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Fuzzy Conditional Inference under Max-$\odot$ Composition

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ABSTRACT

This paper shows that the majority of fuzzy inference methods for a fuzzy conditional proposition "If x is A then y is B," with A and B fuzzy concepts, can infer very reasonable consequences which fit our intuition with respect to several criteria such as *modus ponens* and *modus tollens*, if a new composition called "max-$\odot$ composition" is used in the compositional rule of inference, though reasonable consequences cannot always be obtained when using the max-min composition, which is used usually in the compositional rule of inference. Furthermore, it is shown that a syllogism holds for the majority of the methods under the max-$\odot$ composition, though they do not always satisfy the syllogism under the max-min composition.

1. INTRODUCTION

In our daily life we often make inferences of the form

Ant 1:	If x is A then y is B.
Ant 2:	x is A'.
Cons:	y is B'.

where A, A', B, and B' are fuzzy concepts. In order to make such an inference with fuzzy concepts, Zadeh [1] suggested an inference rule called the "compositional rule of inference," which infers B' of Cons from Ant 1 and Ant 2 by taking the max-min composition of A' and the fuzzy relation which is translated from the fuzzy conditional proposition "If x is A then y is B." In this connection, he [1], Mamdani [2], and Mizumoto et al. [3–7] suggested several translating rules for translating the fuzzy proposition "If x is A then y is B" into a fuzzy relation.

In [4–6] we pointed out that the consequences inferred by Zadeh's and Mamdani's methods do not always fit our intuition, and proposed some new methods which can lead to consequences coinciding with our intuition with respect to several criteria, such as *modus ponens* and *modus tollens*. Moreover, we suggested in [7] new translating rules which are obtained by introducing

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implication rules of many valued logic systems, but these methods were found not to infer reasonable consequences.

In [8], however, we have shown that, although the translating rule called by Zadeh the "arithmetic rule" does not infer reasonable consequences in the compositional rule of inference which uses the max-min composition, the arithmetic rule can infer very reasonable consequences when a new composition named "max-$\odot$ composition" is used in the compositional rule of inference, where $\odot$ is the operation of "bounded product," which is dual to the "bounded sum" introduced by Zadeh [1].

As continuation of our study [8], this paper investigates the inference results of all the translating rules proposed until now under the max-$\odot$ composition, and shows that the majority of the translating rules can infer very reasonable consequences which fit our intuition. Moreover, it is shown that the majority of the translating rules satisfy a syllogism under the max-$\odot$ composition.

2. TRANSLATING RULES

We shall first consider the following form of inference in which a fuzzy conditional proposition is contained:

$$\begin{array}{ll} \text{Ant 1:} & \text{If } x \text{ is } A \text{ then } y \text{ is } B. \\ \text{Ant 2:} & x \text{ is } A'. \\ \hline \text{Cons:} & y \text{ is } B'. \end{array} \tag{1}$$

where x and y are the names of objects, and A, A', B, and B' are fuzzy concepts represented by fuzzy sets in universes of discourse U, U, V, and V, respectively. This form of inference may be viewed as *fuzzy modus ponens*, which reduces to the classical *modus ponens* when $A' = A$ and $B' = B$.

Moreover, the following form of inference is possible, which also contains a fuzzy conditional proposition:

$$\begin{array}{ll} \text{Ant 1:} & \text{If } x \text{ is } A \text{ then } y \text{ is } B. \\ \text{Ant 2:} & y \text{ is } B'. \\ \hline \text{Cons:} & x \text{ is } A'. \end{array} \tag{2}$$

This inference can be considered as *fuzzy modus tollens*, which reduces to the classical *modus tollens* when $B' = \textbf{not } B$ and $A' = \textbf{not } A$.

The fuzzy proposition "If x is A then y is B" in (1) and (2) may represent a certain relationship between A and B. From this point of view, a number of translating rules have been proposed for translating the fuzzy conditional proposition "If x is A then y is B" into a fuzzy relation in $U \times V$.

Let A and B be fuzzy sets in U and V, respectively, which are represented as

$$A=\int_U \mu_A(u)/u, \qquad B=\int_V \mu_B(v)/v,$$

and let $\times$, $\cup$, $\cap$, $\neg$ and $\oplus$ be the cartesian product, union, intersection, complement, and bounded sum for fuzzy sets, respectively. Then the following fuzzy relations in $U\times V$ are translations of the fuzzy conditional proposition "If x is A then y is B." Rm (maximin rule) and Ra (arithmetic rule) were proposed by Zadeh [1], Rc (min rule) by Mamdani [2], and the others were created by Mizumoto et al. [3–7] by introducing the implications of many valued logic systems [9–11].

$$\begin{aligned}\mathrm{Rm}&=(A\times B)\cup(\neg A\times V)\\&=\int_{U\times V}[\mu_A(u)\wedge\mu_B(v)]\vee[1-\mu_A(u)]/(u,v);\end{aligned}\tag{3}$$

$$\begin{aligned}\mathrm{Ra}&=(\neg A\times V)\oplus(U\times B)\\&=\int_{U\times V}1\wedge(1-\mu_A(u)+\mu_B(v))/(u,v);\end{aligned}\tag{4}$$

$$\begin{aligned}\mathrm{Rc}&=A\times B\\&=\int_{U\times V}\mu_A(u)\wedge\mu_B(v)/(u,v);\end{aligned}\tag{5}$$

$$\begin{aligned}\mathrm{Rs}&=A\times V\underset{s}{\Rightarrow}U\times B\\&=\int_{U\times V}\Big[\mu_A(u)\underset{s}{\rightarrow}\mu_B(v)\Big]\Big/(u,v),\end{aligned}\tag{6}$$

where

$$\mu_A(u)\underset{s}{\rightarrow}\mu_B(v)=\begin{cases}1, & \mu_A(u)\leqslant\mu_B(v),\\0, & \mu_A(u)>\mu_B(v);\end{cases}$$

$$\begin{aligned}\mathrm{Rg}&=A\times V\underset{g}{\Rightarrow}U\times B\\&=\int_{U\times V}\Big[\mu_A(u)\underset{g}{\rightarrow}\mu_B(v)\Big]\Big/(u,v),\end{aligned}\tag{7}$$

where

$$\mu_A(u) \underset{g}{\rightarrow} \mu_B(v) = \begin{cases} 1, & \mu_A(u) \leqslant \mu_B(v), \\ \mu_B(v), & \mu_A(u) > \mu_B(v); \end{cases}$$

$$\text{Rsg} = \left(A \times V \underset{s}{\Rightarrow} U \times B\right) \cap \left(\neg A \times V \underset{g}{\Rightarrow} U \times \neg B\right)$$

$$= \int_{U \times V} \left[\mu_A(u) \underset{s}{\rightarrow} \mu_B(v)\right] \wedge \left[1 - \mu_A(u) \underset{g}{\rightarrow} 1 - \mu_B(v)\right] \Big/ (u, v); \quad (8)$$

$$\text{Rgg} = \left(A \times V \underset{g}{\Rightarrow} U \times B\right) \cap \left(\neg A \times V \underset{g}{\Rightarrow} U \times \neg B\right)$$

$$= \int_{U \times V} \left[\mu_A(u) \underset{g}{\rightarrow} \mu_B(v)\right] \wedge \left[1 - \mu_A(u) \underset{g}{\rightarrow} 1 - \mu_B(v)\right] \Big/ (u, v); \quad (9)$$

$$\text{Rgs} = \left(A \times V \underset{g}{\Rightarrow} U \times B\right) \cap \left(\neg A \times V \underset{s}{\Rightarrow} U \times \neg B\right)$$

$$= \int_{U \times V} \left[\mu_A(u) \underset{g}{\rightarrow} \mu_B(v)\right] \wedge \left[1 - \mu_A(u) \underset{s}{\rightarrow} 1 - u_B(v)\right] \Big/ (u, v); \quad (10)$$

$$\text{Rss} = \left(A \times V \underset{s}{\Rightarrow} U \times B\right) \cap \left(\neg A \times V \underset{s}{\Rightarrow} U \times \neg B\right)$$

$$= \int_{U \times V} \left[\mu_A(u) \underset{s}{\rightarrow} \mu_B(v)\right] \wedge \left[1 - \mu_A(u) \underset{s}{\rightarrow} 1 - \mu_B(v)\right] \Big/ (u, v); \quad (11)$$

$$\text{Rb} = (\neg A \times V) \cup (U \times B)$$

$$= \int_{U \times V} [1 - \mu_A(u)] \vee \mu_B(v) / (u, v). \quad (12)$$

$$\text{R}_\triangle = A \times V \underset{\triangle}{\Rightarrow} U \times B$$

$$= \int_{U \times V} \left[\mu_A(u) \underset{\triangle}{\rightarrow} \mu_B(v)\right] \Big/ (u, v), \quad (13)$$

where

$$\mu_A(u) \underset{\triangle}{\rightarrow} \mu_B(v) = \begin{cases} 1, & \mu_A(u) \leqslant \mu_B(v), \\ \dfrac{\mu_B(v)}{\mu_A(u)}, & \mu_A(u) > \mu_B(v); \end{cases}$$

$$\text{R}_\blacktriangle = A \times V \underset{\blacktriangle}{\Rightarrow} U \times B$$

$$= \int_{U \times V} [\mu_A(u) \underset{\blacktriangle}{\rightarrow} \mu_B(v)] / (u, v), \quad (14)$$

where

$$\mu_A(u) \underset{\blacktriangle}{\rightarrow} \mu_B(v) = \left[\mu_A(u) \underset{\vartriangle}{\rightarrow} \mu_B(v)\right] \wedge \left[1-\mu_B(v) \underset{\vartriangle}{\rightarrow} 1-\mu_A(u)\right]$$

$$= \begin{cases} 1 \wedge \dfrac{\mu_B(v)}{\mu_A(u)} \wedge \dfrac{1-\mu_A(u)}{1-\mu_B(v)}, & \mu_A(u) > 0, \quad 1-\mu_B(v) > 0, \\ 1, & \mu_A(u) = 0 \text{ or } 1-\mu_B(v) = 0; \end{cases}$$

$$\mathrm{R}_* = A \times V \underset{*}{\Rightarrow} U \times B$$

$$= \int_{U \times V} \left[\mu_A(u) \underset{*}{\rightarrow} \mu_B(v)\right] / (u, v), \tag{15}$$

where

$$\mu_A(u) \underset{*}{\rightarrow} \mu_B(v) = 1 - \mu_A(u) + \mu_A(u)\mu_B(v);$$

$$\mathrm{R}_\# = A \times V \underset{\#}{\Rightarrow} U \times B$$

$$= \int_{U \times V} \left[\mu_A(u) \underset{\#}{\rightarrow} \mu_B(v)\right] \Big/ (u, v), \tag{16}$$

where

$$\mu_A(u) \underset{\#}{\rightarrow} \mu_B(v) = [\mu_A(u) \wedge \mu_B(v)] \vee [1-\mu_A(u) \wedge 1-\mu_B(v)]$$

$$\vee [\mu_B(v) \wedge 1-\mu_A(u)]$$

$$= [1-\mu_A(u) \vee \mu_B(v)] \wedge [\mu_A(u) \vee 1-\mu_A(u)]$$

$$\wedge [\mu_B(v) \vee 1-\mu_B(v)];$$

$$\mathrm{R}_\square = A \times V \underset{\square}{\Rightarrow} U \times B$$

$$= \int_{U \times V} \left[\mu_A(u) \underset{\square}{\rightarrow} \mu_B(v)\right] \Big/ (u, v), \tag{17}$$

where

$$\mu_A(u) \underset{\square}{\rightarrow} \mu_B(v) = \begin{cases} 1, & \mu_A(u) < 1 \text{ or } \mu_B(v) = 1, \\ 0, & \mu_A(u) = 1, \quad \mu_B(v) < 1. \end{cases}$$

We shall next review the properties of the "bounded product" $\odot$ in order to define a composition, called "max-$\odot$ composition," which is used in the compositional rule of inference.

The operation of *bounded product* $\odot$ is defined as follows: For any x, $y \in [0,1]$,

$$x \odot y = 0 \vee (x + y - 1). \tag{18}$$

This is a dual operation of the "bounded sum" $\oplus$ introduced by Zadeh [1]:

$$x \oplus y = 1 \wedge (x + y). \tag{19}$$

For the bounded product $\odot$, the following properties are obtained. The properties of the bounded sum $\oplus$ are omitted, since it is dual to $\odot$. More detailed properties of $\odot$ and $\oplus$ are found in [12–14].

$$x \leqslant y, \quad z \leqslant w \quad \Rightarrow \quad x \odot z \leqslant y \odot w,$$

$$x \odot x \leqslant x,$$

$$x \odot y = y \odot x,$$

$$x \odot (y \odot z) = (x \odot y) \odot z,$$

$$x \odot (y \oplus z) \neq (x \odot y) \oplus (x \odot z),$$

$$1 - (x \odot y) = (1 - x) \oplus (1 - y),$$

$$x \odot 1 = x, \qquad x \odot 0 = 0,$$

$$x \odot (1 - x) = 0.$$

Moreover, the following properties are also obtained by combining $\odot$ with $\vee$ and $\wedge$:

$$x \odot (y \vee z) = (x \odot y) \vee (x \odot z),$$

$$x \odot (y \wedge z) = (x \odot y) \wedge (x \odot z),$$

$$x \vee (y \odot z) \geqslant (x \vee y) \odot (x \vee z),$$

$$x \wedge (y \odot z) \geqslant (x \wedge y) \odot (x \wedge z).$$

Using the bounded product $\odot$, we can easily define the *max-$\odot$ composition* of a fuzzy set A in U and a fuzzy relation R in $U \times V$:

$$A \square R \Leftrightarrow \mu_{A \square R}(v) = \bigvee_{u} \{\mu_A(u) \odot \mu_R(u,v)\}. \tag{20}$$

From the definition of max-$\odot$ composition $\square$, we have the following properties, which may be useful in discussing the fuzzy conditional inference.

Let A, A_1, and A_2 be fuzzy sets in U, and R, R_1, and R_2 be fuzzy relations in $U \times V$. Then

$$A \square (R_1 \cup R_2) = (A \square R_1) \cup (A \square R_2),$$

$$(A_1 \cup A_2) \square R = (A_1 \square R) \cup (A_2 \square R),$$

$$A \square (R_1 \cap R_2) \subseteq (A \square R_1) \cap (A \square R_2),$$

$$(A_1 \cap A_2) \square R \subseteq (A_1 \square R) \cap (A_2 \square R).$$

Now we shall begin with the fuzzy *modus ponens* of (1). Using the max-$\odot$ composition (20), we can obtain the consequence B' of Cons in (1) from Ant 1 and Ant 2 by taking the max-$\odot$ composition $\square$ of the fuzzy set A' and the fuzzy relation given in (3)–(17). For example, we can have

$$\begin{aligned} \mathrm{Bm}' &= A' \square \mathrm{Rm} \\ &= A' \square [(A \times B) \cup (\neg A \times V)]. \end{aligned} \tag{21}$$

The membership function of the fuzzy set Bm′ in V is given as

$$\begin{aligned} \mu_{\mathrm{Bm}'}(v) &= \bigvee_{u} \{\mu_{A'}(u) \odot \mu_{\mathrm{Rm}}(u,v)\} \\ &= \bigvee_{u} (\mu_{A'}(u) \odot \{[\mu_A(u) \wedge \mu_B(v)] \vee [1 - \mu_A(u)]\}). \end{aligned} \tag{22}$$

In the same way, we have

$$\mathrm{Ba}' = A' \square \mathrm{Ra} = A' \square [(\neg A \times V) \oplus (U \times B)], \tag{23}$$

$$\mathrm{Bc}' = A' \square \mathrm{Rc} = A' \square (A \times B), \tag{24}$$

$$\mathrm{Bs}' = A' \square \mathrm{Rs} = A' \square [A \times V \underset{s}{\Rightarrow} U \times B], \tag{25}$$

$$\vdots$$

Similarly, in the fuzzy *modus tollens* of (2), the consequence A' in Cons can be deduced using the composition $\square$ of the fuzzy relation and the fuzzy set B'. Namely,

$$\begin{aligned} \mathrm{Am}' &= \mathrm{Rm}\square B' \\ &= [(A\times B)\cup(\neg A\times V)]\square B' \qquad (26) \\ &= \int_U \bigvee_v (\{[\mu_A(u)\wedge\mu_B(v)]\vee[1-\mu_A(u)]\}\odot\mu_{B'}(v))/u, \end{aligned}$$

$$\mathrm{Aa}' = \mathrm{Ra}\square B' = [(\neg A\times V)\oplus(U\times B)]\square B', \qquad (27)$$

$$\mathrm{Ac}' = \mathrm{Rc}\square B' = (A\times B)\square B', \qquad (28)$$

$$\mathrm{As}' = \mathrm{Rs}\square B' = \left[A\times V \underset{s}{\Rightarrow} U\times B\right]\square B', \qquad (29)$$

$$\vdots$$

3. COMPARISON OF FUZZY INFERENCE METHODS UNDER MAX-$\odot$ COMPOSITION

In this section we shall make comparisons of the fuzzy inference methods obtained above by applying 15 fuzzy relations of (3)–(17) to the fuzzy *modus ponens* (1) and the fuzzy *modus tollens* (2).

In the fuzzy *modus ponens*, we shall show what the consequences Bm′, Ba′, Bc′,... will be when using the max-$\odot$ composition [as in (21)–(25)] of the fuzzy set A' and the fuzzy relation, where the fuzzy set A' is

$$A' = A = \int_U \mu_A(u)/u,$$

$$A' = \mathbf{very}\,A = A^2 = \int_U \mu_A(u)^2/u,$$

$$A' = \mathbf{more\ or\ less}\,A = \sqrt{A} = \int_U \sqrt{\mu_A(u)}\,/u,$$

$$A' = \mathbf{not}\,A = \neg A = \int_U 1-\mu_A(u)/u,$$

which are typical examples of A'.

Similarly, in the fuzzy *modus tollens* we shall show what the consequences Am′, Aa′, Ac′,... will be when using the max-$\odot$ composition [as in (26)–(29)] of the fuzzy relation and the fuzzy set B', where B' is

$$B' = \textbf{not } B = \neg B = \int_V 1 - \mu_B(v)/v,$$

$$B' = \textbf{not very } B = \neg B^2 = \int_V 1 - \mu_B(v)^2/v,$$

$$B' = \textbf{not more or less } B = \neg\sqrt{B} = \int_V 1 - \sqrt{\mu_B(v)}\,/v,$$

$$B' = B = \int_V \mu_B(v)/v.$$

We shall begin with the fuzzy *modus ponens* in (1). We shall assume in the discussion of the fuzzy *modus ponens* that $\mu_A(u)$ takes all values in $[0,1]$ as u varies over all of U, that is, μ_A is a function onto $[0,1]$. Clearly, from the assumption, the fuzzy set A is a normal fuzzy set.

We shall first discuss Rm and obtain Bm′ of (21). From the above assumption, the expression (22) can be rewritten as

$$b'_m = \bigvee_x \{x' \odot [(x \wedge b) \vee (1-x)]\}, \tag{30}$$

and

$$f(x) = x' \odot [(x \wedge b) \vee (1-x)] \tag{31}$$

by letting

$$\mu_A(u) = x, \quad \mu_{A'}(u) = x', \quad \mu_B(v) = b, \quad \mu_{\mathrm{Bm}'}(v) = b'_m. \tag{32}$$

(i) For $A' = A$: When A' is equal to A (i.e., $\mu_{A'} = \mu_A$), x' becomes x from (32). Thus, we have $f(x)$ of (31) as[1]

$$\begin{aligned} f(x) &= x \odot [(x \wedge b) \vee (1-x)] \\ &= 0 \vee \{x + [(x \wedge b) \vee (1-x)] - 1\} \\ &= 0 \vee \{[x - 1 + (x \wedge b)] \vee [x - 1 + 1 - x]\} \\ &= 0 \vee \{[(x - 1 + x) \wedge (x - 1 + b)] \vee 0\} \\ &= 0 \vee [(2x-1) \wedge (x - 1 + b)] \\ &= [0 \vee (2x-1)] \wedge [0 \vee (x - 1 + b)]. \end{aligned} \tag{33}$$

[1]For any real numbers x, y, and z, we have in general $x + (y \wedge z) = (x+y) \wedge (x+z)$, $x + (y \vee z) = (x+y) \vee (x+z)$, $(x \wedge y) - z = (x-z) \wedge (y-z)$, $(x \vee y) - z = (x-z) \vee (y-z)$, $x - (y \wedge z) = (x-y) \vee (x-z)$, $x - (y \vee z) = (x-y) \wedge (x-z)$,

Figure 1(a) shows partial plots of the expressions $0 \vee (2x-1)$ and $0 \vee (x-1+b)$ of (33) with b as parameter. When b is equal to, say, $0.2, f(x)$ is indicated by the broken line, and thus $b'_m = \vee_x f(x)$ of (30) at $b = 0.2$ is seen to be 0.2 by observing the maximum of this line. In the same way, at $b = 0.6, f(x)$ is shown by the dot-dash line, whose maximum value is 0.6. Thus we have $b'_m = 0.6$ at $b = 0.6$. In general, we can have $b'_m = b$ for any b, that is, $b'_m = b$ at $x' = x$, which leads to $\mu_{\mathrm{Bm}'} = \mu_B$ at $\mu_{A'} = \mu_A$ from (32). Thus, $\mathrm{Bm}' = B$ at $A' = A$. Therefore, from (21),

$$A \square \mathrm{Rm} = B, \tag{34}$$

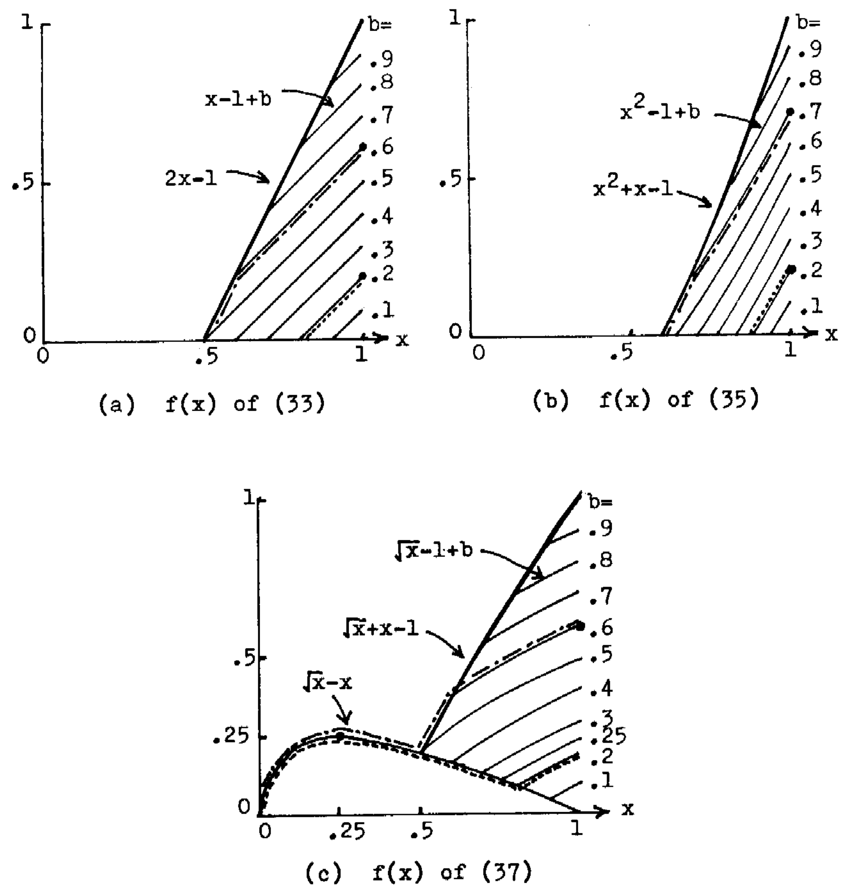

Fig. 1. $f(x) = x' \odot [(x \wedge b) \vee (1-x)]$ at $x' = x$, x^2, and $\sqrt{x}$.

which indicates that the *modus ponens* is satisfied by the method Rm under the max-$\odot$ composition $\square$. It is noted that Rm does not satisfy the *modus ponens* under the max-min composition [4].

(ii) For $A' = \textbf{very}A$: When $A' = \textbf{very}A$ $(=A^2)$, x' becomes x^2. Thus, (31) will be

$$\begin{aligned} f(x) &= x^2 \odot [(x \wedge b) \vee (1-x)] \\ &= 0 \vee \{x^2 + [(x \wedge b) \vee (1-x)] - 1\} \\ &= 0 \vee \{[x^2 - 1 + (x \wedge b)] \vee [(x^2 - 1 + 1 - x)]\} \\ &= 0 \vee [(x^2 - 1 + x) \wedge (x^2 - 1 + b)] \vee (x^2 - x) \\ &= 0 \vee [(x^2 + x - 1) \wedge (x^2 - 1 + b)] \qquad \text{since} \quad x^2 - x \leqslant 0 \\ &= [0 \vee (x^2 + x - 1)] \wedge [0 \vee (x^2 - 1 + b)]. \end{aligned} \tag{35}$$

In Figure 1(b), the expressions $0 \vee (x^2 + x - 1)$ and $0 \vee (x^2 - 1 + b)$ are plotted with b as parameter. For example, at $b = 0.2$, $f(x)$ is shown by the broken line, and its maximum value is 0.2. Thus, $b'_m = \vee_x f(x) = 0.2$. When $b = 0.7$, we have $b'_m = 0.7$. Thus, in general, we can obtain $b'_m = b$ for any b. Therefore, $\text{Bm}' = B$ at $A' = \textbf{very}\text{A}$. Thus,

$$\textbf{very}A \square \text{Rm} = B. \tag{36}$$

(iii) For $A' = \textbf{more or less}\, A$: Since $x' = \sqrt{x}$, $f(x)$ is given by

$$\begin{aligned} f(x) &= \sqrt{x} \odot [(x \wedge b) \vee (1-x)] \\ &= \{[0 \vee (\sqrt{x} + x - 1)] \wedge [0 \vee (\sqrt{x} - 1 + b)]\} \vee (\sqrt{x} - x). \end{aligned} \tag{37}$$

In Figure 1(c), $f(x)$ at $b = 0.2$ $(\leqslant 0.25)$ is shown by the broken line, whose maximum value is equal to the maximum value of $\sqrt{x} - x$. The expression $\sqrt{x} - x$ in fact takes its maximum value 0.25 at $x = 0.25$. Thus, we have $b'_m = \vee_x f(x) = 0.25$ at $b = 0.2$. It is found from this figure that $b'_m = 0.25$ so long as $b \leqslant 0.25$. On the other hand, when $b = 0.6$ $(\geqslant 0.25)$, $f(x)$ is indicated by the dot-dash line. Its maximum value is equal to 0.6. In general, we can obtain

$b'_m = b$ as long as $b \geqslant 0.25$. Thus we conclude that

$$b'_m = \begin{cases} \frac{1}{4}, & b \leqslant \frac{1}{4}, \\ b, & b \geqslant \frac{1}{4}, \end{cases}$$

$$= \tfrac{1}{4} \vee b.$$

Therefore,

$$\textbf{more or less } A \,\Box\, \mathrm{Rm} = B',$$

where

$$\mu_{B'} = \tfrac{1}{4} \vee \mu_B. \tag{38}$$

(iv) For $A' =$ **not** A: Since $x' = 1 - x$, $f(x)$ will be

$$f(x) = (1-x) \odot [(x \wedge b) \vee (1-x)]$$
$$= 0 \vee (-2x + 1).$$

Thus,

$$b'_m = \bigvee_x f(x)$$
$$= \bigvee_x \{0 \vee (-2x+1)\}$$
$$= 1.$$

Therefore,

$$\textbf{not } A \,\Box\, \mathrm{Rm} = \textbf{unknown}. \tag{39}$$

We can obtain the consequences Ba′ (cf. [8]), Bc′,..., $B'_\Box$ in the same way as Bm′, and thus we shall not discuss the details of how to obtain them. Table 1 summarizes the consequences inferred by all the inference methods (3)–(17) under the max-$\odot$ composition.

We shall next discuss the fuzzy *modus tollens* in (2). In the case of the fuzzy *modus tollens*, it is assumed that μ_B is a function onto [0, 1]. Because of the limitation of space, we shall investigate only the case of Rb of (12).

TABLE 1

Inference Results under Max-$\odot$ Composition (Case of Fuzzy *Modus Ponens*)

	A	very A	more or less A	not A
Rm	B	B	$\frac{1}{4}\vee\mu_B$	unknown
Ra	B	B	$\begin{cases}\mu_B+\frac{1}{4}, & \mu_B\leqslant\frac{1}{4}\\ \sqrt{\mu_B}, & \mu_B\geqslant\frac{1}{4}\end{cases}$	unknown
Rc	B	B	B	$\varnothing$
Rs	B	very B	more or less B	unknown
Rg	B	B	more or less B	unknown
Rsg	B	very B	more or less B	not B
Rgg	B	B	more or less B	not B
Rgs	B	B	more or less B	not B
Rss	B	very B	more or less B	not B
Rb	B	B	$\frac{1}{4}\vee\mu_B$	unknown
$R_\vartriangle$	B	B	more or less B	unknown
$R_\blacktriangle$	B	very B	more or less B	unknown
R_*	B	B	$\begin{cases}\dfrac{1}{4(1-\mu_B)}, & \mu_B\leqslant\frac{1}{2}\\ \mu_B, & \mu_B\geqslant\frac{1}{2}\end{cases}$	unknown
$R_\#$	B	B	$\frac{1}{4}\vee\mu_B$	$B\cup$ not B
$R_\square$	unknown	unknown	unknown	unknown

The consequence Ab′ is obtained [see (26)–(29)] by

$$\mathrm{Ab}'=\mathrm{Rb}\,\square\, B',$$

$$\mu_{\mathrm{Ab}'}(u)=\bigvee_{v}\left(\{[1-\mu_A(u)]\vee\mu_B(v)\}\odot\mu_{B'}(v)\right).$$

From the above assumption, this expression can be rewritten as

$$a_b'=\bigvee_{x}\{[(1-a)\vee x]\odot x'\}, \tag{40}$$

$$g(x)=[(1-a)\vee x]\odot x', \tag{41}$$

where

$$a_b'=\mu_{Ab'}(u),\quad a=\mu_A(u),\quad x=\mu_B(v),\quad x'=\mu_{B'}(v). \tag{42}$$

We shall show what the consequence a'_b (or Ab′) will be when $B' =$ **not** B, **not very** B, **not more or less** B, and B under the max-$\odot$ composition.

(i) For $B' =$ **not** B: When $B' =$ **not** B, x' becomes $1-x$, from (42). Thus, $g(x)$ of (41) is given by

$$\begin{aligned} g(x) &= [(1-a)\vee x]\odot(1-x) \\ &= 0\vee\{[(1-a)\vee x]+(1-x)-1\} \\ &= 0\vee\{[(1-a)-x]\vee(x-x)\} \\ &= 0\vee(1-a-x). \end{aligned}$$

Therefore, from (40) we have a'_b as

$$\begin{aligned} a'_b &= \bigvee_x g(x) \\ &= \bigvee_x \{0\vee(1-a-x)\} \\ &= 1-a \qquad \text{at} \quad x=0. \end{aligned}$$

It follows from this result that $a'_b = 1-a$ at $x' = 1-x$, that is, Ab′ = **not** A at $B' =$ **not** B. Hence,

$$\text{Rb}\,\square\,\textbf{not}\,B = \textbf{not}\,A. \tag{43}$$

This identity indicates the satisfaction of *modus tollens* by Rb under the max-$\odot$ composition. Note that Rb does not satisfy the *modus tollens* under the max-min composition [7].

(ii) For $B' =$ **not very** B: Since $x' = 1-x^2$, $g(x)$ will be

$$\begin{aligned} g(x) &= [(1-a)\vee x]\odot(1-x^2) \\ &= \left[0\vee(1-a-x^2)\right]\vee(x-x^2). \end{aligned}$$

Therefore,

$$\begin{aligned} a'_b &= \bigvee_x g(x) \\ &= \bigvee_x \{\left[0\vee(1-a-x^2)\right]\vee(x-x^2)\} \\ &= \bigvee_x \{0\vee(1-a-x^2)\}\vee \bigvee_x \{x-x^2\} \\ &= (1-a)\vee\tfrac{1}{4}. \end{aligned}$$

Hence

$$\text{Rb}\square\textbf{not very } B = A',$$

where

$$\mu_{A'} = \tfrac{1}{4} \vee (1 - \mu_A). \tag{44}$$

(iii) For $B' = \textbf{not more or less } B$:

$$\begin{aligned} g(x) &= [(1-a)\vee x] \odot (1-\sqrt{x}) \\ &= 0 \vee (1 - a - \sqrt{x}), \\ a'_b &= \bigvee_x g(x) \\ &= \bigvee_x \{0 \vee (1 - a - \sqrt{x})\} \\ &= 1 - a \qquad \text{at} \quad x = 0. \end{aligned}$$

Therefore,

$$\text{Rb}\square\textbf{not more or less } B = \textbf{not } A. \tag{45}$$

(iv) For $B' = B$:

$$\begin{aligned} a'_b &= \bigvee_x \{[(1-a)\vee x] \odot x\} \\ &= \bigvee_x [0 \vee (x-a) \vee (2x-1)] \\ &= 0 \vee \Big[\bigvee_x (x-a)\Big] \vee \Big[\bigvee_x (2x-1)\Big] \\ &= 0 \vee (1-a) \vee 1 \\ &= 1. \end{aligned}$$

Thus,

$$\text{Rb}\square\, B = \textbf{unknown}.$$

We can obtain the consequences Am′, Aa′ (cf. [8]), Ac′,…,$A'_\square$ in the same way as Rb′. In Table 2 the inference results obtained by all the methods under the max-⊙ composition are listed.

In the forms of fuzzy conditional inference (1) and (2), it seems according to our intuition that the relations between A' in Ant 2 and B' in Cons of the fuzzy *modus ponens* (1) ought to be satisfied as shown in Table 3 (cf. [4, 5]). Similarly, the relations between B' in Ant 2 and A' in Cons of the fuzzy *modus tollens* (2) ought to be satisfied as in Table 4.

In Table 5, the satisfaction (○) or failure (×) of each criterion of Tables 3 and 4 under each fuzzy inference method is indicated by use of the inference

TABLE 2

Inference Results under Max-⊙ Composition (Case of Fuzzy *Modus Tollens*)

	not B	not very B	not more or less B	B
Rm	not A	$(1-\mu_A)\vee\frac{1}{4}$	not A	$A\cup$ not A
Ra	not A	$\begin{cases}1-\mu_A^2, & \mu_A\leqslant\frac{1}{2}\\ \frac{1}{4}+(1-\mu_A), & \mu_A\geqslant\frac{1}{2}\end{cases}$	not A	unknown
Rc	∅	$\begin{cases}\mu_A-\mu_A^2, & \mu_A\leqslant\frac{1}{2}\\ \frac{1}{4}, & \mu_A\geqslant\frac{1}{2}\end{cases}$	∅	A
Rs	not A	not very A	not more or less A	unknown
Rg	not A	$(1-\mu_A^2)\vee\frac{1}{4}$	not more or less A	unknown
Rsg	not A	not very A	not more or less A	A
Rgg	not A	$(1-\mu_A^2)\vee\frac{1}{4}$	not more or less A	A
Rgs	not A	$(1-\mu_A^2)\vee\frac{1}{4}$	not more or less A	A
Rss	not A	not very A	not more or less A	A
Rb	not A	$(1-\mu_A)\vee\frac{1}{4}$	not A	unknown
$R_\vartriangle$	not A	$\begin{cases}1-\mu_A^2, & \mu_A\leqslant\frac{\sqrt{2}}{2}\\ \frac{1}{4\mu_A^2}, & \mu_A\geqslant\frac{\sqrt{2}}{2}\end{cases}$	not more or less A	unknown
$R_\blacktriangle$	not A	not very A	not more or less A	unknown
R_*	not A	$\left(1-\frac{\mu_A}{2}\right)^2$	not A	unknown
$R_\#$	not A	$(1-\mu_A)\vee\frac{1}{4}$	not A	$A\cup$ not A
$R_\square$	$\begin{cases}1, & \mu_A<1\\ 0, & \mu_A=1\end{cases}$	$\begin{cases}1, & \mu_A<1\\ 0, & \mu_A=1\end{cases}$	$\begin{cases}1, & \mu_A<1\\ 0, & \mu_A=1\end{cases}$	unknown

results in Tables 1 and 2. In order to compare the inference results under the max-⊙ composition and the max-min composition, the inference results under the max-min composition are listed in Table 6 (cf. [7]).

From Tables 1, 2, and 5 it follows that all the inference methods except $R_\square$ can satisfy so-called *modus ponens* under the max-⊙ composition, but only the methods Rc, Rs, ..., Rss can satisfy the *modus ponens* under the max-min composition. Almost the same holds for *modus tollens*. Moreover, it is found that the majority of the methods can infer very reasonable consequences under the max-⊙ composition, though we cannot always get reasonable consequences under the max-min composition, as shown in Table 6.

TABLE 3

Relations between Ant 2 and Cons under Ant 1 for the Fuzzy *Modus Ponens* in (1)

	x is A' (Ant 2)	y is B' (Cons)
Relation I (*modus ponens*)	x is A	y is B
Relation II-1	x is **very** A	y is **very** B
Relation II-2	x is **very** A	y is B
Relation III-1	x is **more or less** A	y is **more or less** B
Relation III-2	x is **more or less** A	y is B
Relation IV-1	x is **not** A	y is **unknown**
Relation IV-2	x is **not** A	y is **not** B

TABLE 4

Relations between Ant 2 and Cons under Ant 1 for the Fuzzy *Modus Tollens* in (2)

	y is B' (Ant 2)	x is A' (Cons)
Relation V (*modus tollens*)	y is **not** B	x is **not** A
Relation VI-1	y is **not very** B	x is **not very** A
Relation VI-2	y is **not very** B	x is **not** A
Relation VII-1	y is **not more or less** B	x is **not more or less** A
Relation VII-2	y is **not more or less** B	x is **not** A
Relation VIII-1	y is B	x is **unknown**
Relation VIII-2	y is B	x is A

TABLE 5

Satisfaction of Each Relation in Tables 3 and 4 under Each Method (Case of Max-⊙ Composition)

	Ant 2	Cons	Rm	Ra	Rc	Rs	Rg	Rsg	Rgg	Rgs	Rss	Rb	$R_\triangle$	$R_\blacktriangle$	R_*	$R_\#$	$R_\square$
Relation I (*modus ponens*)	*A*	*B*	○	○	○	○	○	○	○	○	○	○	○	○	○	○	×
Relation II-1	very *A*	very *B*	×	×	×	○	×	○	×	×	○	×	×	○	×	×	×
Relation II-2	very *A*	*B*	○	○	○	×	○	×	○	○	×	○	○	×	○	○	×
Relation III-1	more or less *A*	more or less *B*	×	×	×	○	○	○	○	○	○	×	○	○	×	×	×
Relation III-2	more or less *A*	*B*	×	×	○	×	×	×	×	×	×	×	×	×	×	×	×
Relative IV-1	not *A*	unknown	○	○	×	○	○	×	×	×	×	○	○	○	○	×	○
Relation IV-2	not *A*	not *B*	×	×	×	×	×	○	○	○	○	×	×	×	×	×	×
Relation V (*modus tollens*)	not *B*	not *A*	○	○	×	○	○	○	○	○	○	○	○	○	○	○	×
Relation VI-1	not very *B*	not very *A*	×	×	×	○	×	○	×	×	○	×	×	○	×	×	×
Relation VI-2	not very *B*	not *A*	×	×	×	×	×	×	×	×	×	×	×	×	×	×	×
Relation VII-1	not more or less *B*	not more or less *A*	×	×	×	○	○	○	○	○	○	×	○	○	×	×	×
Relation VII-2	not more or less *B*	not *A*	○	○	×	×	×	×	×	×	×	○	×	×	○	○	×
Relation VIII-1	*B*	unknown	×	○	×	○	○	×	×	×	×	○	○	○	○	×	○
Relation VIII-2	*B*	*A*	×	×	○	×	×	○	○	○	○	×	×	×	×	×	×

TABLE 6

Satisfaction of Each Relation in Tables 3 and 4 under Each Method (The Case of Max-Min Composition)[a]

	Ant 2	Cons	Rm	Ra	Rc	Rs	Rg	Rsg	Rgg	Rgs	Rss	Rb	$R_{\triangle}$	$R_{\blacktriangle}$	R_{*}	$R_{\#}$	$R_{\square}$
Relation I (*modus ponens*)	*A*	*B*	×	×	○	○	○	○	○	○	○	×	×	×	×	×	×
Relation II-1	very *A*	very *B*	×	×	×	○	×	○	×	×	○	×	×	×	×	×	×
Relation II-2	very *A*	*B*	×	×	○	×	○	×	○	○	×	×	×	×	×	×	×
Relation III-1	more or less *A*	more or less *B*	×	×	×	○	○	○	○	○	○	×	×	×	×	×	×
Relation III-2	more or less *A*	*B*	×	×	○	×	×	×	×	×	×	×	×	×	×	×	×
Relation IV-1	not *A*	unknown	○	○	×	○	○	×	×	×	×	○	○	○	○	×	○
Relation IV-2	not *A*	not *B*	×	×	×	×	×	○	○	○	○	×	×	×	×	×	×
Relation V (*modus tollens*)	not *B*	not *A*	×	×	×	○	×	○	×	×	○	×	×	×	×	×	×
Relation VI-1	not very *B*	not very *A*	×	×	×	○	×	○	×	×	○	×	×	×	×	×	×
Relation VI-2	not very *B*	not *A*	×	×	×	×	×	×	×	×	×	×	×	×	×	×	×
Relation VII-1	not more or less *B*	not more or less *A*	×	×	×	○	×	○	×	×	○	×	×	×	×	×	×
Relation VII-2	not more or less *B*	not *A*	×	×	×	×	×	×	×	×	×	×	×	×	×	×	×
Relation VIII-1	*B*	unknown	×	○	×	○	○	×	×	×	×	○	○	○	○	×	○
Relation VIII-2	*B*	*A*	×	×	○	×	×	×	×	○	○	×	×	×	×	×	×

[a] Cf. [7].

4. SYLLOGISM BY EACH METHOD UNDER MAX-$\odot$ COMPOSITION

In this section we shall investigate a syllogism by each method under the max-$\odot$ composition $\Box$.

Let P_1, P_2 and P_3 be the fuzzy conditional propositions

$$P_1: \text{If } x \text{ is } A \text{ then } y \text{ is } B.$$
$$P_2: \text{If } y \text{ is } B \text{ then } z \text{ is } C.$$
$$P_3: \text{If } x \text{ is } A \text{ then } z \text{ is } C.$$

where A, B, and C are fuzzy sets in U, V, and W, respectively. If the proposition P_3 is deduced from the propositions P_1 and P_2—that is, the following holds:

$$\begin{array}{l} P_1: \text{If } x \text{ is } A \text{ then } y \text{ is } B. \\ P_2: \text{If } y \text{ is } B \text{ then } z \text{ is } C. \\ \hline P_3: \text{If } x \text{ is } A \text{ then } z \text{ is } C. \end{array}$$

—then it is said that a syllogism holds.

Let $R(A,B)$, $R(B,C)$, and $R(A,C)$ be fuzzy relations in $U \times V$, $V \times W$, and $U \times W$, respectively, which are obtained from the propositions P_1, P_2, and P_3. If the following equality holds, the syllogism holds under the max-$\odot$ composition $\Box$:

$$R(A,B) \Box R(B,C) = R(A,C). \tag{46}$$

That is to say,

$$\begin{array}{l} P_1: \text{If } x \text{ is } A \text{ then } y \text{ is } B \rightarrow R(A,B) \\ P_2: \text{If } y \text{ is } B \text{ then } z \text{ is } C \rightarrow R(B,C) \\ \hline P_3: \text{If } x \text{ is } A \text{ then } z \text{ is } C \leftarrow R(A,B) \Box R(B,C) \end{array} \tag{47}$$

The membership function of $R(A,B) \Box R(B,C)$ is given by

$$\mu_{R(A,B)\Box R(B,C)}(u,w) = \bigvee_{v} \{\mu_{R(A,B)}(u,v) \odot \mu_{R(B,C)}(v,w)\}. \tag{48}$$

Now we shall obtain $R(A,B) \Box R(B,C)$ under each fuzzy inference method and show whether the syllogism holds or not. In the discussion of the syllogism it is assumed that the membership function μ_B of the fuzzy set B is a function onto $[0,1]$.

We shall discuss only the case of R_Δ of (13). The membership functions of the fuzzy relations $R_\Delta(A,B)$ and $R_\Delta(B,C)$ are obtained from the propositions P_1 and P_2 by using (13).

$$
\begin{aligned}
\mu_{R_\Delta(A,B)}(u,v) &= \mu_A(u) \underset{\Delta}{\rightarrow} \mu_B(v) \\
&= \begin{cases} 1, & \mu_A(u) \leqslant \mu_B(v), \\ \dfrac{\mu_B(v)}{\mu_A(u)}, & \mu_A(u) > \mu_B(v), \end{cases} \\
&= \begin{cases} 1 \wedge \dfrac{\mu_B(v)}{\mu_A(u)}, & \mu_A(u) > 0, \\ 1, & \mu_A(u) = 0, \end{cases}
\end{aligned}
\tag{49}
$$

$$
\begin{aligned}
\mu_{R_\Delta(B,C)}(v,w) &= \mu_B(v) \underset{\Delta}{\rightarrow} \mu_C(w) \\
&= \begin{cases} 1, & \mu_B(v) \leqslant \mu_C(w), \\ \dfrac{\mu_C(w)}{\mu_B(v)}, & \mu_B(w) > \mu_C(u), \end{cases} \\
&= \begin{cases} 1 \wedge \dfrac{\mu_C(w)}{\mu_B(v)}, & \mu_B(v) > 0, \\ 1, & \mu_B(v) = 0. \end{cases}
\end{aligned}
\tag{50}
$$

Then the membership functions of the max-$\odot$ composition of $R_\Delta(A,B)$ and $R_\Delta(B,C)$ will be given by

$$
\mu_{R_\Delta(A,B) \square R_\Delta(B,C)}(u,w) = \bigvee_v \left\{ [\mu_A(u) \underset{\Delta}{\rightarrow} \mu_B(v)] \odot [\mu_B(v) \underset{\Delta}{\rightarrow} \mu_C(w)] \right\}. \tag{51}
$$

Under the assumption that μ_B is a function onto $[0,1]$, (51) is rewritten as

$$
d = \bigvee_x \left\{ [a \underset{\Delta}{\rightarrow} x] \odot [x \underset{\Delta}{\rightarrow} c] \right\}, \tag{52}
$$

where

$$
d = \mu_{R_\Delta(A,B) \square R_\Delta(B,C)}(u,w), \quad a = \mu_A(u), \quad x = \mu_B(v), \quad c = \mu_C(w) \tag{53}
$$

and

$$a \underset{\Delta}{\rightarrow} x = \begin{cases} 1 \wedge \dfrac{x}{a}, & a > 0, \\ 1, & a = 0, \end{cases}$$

$$x \underset{\Delta}{\rightarrow} c = \begin{cases} 1 \wedge \dfrac{c}{x}, & x > 0, \\ 1, & x = 0. \end{cases}$$

Then, $[a \underset{\Delta}{\rightarrow} x] \odot [x \underset{\Delta}{\rightarrow} c]$ is given as

$$[a \underset{\Delta}{\rightarrow} x] \odot [x \underset{\Delta}{\rightarrow} c]$$

$$= \begin{cases} 1 \wedge \dfrac{x}{a} \wedge \dfrac{c}{x} \wedge \left(\dfrac{x}{a} + \dfrac{c}{x} - 1\right), & a, x > 0, \\ 1, & a, x = 0, \\ 0, & (a = 0, x > 0) \text{ or } (a > 0, x = 0). \end{cases} \tag{54}$$

When $a > c$, the expression (54) is represented by the solid line in Figure 2(a) with parameters a and c. The maximum value of this line is c/a at $x = a$ and c. Thus, we have d of (52) as

$$d = \frac{c}{a}, \qquad a > c. \tag{55}$$

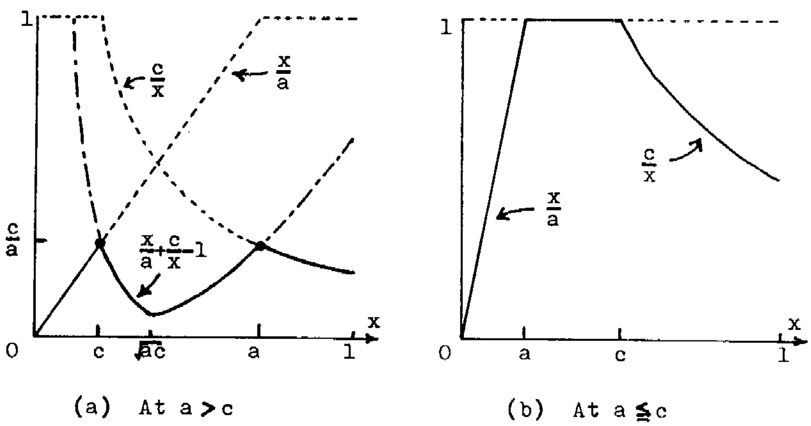

Fig. 2. $[a \underset{\Delta}{\rightarrow} x] \odot [x \underset{\Delta}{\rightarrow} c]$ of (54) (solid line).

On the other hand, when $a \leqslant c$, (54) is shown by the solid line in Figure 2(b), whose maximum value is 1. Thus,

$$d = 1, \qquad a \leqslant c. \tag{56}$$

From (55) and (56), d is given by

$$d = \begin{cases} 1, & a \leqslant c, \\ \dfrac{c}{a}, & a > c, \end{cases}$$

which leads to

$$\begin{aligned} \mu_{R_\Delta(A,B) \Box R_\Delta(B,C)}(u,v) &= \begin{cases} 1, & \mu_A(u) \leqslant \mu_C(w), \\ \dfrac{\mu_C(w)}{\mu_A(u)}, & \mu_A(u) > \mu_C(w) \end{cases} \\ &= \mu_A(u) \underset{\Delta}{\rightarrow} \mu_C(w) \\ &= \mu_{R_\Delta(A,C)}(u,w). \end{aligned} \tag{57}$$

Thus, we have

$$R_\Delta(A,B) \Box R_\Delta(B,C) = R_\Delta(A,C). \tag{58}$$

Therefore, the syllogism holds for R_Δ under the max-$\odot$ composition $\Box$. Note that R_Δ does not satisfy the syllogism under the max-min composition [7].

In the same way, we can obtain $R(A,B) \Box R(B,C)$ by the other methods; the results are as follows:

$$\begin{aligned} &\mathrm{Rm}(A,B) \Box \mathrm{Rm}(B,C) \\ &\quad = \int_{U \times W} [\mu_A(u) + \mu_C(w) - 1] \vee [1 - \mu_A(u)] / (u,w) \\ &\quad \neq \mathrm{Rm}(A,C) \left(= \int_{U \times W} [\mu_A(u) \wedge \mu_C(w)] \vee [1 - \mu_A(u)] / (u,w) \right). \end{aligned} \tag{59}$$

$$\mathrm{Ra}(A,B)\square\mathrm{Ra}(B,C)=\int_{U\times W}1\wedge[1-\mu_A(u)+\mu_C(w)]/(u,w)$$
$$=\mathrm{Ra}(A,C),\tag{60}$$

$$\mathrm{Rc}(A,B)\square\mathrm{Rc}(B,C)=\int_{U\times W}0\vee[\mu_A(u)+\mu_C(w)-1]/(u,w)$$
$$\neq\mathrm{Rc}(A,C)\left(=\int_{U\times W}\mu_A(u)\wedge\mu_C(w)/(u,w)\right),\tag{61}$$

$$\mathrm{Rs}(A,B)\square\mathrm{Rs}(B,C)=\int_{U\times W}\mu_A(u)\underset{s}{\rightarrow}\mu_C(w)/(u,w)$$
$$=\mathrm{Rs}(A,C),\tag{62}$$

$$\mathrm{Rg}(A,B)\square\mathrm{Rg}(B,C)=\int_{U\times W}\mu_A(u)\underset{g}{\rightarrow}\mu_C(w)/(u,w)$$
$$=\mathrm{Rg}(A,C),\tag{63}$$

$$\mathrm{Rsg}(A,B)\square\mathrm{Rsg}(B,C)=\int_{U\times W}[\mu_A(u)\underset{s}{\rightarrow}\mu_C(w)]$$
$$\wedge\left[1-\mu_A(u)\underset{g}{\rightarrow}1-\mu_C(w)\right]\Big/(u,w)\tag{64}$$
$$=\mathrm{Rsg}(A,C),$$

$$\mathrm{Rgg}(A,B)\square\mathrm{Rgg}(B,C)=\int_{U\times W}\left[\mu_A(u)\underset{g}{\rightarrow}\mu_C(w)\right]$$
$$\wedge\left[1-\mu_A(u)\underset{g}{\rightarrow}1-\mu_C(w)\right]\Big/(u,w)\tag{65}$$
$$=\mathrm{Rgg}(A,C),$$

$$\mathrm{Rgs}(A,B)\square\mathrm{Rgs}(B,C)=\int_{U\times W}\left[\mu_A(u)\underset{g}{\rightarrow}\mu_C(w)\right]$$
$$\wedge\left[1-\mu_A(u)\underset{s}{\rightarrow}1-\mu_C(w)\right]\Big/(u,w)\tag{66}$$
$$=\mathrm{Rgs}(A,C),$$

$$\mathrm{Rss}(A,B)\square\mathrm{Rss}(B,C)=\int_{U\times W}\left[\mu_A(u)\underset{s}{\rightarrow}\mu_C(w)\right]$$
$$\wedge\left[1-\mu_A(u)\underset{s}{\rightarrow}1-\mu_C(w)\right]\Big/(u,w)\tag{67}$$
$$=\mathrm{Rss}(A,C),$$

$$\mathrm{Rb}(A,B)\square\mathrm{Rb}(B,C)=\int_{U\times W}[1-\mu_A(u)]\vee\mu_C(w)/(u,w)$$

$$=\mathrm{Rb}(A,C), \tag{68}$$

$$\mathrm{R}_{\vartriangle}(A,B)\square\mathrm{R}_{\vartriangle}(B,C)=\int_{U\times W}\mu_A(u)\underset{\vartriangle}{\rightarrow}\mu_C(w)/(u,w)$$

$$=\mathrm{R}_{\vartriangle}(A,C), \tag{69}$$

$$\mathrm{R}_{\blacktriangle}(A,B)\square\mathrm{R}_{\blacktriangle}(B,C)=\int_{U\times W}\mu_A(u)\underset{\blacktriangle}{\rightarrow}\mu_C(w)/(u,w)$$

$$=\mathrm{R}_{\blacktriangle}(A,C), \tag{70}$$

$$\mathrm{R}_{*}(A,B)\square\mathrm{R}_{*}(B,C)=\int_{U\times W}[1-\mu_A(u)]\vee\mu_C(w)/(u,w)$$

$$\neq\mathrm{R}_{*}(A,C)\Big(=\int_{U\times W}1-\mu_A(u)$$

$$+\mu_A(u)\mu_C(w)/(u,w)\Big), \tag{71}$$

$$\mathrm{R}_{\#}(A,B)\square\mathrm{R}_{\#}(B,C)=\int_{U\times W}[\mu_A(u)+\mu_C(w)-1]\vee[1-\mu_A(u)-\mu_C(w)]$$

$$\vee[\mu_C(w)-\mu_A(u)]/(u,w)\neq\mathrm{R}_{\#}(A,C)$$

$$\Big(=\int_{U\times W}[\mu_A(u)\wedge\mu_C(w)]$$

$$\vee[1-\mu_A(u)\wedge 1-\mu_C(w)]$$

$$\vee[1-\mu_A(u)\wedge\mu_C(w)]/(u,w)\Big), \tag{72}$$

$$\mathrm{R}_{\square}(A,B)\square\mathrm{R}_{\square}(B,C)=\int_{U\times W}\mu_A(u)\underset{\square}{\rightarrow}\mu_C(w)/(u,w)$$

$$=\mathrm{R}_{\square}(A,C). \tag{73}$$

TABLE 7

Satisfaction of Syllogism under Max-⊙ Composition and Max-Min Composition

	Rm	Ra	Rc	Rs	Rg	Rsg	Rgg	Rgs	Rss	Rb	$R_{\triangle}$	$R_{\blacktriangle}$	R_{*}	$R_{\#}$	$R_{\square}$
Max-⊙ composition	×	○	×	○	○	○	○	○	○	○	○	○	×	×	○
Max-min composition	×	×	○	○	○	○	○	○	○	×	×	×	×	×	○

Using these results, the satisfaction (○) or failure (×) of syllogism by each method under the max-⊙ composition is listed in Table 7. This table also contains the results under the max-min composition (cf. [7]).

It follows from Table 7 that the methods Ra, Rb, $R_{\triangle}$, and $R_{\blacktriangle}$ can satisfy the syllogism under the max-⊙ composition, though they do not satisfy it under the max-min composition. But the converse holds for Rc.

5. CONCLUSION

We have shown that, when the max-⊙ composition is used in the compositional rule of inference, the majority of fuzzy inference methods can lead to very reasonable consequences which coincide with our intuition with respect to several criteria such as *modus ponens*, *modus tollens*, and syllogism.

It will be of interest to apply the max-⊙ composition to fuzzy inferences which are of the more complicated form, such as

$$\begin{array}{l} \text{If } x \text{ is } A \text{ then } y \text{ is } B \text{ else } y \text{ is } C. \\ \underline{x \text{ is } A'.\qquad\qquad\qquad\qquad\qquad} \\ y \text{ is } D. \end{array}$$

$$\begin{array}{l} \text{If } x \text{ is } A_1 \text{ then } y \text{ is } B_1 \text{ else} \\ \text{if } x \text{ is } A_2 \text{ then } y \text{ is } B_2 \text{ else} \\ \vdots \\ \text{if } x \text{ is } A_n \text{ then } y \text{ is } B_n. \\ \underline{x \text{ is } A'.\qquad\qquad\qquad\qquad\qquad} \\ y \text{ is } B'. \end{array}$$

These results will be presented in subsequent papers.

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