$= \int_{U\times V} [\mu_A(u) \underset{\#}{\to} \mu_B(v)] \;/\; (u,v),$$
$$\mu_A(u) \underset{\#}{\to} \mu_B(v)$$
$$= (\mu_A(u) \wedge \mu_B(v)) \vee (1-\mu_A(u) \wedge 1-\mu_B(v)) \vee (\mu_B(v) \wedge 1-\mu_A(u))$$
$$= (1-\mu_A(u) \vee \mu_B(v)) \wedge (\mu_A(u) \vee 1-\mu_A(u)) \wedge (\mu_B(v) \vee 1-\mu_B(v)).$$

$$R_\square = A \times V \underset{\square}{\Longrightarrow} U \times B \tag{17}$$
$$= \int_{U\times V} [\mu_A(u) \underset{\square}{\to} \mu_B(v)] \;/\; (u,v),$$
$$\mu_A(u) \underset{\square}{\to} \mu_B(v) = \begin{cases} 1 & \ldots \; \mu_A(u)<1 \text{ or } \mu_B(v)=1, \\ 0 & \ldots \; \mu_A(u)=1,\ \mu_B(v)<1. \end{cases}$$

In order to define a new composition "max-$\dot{\wedge}$ composition" which will be used in the compositional rule of inference, we shall introduce a new binary operation $\dot{\wedge}$ called "drastic product" which is the operation Tw(x,y) by Dubois [8].

For $x,\ y \in [0,1]$,

$$x \,\dot{\wedge}\, y = Tw(x,y) = \begin{cases} x & \ldots \; y = 1, \\ y & \ldots \; x = 1, \\ 0 & \ldots \; x,\ y < 1. \end{cases} \tag{18}$$

Using this operation, we can easily define a new composition of "max-$\dot{\wedge}$ composition" of a fuzzy set A in U and a fuzzy relation R in UxV.

Max-$\vartriangle$ Composition:

$$\mu_{A\blacktriangle R}(v) = \bigvee_u \{\mu_A(u) \vartriangle \mu_R(u,v)\}. \tag{19}$$

From the definition of max-$\vartriangle$ composition "$\blacktriangle$" we can have the following properties which will be useful to discuss the fuzzy conditional inference. The more detailed properties of max-$\vartriangle$ composition $\blacktriangle$ and drastic product $\vartriangle$ are found in [8,12].

Let A, A_1 and A_2 be fuzzy sets in U, and R, R_1 and R_2 be fuzzy relations in U x V, then

$$A \blacktriangle (R_1 \cup R_2) = (A \blacktriangle R_1) \cup (A \blacktriangle R_2), \tag{20}$$

$$(A_1 \cup A_2) \blacktriangle R = (A_1 \blacktriangle R) \cup (A_2 \blacktriangle R), \tag{21}$$

$$A \blacktriangle (R_1 \cap R_2) \subseteq (A \blacktriangle R_1) \cap (A \blacktriangle R_2), \tag{22}$$

$$(A_1 \cap A_2) \blacktriangle R \subseteq (A_1 \blacktriangle R) \cap (A_2 \blacktriangle R). \tag{23}$$

Now, we shall begin with the fuzzy modus ponens in (1). Using the max-$\vartriangle$ composition (19), we can deduce the consequence B' of Cons in (1) from Ant 1 and Ant 2 by taking the max-$\vartriangle$ composition "$\blacktriangle$" of the fuzzy set A' and the fuzzy relation given in (3)-(17). For example, we can have for the method Rm of (3)

$$\begin{aligned} Bm' &= A' \blacktriangle Rm \\ &= A' \blacktriangle [(A \times B) \cup (\neg A \times V)]. \end{aligned} \tag{24}$$

The membership function of the fuzzy set Bm' in V is given as

$$\begin{aligned} \mu_{Bm'}(v) &= \bigvee_u \{\mu_{A'}(u) \vartriangle \mu_{Rm}(u,v)\} \\ &= \bigvee_u \{\mu_{A'}(u) \vartriangle [(\mu_A(u) \wedge \mu_B(v)) \vee (1-\mu_A(u))]\}. \end{aligned} \tag{25}$$

In the same way, we have

$$Ba' = A' \blacktriangle Ra = A' \blacktriangle [(\neg A \times V) \oplus (U \times B)]. \tag{26}$$

$$Bc' = A' \blacktriangle Rc = A' \blacktriangle (A \times B). \tag{27}$$

$$Bs' = A' \blacktriangle Rs = A' \blacktriangle [A \times V \underset{s}{\Longrightarrow} U \times B]. \tag{28}$$

$$\vdots$$

Similarly, in the fuzzy modus tollens of (2), the consequence A' in Cons can be deduced using the composition "$\blacktriangle$" of the fuzzy relation and the fuzzy set B'. Namely,

$$\begin{aligned} Am' &= Rm \blacktriangle B' \\ &= [(A \times B) \cup (\neg A \times V)] \blacktriangle B' \\ &= \int_U \bigvee_v \{[(\mu_A(u) \wedge \mu_B(v)) \vee (1-\mu_A(u))] \vartriangle \mu_{B'}(v)\}/u. \end{aligned} \tag{29}$$

$$Aa' = Ra \blacktriangle B' = [(\neg A \times V) \oplus (U \times B)] \blacktriangle B'. \tag{30}$$

$$Ac' = Rc \blacktriangle B' = (A \times B) \blacktriangle B'. \tag{31}$$

$$As' = Rs \blacktriangle B' = [A \times V \underset{s}{\Longrightarrow} U \times B] \blacktriangle B'. \tag{32}$$

$$\vdots$$

3. <u>COMPARISON OF FUZZY INFERENCE METHODS UNDER MAX-$\vartriangle$ COMPOSITION</u>

In this section we shall make comparison of the fuzzy inference methods obtained above by applying 15 fuzzy relations (3)-(17) to the fuzzy modus ponens (1) and the fuzzy modus tollens (2).

In the fuzzy modus ponens, we shall show what the consequences Bm', Ba', Bc', ... will be when using the max-$\vartriangle$ composition of the fuzzy set A' and the fuzzy relation, where the fuzzy set A' is

$$A' = A = \int_U \mu_A(u)/u,$$

$$A' = \underline{\text{very}}\ A = A^2 = \int_U \mu_A(u)^2/u,$$

$$A' = \underline{\text{more or less}}\ A = A^{0.5} = \int_U \mu_A(u)^{0.5}/u,$$

$$A' = \underline{\text{not}}\ A = \neg A = \int_U 1-\mu_A(u)/u,$$

which are typical examples of A'.

Similarly, in the fuzzy modus tollens we shall show the consequences Am', Aa', Ac', ... under the max-$\vartriangle$ composition (as in (29)-(32)) of the fuzzy relation and the fuzzy set B', where B' is

$$B' = \underline{\text{not}}\ B = \neg B = \int_V 1-\mu_B(v)/v,$$

$$B' = \underline{\text{not very}}\ B = \neg B^2 = \int_V 1-\mu_B(v)^2/v,$$

$$\begin{aligned} B' &= \underline{\text{not more or less}}\ B = \neg B^{0.5} \\ &= \int_V 1-\mu_B(v)^{0.5}/v, \end{aligned}$$

$$B' = B = \int_V \mu_B(v)/v.$$

We shall begin with the fuzzy modus ponens in (1). We assume in the discussion of fuzzy modus ponens that $\mu_A(u)$ takes all values in [0,1] according to u varying all over U, that is, μ_A is a function onto [0,1]. Clearly, from this assumption, the fuzzy set A is a normal fuzzy set.

We shall first discuss Rm and obtain Bm' of (24). From the above assumption, the expression (25) can be rewritten as

$$b_{m'} = \bigvee_x \{x' \vartriangle [(x \wedge b) \vee (1-x)]\}, \tag{33}$$

and $f(x) = x' \vartriangle [(x \wedge b) \vee (1-x)]$, (34)

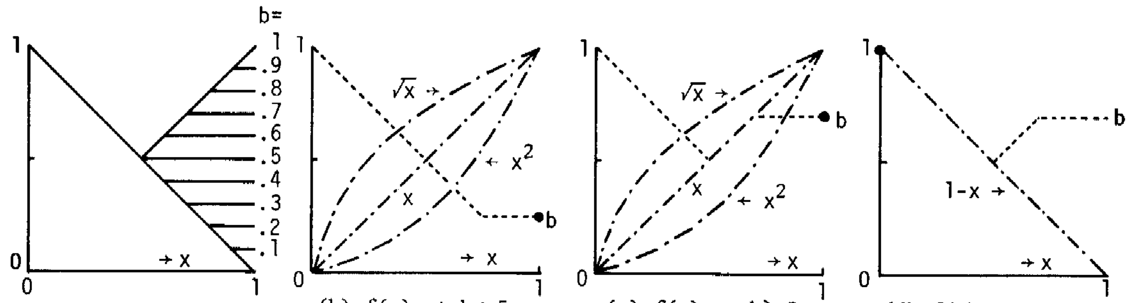
by letting

$$\mu_A(u) = x,\ \mu_{A'}(u) = x',\ \mu_B(v) = b,\ \mu_{Bm'}(v) = b_{m'}. \tag{35}$$

The expression $(x \wedge b) \vee (1-x)$ in (33) can be shown in Fig.1(a) by using a parameter b.

When A' is A, x' becomes x. Thus, f(x) of (34) is

$$f(x) = x \vartriangle [(x \wedge b) \vee (1-x)].$$



(a) $(x \wedge b) \vee (1-x)$ (b) $f(x)$ at $b \leq .5$ (black circle) (c) $f(x)$ at $b \geq .5$ (black circle) (d) $f(x)$ at $x' = 1-x$ (black circle)

Fig.1 $f(x) = x' \mathbin{\Delta} [(x \wedge b) \vee (1-x)]$ of (34) at $x' = x$, x^2, $\sqrt{x}$ and $1-x$

For example, at $b \leq 0.5$, $f(x)$ is shown by the black circle in Fig.1(b), i.e.,

$$f(x) = \begin{cases} b & \ldots \quad x = 1, \\ 0 & \ldots \quad x < 1. \end{cases}$$

Hence,

$$b_{m'} = \bigvee_x f(x) = b.$$

In the same way, at $b \geq 0.5$, $f(x)$ is shown by the black circle in Fig.1(c). Thus,

$$b_{m'} = \bigvee_x f(x) = b.$$

In general, we can obtain $b_{m'} = b$ for any b. In other words, we have $b_{m'} = b$ when $x' = x$, which leads to $Bm' = B$ at $A' = A$. Hence, from (24) the following is obtained. This shows that the so-called modus ponens is satisfied under the max-Δ composition "▲". Note that modus ponens does not hold for the max-min composition [4].

$$Bm' = A \blacktriangle Rm = B. \tag{36}$$

Similarly, at A' = very A (i.e., $x' = x^2$) and A' = more or less A (i.e., $x' = \sqrt{x}$), we can have $b_{m'} = b$ for any b from Fig.1(b) and (c). Therefore,

$$Bm' = \underline{\text{very}}\ A \blacktriangle Rm = B, \tag{37}$$

$$Bm' = \underline{\text{more or less}}\ A \blacktriangle Rm = B. \tag{38}$$

Finally, when A' = not A (i.e., $x' = 1-x$), $f(x)$ of (34) is

$$f(x) = (1-x) \mathbin{\Delta} [(x \wedge b) \vee (1-x)]$$

and given from Fig.1(d) by

$$f(x) = \begin{cases} 1 & \ldots \quad x = 0, \\ 0 & \ldots \quad x > 0, \end{cases}$$

for any b. Thus,

$$b_{m'} = \bigvee_x f(x) = 1,$$

which leads to $b_{m'} = 1$ at $x' = 1-x$, that is, Bm' = unknown at A' = not A. Therefore,

$$\underline{\text{not}}\ A \blacktriangle Rm = \underline{\text{unknown}}. \tag{39}$$

We can obtain the consequences Ba', Bc', Bs', ..., $B_\square'$ in the same way as Bm', and thus we omit the ways of how to obtain them. Table I summarizes the consequences inferred by all the fuzzy inference methods (3)-(17).

We shall next discuss the fuzzy modus tollens in (2). In the case of fuzzy modus tollens, we shall assume that μ_B is a function onto [0,1]. We shall investigate only the case of R_Δ of (13) because of the limitation of space.

The consequence A_Δ' is obtained by

$$A_\Delta' = R_\Delta \blacktriangle B',$$

$$\mu_{A_\Delta'}(u) = \bigvee_v \{[\mu_A(u) \underset{\Delta}{\rightarrow} \mu_B(v)] \mathbin{\Delta} \mu_{B'}(v)\}. \tag{40}$$

From the assumption, this expression can be rewritten as

$$a_\Delta' = \bigvee_x [a \underset{\Delta}{\rightarrow} x] \mathbin{\Delta} x', \tag{41}$$

$$g(x) = [a \underset{\Delta}{\rightarrow} x] \mathbin{\Delta} x', \tag{42}$$

where

$$x = \mu_B(v), \quad x' = \mu_{B'}(v), \quad a = \mu_A(u),$$

$$a_\Delta' = \mu_{A_\Delta'}(u),$$

$$a \underset{\Delta}{\rightarrow} x = \begin{cases} 1 & \ldots \quad a \leq x, \\ \frac{x}{a} & \ldots \quad a > x. \end{cases} \tag{43}$$

The expression (43) with parameter a is depicted in Fig.2(a).

When B' is not B, x' becomes 1-x. Thus, $g(x)$ of (42) becomes

$$g(x) = [a \underset{\Delta}{\rightarrow} x] \mathbin{\Delta} (1-x)$$

and is obtained from Fig.2(b) as

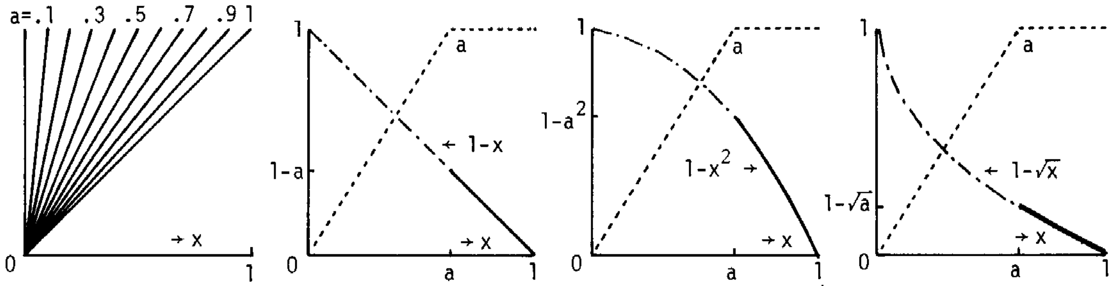
$$g(x) = \begin{cases} 1-x & \ldots \quad a \leq x \leq 1, \\ 0 & \ldots \quad \text{otherwise}. \end{cases}$$

Table I Inference results by each method
(Case of fuzzy modus ponens)

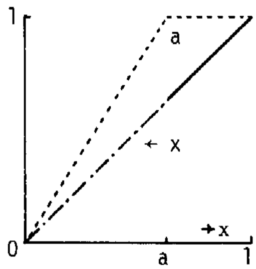
	A	very A	more or less A	not A
Rm	B	B	B	unknown
Ra	B	B	more or less B	unknown
Rc	B	B	B	Ø
Rs	B	very B	more or less B	unknown
Rg	B	B	more or less B	unknown
Rsg	B	very B	more or less B	not B
Rgg	B	B	more or less B	not B
Rgs	B	B	more or less B	not B
Rss	B	very B	more or less B	not B
Rb	B	B	B	unknown
R_Δ	B	B	more or less B	unknown
$R_\blacktriangle$	B	very B	more or less B	unknown
R_*	B	B	B	unknown
$R_\#$	B	B	B	B ∪ not B
$R_\Box$	unknown	unknown	unknown	unknown

Table II Inference results by each method
(Case of fuzzy modus tollens)

	not B	not very B	not more or less B	B
Rm	not A	not A	not A	A ∪ not A
Ra	not A	not very A	not A	unknown
Rc	Ø	Ø	Ø	A
Rs	not A	not very A	not more or less A	unknown
Rg	not A	not very A	not more or less A	unknown
Rsg	not A	not very A	not more or less A	A
Rgg	not A	not very A	not more or less A	A
Rgs	not A	not very A	not more or less A	A
Rss	not A	not very A	not more or less A	A
Rb	not A	not A	not A	unknown
R_Δ	not A	not very A	not more or less A	unknown
$R_\blacktriangle$	not A	not very A	not more or less A	unknown
R_*	not A	not A	not A	unknown
$R_\#$	not A	not A	not A	A ∪ not A
$R_\Box$	$\{1..\mu_A<1;\ 0..\mu_A=1$	$\{1..\mu_A<1;\ 0..\mu_A=1$	$\{1\ \dots\ \mu_A<1;\ 0\ \dots\ \mu_A=1$	unknown



(a) $a \underset{\Delta}{\rightarrow} x$ (b) g(x) at x'=1-x (c) g(x) at x'=1-x² (d) g(x) at x'=1-√x



(e) g(x) at x'=x

Fig.2 $g(x) = [a \underset{\Delta}{\rightarrow} x] \wedge x'$ of (42) (solid line)

Hence,

$$a_\Delta' = \bigvee_x g(x) = \bigvee_{x\in[a,1]} 1-x = 1-a$$

for any a. Therefore, $a_\Delta' = 1-a$ at $x' = 1-x$, that is, A_Δ' = not A at B' = not B. Stated alternatively,

$$A_\Delta' = R_\Delta \blacktriangle \text{ not } B = \text{not } A. \qquad (44)$$

When B' = not very B (i.e., $x' = 1 - x^2$), g(x) of (42) will be

$$g(x) = \begin{cases} 1 - x^2 & \dots\ a \leq x \leq 1, \\ 0 & \dots\ \text{otherwise}, \end{cases}$$

from Fig.2(c). Thus,

$$a_\Delta' = \bigvee_x g(x) = \bigvee_{x\in[a,1]} 1 - x^2 = 1 - a^2.$$

Therefore, we have $a_\Delta' = 1 - a^2$ at $x' = 1 - x^2$. Namely,

$$A_\Delta' = R_\Delta \blacktriangle \text{ not very } B = \text{not very } A. \qquad (45)$$

At B' = not more or less B ($x' = 1 - \sqrt{x}$), g(x) is given by

Table III Relations between Ant 2 and Cons under Ant 1 for the fuzzy modus ponens in (1)

	x is A'	y is B'
Relation I (modus ponens)	x is A	y is B
Relation II-1	x is very A	y is very B
Relation II-2	x is very A	y is B
Relation III-1	x is more or less A	y is more or less B
Relation III-2	x is more or less A	y is B
Relation IV-1	x is not A	y is unknown
Relation IV-2	x is not A	y is not B

Table IV Relations between Ant 2 and Cons under Ant 1 for the fuzzy modus tollens in (2)

	y is B'	x is A'
Relation V (modus tollens)	y is not B	x is not A
Relation VI-1	y is not very B	x is not very A
Relation VI-2	y is not very B	x is not A
Relation VII-1	y is not more or less B	x is not more or less A
Relation VII-2	y is not more or less B	x is not A
Relation VIII-1	y is B	x is unknown
Relation VIII-2	y is B	x is A

$$g(x) = \begin{cases} 1 - \sqrt{x} & \ldots\ a \leq x \leq 1, \\ 0 & \ldots\ \text{otherwise}, \end{cases}$$

and

$$a_\Delta' = \bigvee_{x \in [a,1]} 1 - \sqrt{x} = 1 - \sqrt{a}.$$

Therefore,

$$A_\Delta' = R_\Delta \blacktriangle \underline{\text{not more or less}}\ B$$
$$= \underline{\text{not more or less}}\ A. \qquad (46)$$

Finally, at $B' = B$ $(x' = x)$

$$g(x) = \begin{cases} x & \ldots\ a \leq x \leq 1, \\ 0 & \ldots\ \text{otherwise}. \end{cases}$$

$$a_\Delta' = \bigvee_{x \in [a,1]} x = 1.$$

Therefore,

$$A_\Delta' = R_\Delta \blacktriangle B = \underline{\text{unknown}}. \qquad (47)$$

We can obtain the consequences Am', Aa', Ac', ..., $A_\square'$ in the same way as R_Δ'. In Table II the inference results by all the methods are listed.

In the forms of fuzzy conditional inferences (1) and (2), it seems according to our intuition that the relations between A' in Ant 2 and B' in Cons of the fuzzy modus ponens (1) ought to be satisfied as shown in Table III (cf. [4,5]). Similarly, the relations between B' in Ant 2 and A' in Cons of the fuzzy modus tollens (2) ought to be satisfied as in Table IV.

In Table V, the satisfaction (O) or failure (X) of each relation in Tables III and IV under each fuzzy inference method is indicated by using the consequence results of Tables I and II. In order to compare the inference results under the max-Δ composition and the max-min composition, the inference results under the max-min composition is listed in Table VI [7].

From Tables I, II and V it follows that all the inference methods except $R_\square$ can satisfy the so-called modus ponens under the max-Δ composition, but only the methods Rc, Rs, ..., Rss can satisfy the modus ponens under the max-min composition. The same holds for modus tollens. Moreover, it is found that almost all the methods can infer quite reasonable consequences under the max-Δ composition, though we can not always get reasonable consequences under the max-min composition as shown in Table VI.

4. SYLLOGISM BY EACH METHOD UNDER MAX-Δ COMPOSITION

In this section we shall investigate a syllogism by each fuzzy inference method under the max-Δ composition.

Let P_1, P_2 and P_3 be fuzzy conditional propositions such as

P_1: If x is A then y is B
P_2: If y is B then z is C
P_3: If x is A then z is C

where A, B and C are fuzzy sets in U, V and W, respectively. If the proposition P_3 is deduced from the propositions P_1 and P_2, that is, the following holds:

P_1: If x is A then y is B
P_2: If y is B then z is C
―――――――――――――――――
P_3: If x is A then z is C

then it is said that a syllogism holds.

Let R(A,B), R(B,C) and R(A,C) be fuzzy relations in UxV, VxW and UxW, respectively, which are obtained from the propositions P_1, P_2 and P_3, respectively. If the following equality holds, the syllogism holds under the max-Δ composition.

$$R(A,B) \blacktriangle R(B,C) = R(A,C). \qquad (48)$$

That is to say,

P_1: If x is A then y is B ⟶ R(A,B)
P_2: If y is B then z is C ⟶ R(B,C) (49)
―――――――――――――――――
P_3: If x is A then z is C ⟵ R(A,B)▲R(B,C)

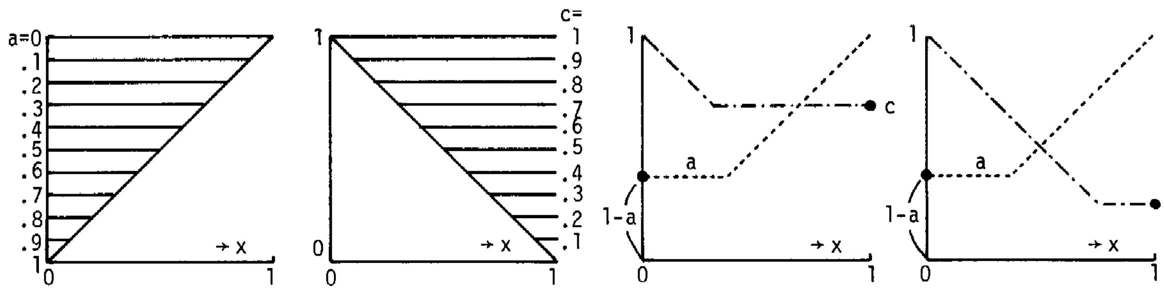
where "▲" is the max-Δ composition of R(A,B) and R(B,C), and the membership function of R(A,B) ▲ R(B,C) is given by

Table V Satisfaction of each relation in Tables III and IV under each method (The case of max-$\wedge$ composition)

	Ant 2	Cons	Rm	Ra	Rc	Rs	Rg	Rsg	Rgg	Rgs	Rss	Rb	$R_\triangle$	$R_\blacktriangle$	R_*	$R_\#$	$R_\square$
Relation I (modus ponens)	A	B	0	0	0	0	0	0	0	0	0	0	0	0	0	0	X
Relation II-1	very A	very B	X	X	X	0	X	0	X	X	0	X	X	0	X	X	X
Relation II-2	very A	B	0	0	0	X	0	X	0	0	X	0	0	X	0	0	X
Relation III-1	more or less A	more or less B	X	0	X	0	0	0	0	0	0	X	0	0	X	X	X
Relation III-2	more or less A	B	0	X	0	X	X	X	X	X	X	0	X	X	0	0	X
Relation IV-1	not A	unknown	0	0	X	0	0	X	X	X	X	0	0	0	0	X	0
Relation IV-2	not A	not B	X	X	X	X	X	0	0	0	0	X	X	X	X	X	X
Relation V (modus tollens)	not B	not A	0	0	X	0	0	0	0	0	0	0	0	0	0	0	X
Relation VI-1	not very B	not very A	X	0	X	0	0	0	0	0	0	X	0	0	X	X	X
Relation VI-2	not very B	not A	0	X	X	X	X	X	X	X	X	0	X	X	0	0	X
Relation VII-1	not more or less B	not more or less A	X	X	X	0	0	0	0	0	0	X	0	0	X	X	X
Relation VII-2	not more or less B	not A	0	0	X	X	X	X	X	X	X	0	X	X	0	0	X
Relation VIII-1	B	unknown	X	0	X	0	0	X	X	X	X	0	0	0	0	X	0
Relation VIII-2	B	A	X	X	0	X	X	0	0	0	0	X	X	X	X	X	X

Table VI Satisfaction of each relation in Tables III and IV under each method (The case of max-min composition) (cf. [7])

	Ant 2	Cons	Rm	Ra	Rc	Rs	Rg	Rsg	Rgg	Rgs	Rss	Rb	$R_\triangle$	$R_\blacktriangle$	R_*	$R_\#$	$R_\square$
Relation I (modus ponens)	A	B	X	X	0	0	0	0	0	0	0	X	X	X	X	X	X
Relation II-1	very A	very B	X	X	X	0	X	0	X	X	0	X	X	X	X	X	X
Relation II-2	very A	B	X	X	0	X	0	X	0	0	X	X	X	X	X	X	X
Relation III-1	more or less A	more or less B	X	X	X	0	0	0	0	0	0	X	X	X	X	X	X
Relation III-2	more or less A	B	X	X	0	X	X	X	X	X	X	X	X	X	X	X	X
Relation IV-1	not A	unknown	0	0	X	0	0	X	X	X	X	0	0	0	0	X	0
Relation IV-2	not A	not B	X	X	X	X	X	0	0	0	0	X	X	X	X	X	X
Relation V (modus tollens)	not B	not A	X	X	X	0	X	0	X	X	0	X	X	X	X	X	X
Relation VI-1	not very B	not very A	X	X	X	0	X	0	X	X	0	X	X	X	X	X	X
Relation VI-2	not very B	not A	X	X	X	X	X	X	X	X	X	X	X	X	X	X	X
Relation VII-1	not more or less B	not more or less A	X	X	X	0	X	0	X	X	0	X	X	X	X	X	X
Relation VII-2	not more or less B	not A	X	X	X	X	X	X	X	X	X	X	X	X	X	X	X
Relation VIII-1	B	unknown	X	0	X	0	0	X	X	X	X	0	0	0	0	X	0
Relation VIII-2	B	A	X	X	0	X	X	X	X	0	0	X	X	X	X	X	X



(a) $(1-a) \vee x$ (b) $(1-x) \vee c$ (c) $h(x)$ at $1-a \leq c$ (d) $h(x)$ at $1-a \geq c$

Fig.3 $h(x) = [(1-a) \vee x] \dot{\wedge} [(1-x) \vee c]$ of (52) (black circles)

$$\mu_{R(A,B)\blacktriangle R(B,C)}(u,w)$$

$$= \bigvee_{v} \{\mu_{R(A,B)}(u,v) \dot{\wedge} \mu_{R(B,C)}(v,w)\}. \quad (50)$$

Now we shall obtain $R(A,B) \blacktriangle R(B,C)$ under each fuzzy inference method and show whether the syllogism holds or not. In the discussion of the syllogism it is assumed that the membership function μ_B of the fuzzy set B is a function onto [0,1].

We shall discuss only the case of Rb of (12). The fuzzy relations Rb(A,B) and Rb(B,C) are obtained from the propositions P_1 and P_2 by using (12).

$$Rb(A,B) = (\neg A \times V) \cup (U \times B),$$

$$Rb(B,C) = (\neg B \times W) \cup (V \times C).$$

Thus, the max-$\dot{\wedge}$ composition of Rb(A,B) and Rb(B,C) will be

$$Rb(A,B) \blacktriangle Rb(B,C)$$

$$= [(\neg A \times V) \cup (U \times B)] \blacktriangle [(\neg B \times W) \cup (V \times C)]$$

and its membership function is as follows.

$$\mu_{Rb(A,B)\blacktriangle Rb(B,C)}(u,w)$$

$$= \bigvee_{v}\{[(1-\mu_A(u)) \vee \mu_B(v)] \dot{\wedge} [(1-\mu_B(v)) \vee \mu_C(w)]\}.$$

Moreover, this expression can be rewritten as

$$d = \bigvee_{x} \{[(1-a) \vee x] \dot{\wedge} [(1-x) \vee c]\}, \quad (51)$$

$$h(x) = [(1-a) \vee x] \dot{\wedge} [(1-x) \vee c] \quad (52)$$

under the above assumption that μ_B is a function onto [0,1], where

$$d = \mu_{Rb(A,B)\blacktriangle Rb(B,C)}(u,w), \quad a = \mu_A(u),$$

$$x = \mu_B(v), \quad c = \mu_C(w).$$

The expression $(1-a) \vee x$ of (52) can be depicted in Fig.3(a) by using parameter a, and the expression $(1-x) \vee c$ is shown by using parameter c as in Fig.3(b).

When $1-a \leq c$, the function h(x) of (52) is given by the black circles in Fig.3(c), i.e.,

$$h(x) = \begin{cases} 1-a & \ldots \quad x = 0, \\ c & \ldots \quad x = 1, \\ 0 & \ldots \quad \text{otherwise}. \end{cases}$$

Thus, d of (51) is obtained by

$$d = \bigvee_{x} h(x) = c \quad \ldots \text{ at } 1-a \leq c. \quad (53)$$

On the other hand, when $1-a \geq c$, h(x) is given by the black circles in Fig.3(d). Thus,

$$d = 1 - a \quad \ldots \text{ at } 1-a \geq c. \quad (54)$$

Therefore, from (53) and (54) we have

$$d = (1-a) \vee c$$

for any a and c, i.e.,

$$\mu_{Rb(A,B)\blacktriangle Rb(B,C)}(u,w)$$

$$= (1 - \mu_A(u)) \vee \mu_C(w), \quad (55)$$

which indicates

$$Rb(A,B) \blacktriangle Rb(B,C) = Rb(A,C). \quad (56)$$

Therefore, the syllogism holds for Rb under the max-$\dot{\wedge}$ composition "$\blacktriangle$". Note that the syllogism does not hold under the max-min composition [7].

In the same way, we can obtain $R(A,B) \blacktriangle R(B,C)$ by the other fuzzy inference methods and thus we shall list the results in the following.

$$Rm(A,B) \blacktriangle Rm(B,C)$$

$$= \int_{U\times W} f(\mu_A(u),\mu_C(w)) \,/\, (u,w) \quad (57)$$

$$\neq Rm(A,C) \quad (= \int_{U\times W} (\mu_A(u) \wedge \mu_C(w)) \vee (1-\mu_A(u)) / (u,w))$$

where

$$f(\mu_A(u),\mu_C(w)) = \begin{cases} \mu_C(w) & \ldots \quad \mu_A(u)=1, \\ \mu_A(u) \vee (1-\mu_A(u)) & \ldots \quad \mu_C(w)=1, \\ 0 & \ldots \quad \text{otherwise}. \end{cases}$$

Table VII Satisfaction of syllogism under the max-▲ composition and max-min composition

	Rm	Ra	Rc	Rs	Rg	Rsg	Rgg	Rgs	Rss	Rb	$R_{\triangle}$	$R_{\blacktriangle}$	R_*	$R_\#$	$R_{\square}$
Max-▲ composition	X	O	X	O	O	O	O	O	O	O	O	O	X	X	O
Max-min composition	X	X	O	O	O	O	O	O	O	X	X	X	X	X	O

$$Ra(A,B) \blacktriangle Ra(B,C) = \int_{U\times W} 1 \wedge (1 - \mu_A(u) + \mu_C(w)) \,/\, (u,w) = Ra(A,C). \quad (58)$$

$$Rc(A,B) \blacktriangle Rc(B,C) = \int_{U\times W} f(\mu_A(u),\mu_C(w)) \,/\, (u,w) \neq Rc(A,C) \quad \left(= \int_{U\times W} \mu_A(u) \wedge \mu_C(w) \,/\, (u,w)\right), \quad (59)$$

where

$$f(\mu_A(u),\mu_C(w)) = \begin{cases} \mu_C(w) & \ldots \ \mu_A(u) = 1, \\ \mu_A(u) & \ldots \ \mu_C(w) = 1, \\ 0 & \ldots \ \text{otherwise.} \end{cases}$$

$$Rs(A,B) \blacktriangle Rs(B,C) = \int_{U\times W} \mu_A(u) \underset{s}{\rightarrow} \mu_C(w) \,/\, (u,w) = Rs(A,C). \quad (60)$$

$$Rg(A,B) \blacktriangle Rg(B,C) = \int_{U\times W} \mu_A(u) \underset{g}{\rightarrow} \mu_C(w) \,/\, (u,w) = Rg(A,C). \quad (61)$$

$$Rsg(A,B) \blacktriangle Rsg(B,C) = \int_{U\times W} [\mu_A(u) \underset{s}{\rightarrow} \mu_C(w)] \wedge [1-\mu_A(u) \underset{g}{\rightarrow} 1-\mu_C(w)]/(u,w) = Rsg(A,C). \quad (62)$$

$$Rgg(A,B) \blacktriangle Rgg(B,C) = \int_{U\times W} [\mu_A(u) \underset{g}{\rightarrow} \mu_C(w)] \wedge [1-\mu_A(u) \underset{g}{\rightarrow} 1-\mu_C(w)]/(u,w) = Rgg(A,C). \quad (63)$$

$$Rgs(A,B) \blacktriangle Rgs(B,C) = \int_{U\times W} [\mu_A(u) \underset{g}{\rightarrow} \mu_C(w)] \wedge [1-\mu_A(u) \underset{s}{\rightarrow} 1-\mu_C(w)]/(u,w) = Rgs(A,C). \quad (64)$$

$$Rss(A,B) \blacktriangle Rss(B,C) = \int_{U\times W} [\mu_A(u) \underset{s}{\rightarrow} \mu_C(w)] \wedge [1-\mu_A(u) \underset{s}{\rightarrow} 1-\mu_C(w)]/(u,w) = Rss(A,C). \quad (65)$$

$$Rb(A,B) \blacktriangle Rb(B,C) = \int_{U\times W} (1 - \mu_A(u)) \vee \mu_C(w) \,/\, (u,w) = Rb(A,C). \quad (66)$$

$$R_{\triangle}(A,B) \blacktriangle R_{\triangle}(B,C) = \int_{U\times W} \mu_A(u) \underset{\triangle}{\rightarrow} \mu_C(w) \,/\, (u,w) = R_{\triangle}(A,C). \quad (67)$$

$$R_{\blacktriangle}(A,B) \blacktriangle R_{\blacktriangle}(B,C) = \int_{U\times W} \mu_A(u) \underset{\blacktriangle}{\rightarrow} \mu_C(w) \,/\, (u,w) = R_{\blacktriangle}(A,C). \quad (68)$$

$$R_*(A,B) \blacktriangle R_*(B,C) = \int_{U\times W} (1 - \mu_A(u)) \vee \mu_C(w) \,/\, (u,w) \quad (69)$$

$$\neq R_*(A,C) \quad \left(= \int_{U\times W} 1-\mu_A(u)+\mu_A(u)\mu_C(w)/(u,w)\right).$$

$$R_\#(A,B) \blacktriangle R_\#(B,C) = \int_{U\times W} f(\mu_A(u),\mu_C(w)) \,/\, (u,w)$$

$$\neq R_\#(A,C) \quad (70)$$

$$\left(= \int_{U\times W} (1-\mu_A(u) \vee \mu_C(w)) \wedge (\mu_A(u) \vee 1-\mu_A(u)) \wedge (1-\mu_C(w) \vee \mu_C(w)) \,/\, (u,w)\right),$$

where

$$f(\mu_A(u),\mu_C(w)) = \begin{cases} \mu_C(w) \vee (1-\mu_C(w)) & \ldots \ \mu_A(u) = 0, \\ \mu_C(w) & \ldots \ \mu_A(u) = 1, \\ 1 - \mu_A(u) & \ldots \ \mu_C(w) = 0, \\ \mu_A(u) \vee (1-\mu_A(u)) & \ldots \ \mu_C(w) = 1, \\ 0 & \ldots \ \text{otherwise.} \end{cases}$$

$$R_{\square}(A,B) \blacktriangle R_{\square}(B,C) = \int_{U\times W} \mu_A(u) \underset{\square}{\rightarrow} \mu_C(w) \,/\, (u,w) = R_{\square}(A,C). \quad (71)$$

Using these results, the satisfaction (O) or failure (X) of the syllogism by each method under the max-$\triangle$ composition is listed in Table VII. This table also contains the results under the max-min composition (cf. [7]).

It follows from Table VII that the methods Ra, Rb, $R_{\triangle}$ and $R_{\blacktriangle}$ can satisfy the syllogism under the max-$\triangle$ composition, though they do not satisfy it under the max-min composition. But the converse holds for Rc.

5. CONCLUSION

We have shown that, when the max-$\triangle$ composition is used in the compositional rule of inference, most of the fuzzy inference methods can get quite reasonable consequences which coincide with our intuition with respect to several criteria such as modus ponens, modus tollens and syllogism.

It will be of interest to apply the max-$\triangle$ composition to fuzzy inferences which are of the more complicated form such as

If x is A then y is B else y is C.
x is A'.
―――――――――――――――――
y is D.

If x is A_1 then y is B_1 else
If x is A_2 then y is B_2 else
⋮
If x is A_n then y is B_n.
x is A'.
―――――――――――――――――
y is B'.

These results will be presented in subsequent papers.

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