# FUZZY REASONING WITH A FUZZY CONDITIONAL PROPOSITION "IF ... THEN ... ELSE ..."

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## ABSTRACT

We investigate the properties of fuzzy reasoning methods by Zadeh with a fuzzy conditional proposition "If x is A then y is B else y is C," with A, B and C being fuzzy concepts, and point out that the consequences inferred by his methods do not always fit our intuition, and suggest a new method which fits our intuition under several criteria.

KEYWORDS

fuzzy set; fuzzy relation; fuzzy conditional proposition; fuzzy reasoning

## INTRODUCTION

In much of human reasoning, the form of reasoning is approximate rather than exact as in the statement:

If a tomato is <u>red</u> then the tomato is <u>ripe</u>.
This tomato is <u>very red</u>.

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This tomato is <u>very ripe</u>.

Zadeh (1975), Mamdani (1977), and Mizumoto et al. (1979a, 1979b, 1979c, 1980, 1981) suggested methods for such reasoning in which the antecedent involves a fuzzy conditional proposition such as "If x is A then y is B," where A and B are fuzzy concepts. In Mizumoto we investigated the properties of their methods.

As a generalization of such a fuzzy conditional inference containing the proposition "If x is A then y is B," Zadeh (1975) also proposed a fuzzy conditional inference of the form:

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Ant 1: If x is A then y is B else y is C.
Ant 2: x is A'. (1)

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Cons: y is D.

For this form of inference, Zadeh proposed methods for obtaining the consequence (Cons) from two antecedents (Ant 1 and Ant 2) in (1).

This paper investigates the properties of his methods and points out that the consequences inferred by his methods do not always fit our intuition, and suggests a new method which fits our intuition under several criteria.

## FUZZY REASONING WITH "IF ... THEN ... ELSE ..."

We shall focus our attention on the following form of inference in which a fuzzy conditional proposition "If ... then ... else ..." is contained.

Ant 1: If x is A then y is B else y is C.
Ant 2: x is A'. (2)

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Cons: y is D.

where x and y are the names of objects, and A, A', B, C and D are fuzzy concepts which are represented by fuzzy sets in universes of discourse U, U, V, V and V, respectively.

An example of such a form of inference is the following:

Ant 1: If x is tall then y is fairly heavy else y is pretty light.
Ant 2: x is pretty tall.

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Cons: y is very heavy.

The Ant 1 of the form "If x is A then y is B else y is C" in (2) may represent a certain relationship between A and B, C. From this point of view, Zadeh (1975) gave the translation rules (Maximin Rule and Arithmetic Rule) for translating the fuzzy conditional proposition "If x is A then y is B else y is C" into a fuzzy relation in U x V.

Let A, B and C be fuzzy sets in U, V and V, respectively, which are written as

$$A = \int_U \mu_A(u)/u \; ; \qquad B = \int_V \mu_B(v)/v \; ; \qquad C = \int_V \mu_C(v)/v \qquad (3)$$

then we have:

(i) Maximin Rule Rm':

$$Rm' = (A \times B) \cup (\neg A \times C) \qquad (4)$$

$$= \int_{U\times V} (\mu_A(u) \wedge \mu_B(v)) \vee ((1-\mu_A(u)) \wedge \mu_C(v)) \;/\; (u,v).$$

(ii) Arithmetic Rule Ra':

$$Ra' = (\neg A \times V \oplus U \times B) \cap (A \times V \oplus U \times C) \quad (5)$$

$$= \int_{U \times V} 1 \wedge (1-\mu_A(u)+\mu_B(v)) \wedge (\mu_A(u)+\mu_C(v)) \;/\; (u,v).$$

where $\times$, $\cup$, $\cap$, $\neg$ and $\oplus$ denote cartesian product, union, intersection, complement and bounded-sum for fuzzy sets, respectively.

Remark: If C in the above translation rules is replaced by V (the universe of discourse of C) which is interpreted as "unknown," then the fuzzy conditional proposition "If x is A then y is B else y is C" is reduced to a proposition "If x is A then y is B else y is unknown," namely, "If x is A then y is B." Thus, the above rules are the generalization of the well-known translation rules for the fuzzy conditional proposition "If x is A then y is B" (Zadeh, 1975). For example, Rm' in (4) and Ra' in (5) become as follows at C = V.

$$Rm = (A \times B) \cup (\neg A \times V) \quad (6)$$

$$= \int_{U \times V} (\mu_A(u) \wedge \mu_B(v)) \vee (1-\mu_A(u)) \;/\; (u,v).$$

$$Ra = (\neg A \times V) \oplus (U \times B) \quad (7)$$

$$= \int_{U \times V} 1 \wedge (1-\mu_A(u)+\mu_B(v)) \;/\; (u,v).$$

In addition, as another case, when C = $\emptyset$ (empty set), Rm' is reduced to the "Mini Rule" Rc by Mamdani (1977). Namely,

$$Rc = A \times B \quad (8)$$

$$= \int_{U \times V} \mu_A(u) \wedge \mu_B(v) \;/\; (u,v).$$

For the proposition "If x is A then y is B else y is C," it is also possible to define a translation rule Rb' which is based on the implication in binary logic.

(iii) Fuzzified Binary Rule Rb':

$$Rb' = (\neg A \times V \cup U \times B) \cap (A \times V \cup U \times C) \quad (9)$$

$$= \int_{U \times V} (1-\mu_A(u) \vee \mu_B(v)) \wedge (\mu_A(u) \vee \mu_C(v)) \;/\; (u,v).$$

Furthermore, we shall also introduce a new method Rgg' for the proposition "If x is A then y is B else y is C." The implication $a \underset{g}{\rightarrow} b$ is based on the implication rule in $G_{aleph}$ logic system by Gödel (Rescher, 1969).

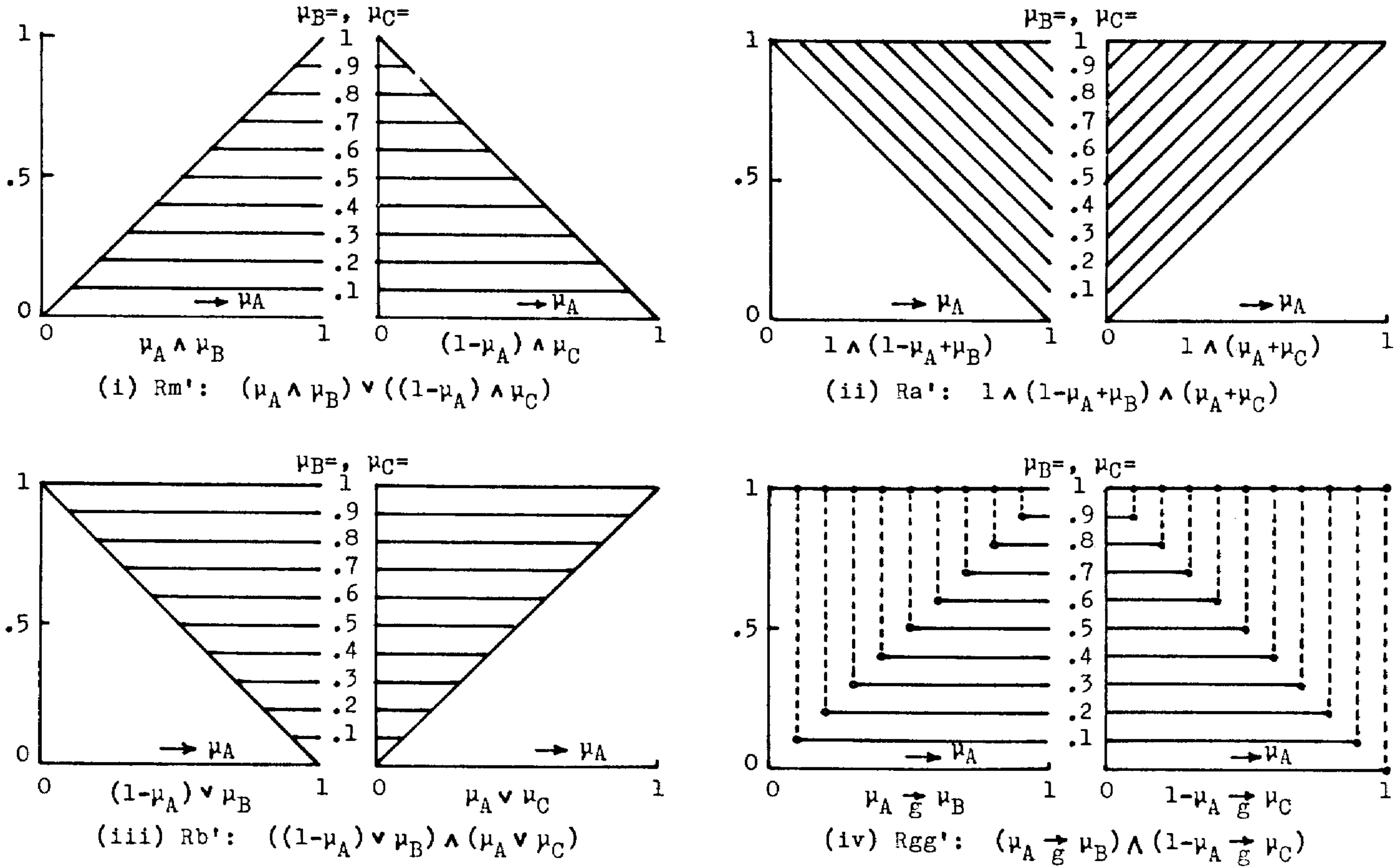


Fig. 1. Diagram for each rule of (4), (5), (9) and (10)

(iv) Rule Rgg':

$$Rgg' = (A \times V \underset{g}{\Rightarrow} U \times B) \cap (\neg A \times V \underset{g}{\Rightarrow} U \times C) \quad (10)$$
$$= \int_{U\times V} (\mu_A(u) \underset{g}{\rightarrow} \mu_B(v)) \wedge (1-\mu_A(u) \underset{g}{\rightarrow} \mu_C(v)) \; / \; (u,v),$$

where

$$\mu_A(u) \underset{g}{\rightarrow} \mu_B(v) = \begin{cases} 1 & \ldots \; \mu_A(u) \leq \mu_B(v), \\ \mu_B(v) & \ldots \; \mu_A(u) > \mu_B(v). \end{cases}$$

In Fig.1 each rule of (4), (5), (9) and (10) for the proposition "If x is A then y is B else y is C" is illustrated by a diagram which will be useful in the later discussion. In each figure, the part of expression of each rule which contains $\mu_B(v)$ (say, $\mu_A(u) \vee \mu_B(v)$ in Rm') is depicted using a parameter $\mu_B(v)$. The other part containing $\mu_C(v)$ (say, $(1-\mu_A(u)) \wedge \mu_C(v)$ in Rm') is illustrated by a parameter $\mu_C(v)$. In the figures the symbols $\mu_A$, $\mu_B$, $\mu_C$ are used instead of $\mu_A(u)$, $\mu_B(v)$, $\mu_C(v)$, respectively, for convenience.

The consequence D in Cons of (2) can be deduced from Ant 1 and Ant 2 using the max-min composition "o" of the fuzzy set A' in U and the fuzzy relation in U x V obtained above. Thus, we can have for each translation rule by the following.

$$Dm = A' \circ Rm' = A' \circ \left[ (A \times B) \cup (\neg A \times C) \right] \quad (11)$$
$$= \int_V \bigvee_u \left\{ \mu_{A'}(u) \wedge \left[ (\mu_A(u) \wedge \mu_B(v)) \vee (1-\mu_A(u) \wedge \mu_C(v)) \right] \right\} / \; v.$$

$$Da = A' \circ Ra' = A' \circ \left[ (\neg A \times V \oplus U \times B) \cap (A \times V \oplus U \times C) \right] \quad (12)$$

$$Db = A' \circ Rb' = A' \circ \left[ (\neg A \times V \cup U \times B) \cap (A \times V \cup U \times C) \right] \quad (13)$$

$$Dgg = A' \circ Rgg' = A' \circ \left[ (A \times V \underset{g}{\Rightarrow} U \times B) \cap (\neg A \times V \underset{g}{\Rightarrow} U \times C) \right] \quad (14)$$

## COMPARISON BETWEEN FUZZY REASONING METHODS

Using (11)-(14), we shall show what the consequences Dm, Da, Db and Dgg will be when A' is

$A' = A$ (15)

$A' =$ very A $(= A^2)$ (16)

$A' =$ more or less A $(= A^{0.5})$ (17)

$A' =$ not A $(= \neg A)$ (18)

$A' =$ not very A $(= \neg A^2)$ (19)

$A' =$ not more or less A $(= \neg A^{0.5})$ (20)

which are typical examples of A'.

TABLE 1 Inference Results by Each Rule

| A' \ D | Dm | Da | Db | Dgg |
|---|---|---|---|---|
| A | $\mu_B \vee (.5 \wedge \mu_C)$ | $\frac{1+\mu_B}{2}$ | $\mu_B \vee .5$ | $\mu_B$ |
| very A | $\mu_B \vee (\frac{3-\sqrt{5}}{2} \wedge \mu_C)$ | $\frac{3+2\mu_B-\sqrt{5+4\mu_B}}{2}$ | $\mu_B \vee \frac{3-\sqrt{5}}{2}$ | $\mu_B$ |
| more or less A | $\mu_B \vee (\frac{\sqrt{5}-1}{2} \wedge \mu_C)$ | $\frac{\sqrt{5+4\mu_B}-1}{2} \wedge \frac{1+\mu_B+\mu_C}{2}$ | $\begin{cases} \mu_B & \cdots \ \mu_B \geq \frac{\sqrt{5}-1}{2} \\ (\mu_B \vee \mu_C \vee .5) \wedge \frac{\sqrt{5}-1}{2} & \cdot\cdot \ \mu_B < \frac{\sqrt{5}-1}{2} \end{cases}$ | $\begin{cases} \sqrt{\mu_B} & \cdots \ \mu_B+\mu_C \geq 1 \\ \mu_B \vee (\sqrt{\mu_B} \wedge \mu_C) & \cdots \ \mu_B+\mu_C < 1 \end{cases}$ |
| not A | $\mu_C \vee (.5 \wedge \mu_B)$ | $\frac{1+\mu_C}{2}$ | $\mu_C \vee .5$ | $\mu_C$ |
| not very A | $\mu_C \vee (\frac{\sqrt{5}-1}{2} \wedge \mu_B)$ | $\frac{2\mu_C-1+\sqrt{5-4\mu_C}}{2} \wedge \frac{1+\mu_C+\mu_B}{2}$ | $\begin{cases} \mu_C & \cdots \ \mu_C \geq \frac{\sqrt{5}-1}{2} \\ (\mu_C \vee \mu_B \vee .5) \wedge \frac{\sqrt{5}-1}{2} & \cdot\cdot \ \mu_C < \frac{\sqrt{5}-1}{2} \end{cases}$ | $\begin{cases} 1-(1-\mu_C)^2 & \cdots \ \mu_B+\mu_C \geq 1 \\ [1-(1-\mu_C)^2 \wedge \mu_B] \vee \mu_C & \cdot\cdot \ \mu_B+\mu_C < 1 \end{cases}$ |
| not more or less A | $\mu_C \vee (\frac{3-\sqrt{5}}{2} \wedge \mu_B)$ | $\frac{3-\sqrt{5-4\mu_C}}{2}$ | $\mu_C \vee \frac{3-\sqrt{5}}{2}$ | $\mu_C$ |

The consequences inferred by all the fuzzy reasoning methods are summarized in Table 1, in which $\mu_B$ and $\mu_C$ stand for $\mu_B(v)$ and $\mu_C(v)$, respectively.

Using Fig.1 we shall show how to obtain the consequences D in Table 1 under each method (Rm', Ra', Rb', Rgg') when A' is A, very A, ..., not more or less A. Because of limitations of space, however, we shall discuss only the case of A' = very A.

(i) The Case of $Rm' = (A \times B) \cup (\neg A \times C)$:

Let A be a fuzzy set and R1, R2 be fuzzy relations, then, in general, the following identity holds for the max-min composition "o".

$$A \circ (R_1 \cup R_2) = (A \circ R_1) \cup (A \circ R_2).$$

Using this fact, (11) will be

$$A' \circ \left[(A \times B) \cup (\neg A \times C)\right]$$
$$= \left[A' \circ (A \times B)\right] \cup \left[A' \circ (\neg A \times C)\right]. \quad (21)$$

Therefore, at A' = very A (= $A^2$), the membership function of $A^2 \circ (A \times B)$ is given as

$$\mu_{A^2 \circ (A \times B)}(v) = \bigvee_u \left[\mu_A(u)^2 \wedge (\mu_A(u) \wedge \mu_B(v))\right]. \quad (22)$$

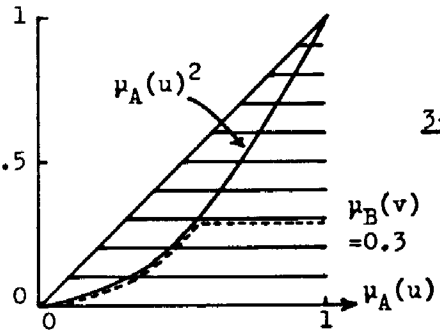


Fig.2(a) $\mu_{A^2 \circ (A \times B)}(v)$

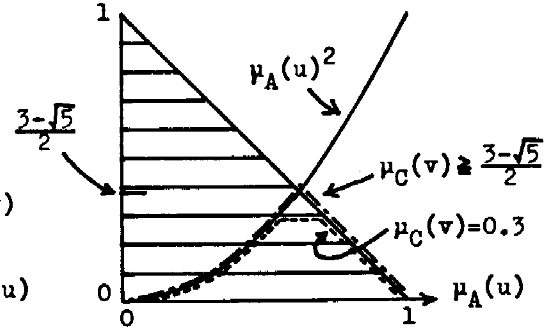


Fig.2(b) $\mu_{A^2 \circ (\neg A \times C)}(v)$

If $\mu_B(v)$ is, say, 0.3, then the expression in [...] of (22) will be shown by the dotted line in Fig.2(a) whose figure comes from the left figure of Fig.1(i). The value $\mu_{A^2 \circ (A \times B)}(v)$ at $\mu_B(v) = 0.3$ becomes 0.3 by taking maximum of this line by virtue of (22). In general, we can have for any $\mu_B(v)$

$$\mu_{A^2 \circ (A \times B)}(v) = \mu_B(v). \quad (23)$$

On the other hand, the membership function of $A^2 \circ (\neg A \times C)$ in (21) is as follows.

$$\mu_{A^2 o(\neg A \times C)}(v) = \bigvee_u \left[\mu_A(u)^2 \wedge (1-\mu_A(u) \wedge \mu_C(v))\right]. \qquad (24)$$

For example, at $\mu_C(v) = 0.3$ ($\leq \frac{3-\sqrt{5}}{2}$) the expression in [...] of (24) is shown by the dotted line '----' in Fig.2(b), and at $\mu_C(v) = 0.7$ ($\geq \frac{3-\sqrt{5}}{2}$) it is shown by the line '—·—·'. Thus, from this figure

$$\mu_{A^2 o(\neg A \times C)}(v) = \begin{cases} \mu_C(v) & \cdots \quad \mu_C(v) \leq \frac{3-\sqrt{5}}{2} (= 0.3819\ldots), \\ \frac{3-\sqrt{5}}{2} & \cdots \quad \mu_C(v) \geq \frac{3-\sqrt{5}}{2}. \end{cases}$$

Stated alternatively,

$$\mu_{A^2 o(\neg A \times C)}(v) = \frac{3-\sqrt{5}}{2} \wedge \mu_C(v). \qquad (25)$$

Finally, we can obtain the membership function of $Dm = A^2 \circ [(A \times B) \cup (\neg A \times C)]$ as follows by taking max ($\vee$) of (23) and (25) by virtue of (21).

$$\mu_{Dm}(v) = \mu_B(v) \vee \left(\frac{3-\sqrt{5}}{2} \wedge \mu_C(v)\right).$$

(ii) <u>The Case of $Ra' = (\neg A \times V \oplus U \times B) \cap (A \times V \oplus U \times C)$</u>:
The membership function $\mu_{Da}$ of (12) is written as

$$\mu_{Da} = \bigvee_u \left\{ \mu_A{}^2 \wedge \left[1 \wedge (1-\mu_A+\mu_B) \wedge (\mu_A+\mu_C)\right] \right\} \qquad (26)$$

by omitting "(u)" and "(v)". In general, the inequality

$$\mu_A{}^2 \leq \mu_A+\mu_C$$

holds for any $\mu_A$ and $\mu_C$, and thus (26) will be

$$\mu_{Da} = \bigvee_u \left\{\mu_A{}^2 \wedge \left[1 \wedge (1-\mu_A+\mu_B)\right]\right\}.$$

This corresponds to the composition of $A^2$ and the fuzzy relation (7). Hence from Mizumoto (1979b) we have $\mu_{Da}$ as

$$\mu_{Da} = \frac{3+2\mu_B-\sqrt{5+4\mu_B}}{2}.$$

(iii) <u>The Case of $Rb' = (\neg A \times V \cup U \times B) \cap (A \times V \cup U \times C)$</u>:
The membership function of Rb' is

$$((1-\mu_A) \vee \mu_B) \wedge (\mu_A \vee \mu_C)$$

from (9). The diagram with both parameters $\mu_B$ and $\mu_C$ is depicted by taking min ($\wedge$) of the left and right figures of Fig.1(iii). For example, in the case of $\mu_C = 0.5$ and $\mu_B = 0.2$, the part ① in Fig.3

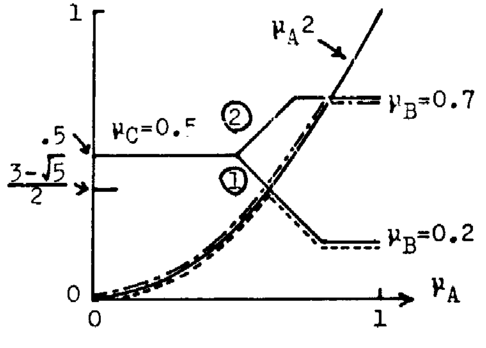


Fig.3 $\mu_{Db}$

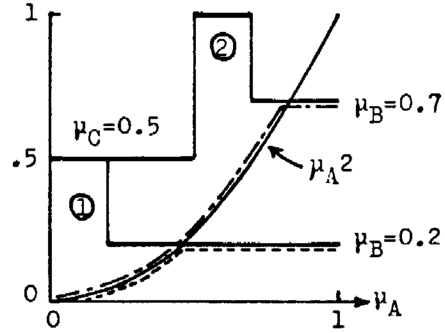


Fig.4 $\mu_{Dgg}$

is obtained, and the part ② is obtained at $\mu_C = 0.5$ and $\mu_B = 0.7$. Therefore, when $\mu_C = 0.5$ and $\mu_B = 0.2$ ($\leq \frac{3-\sqrt{5}}{2}$), the membership function $\mu_{Db}$ in (13) takes $\frac{3-\sqrt{5}}{2}$ as the maximal value in the dotted line '----' in Fig.3. Similarly, when $\mu_C = 0.5$ and $\mu_B = 0.7$ ($\geq \frac{3-\sqrt{5}}{2}$), $\mu_{Db}$ takes 0.7 as the maximum value in the line '—·—'. Thus, in general

$$\mu_{Db} = \begin{cases} \frac{3-\sqrt{5}}{2} & \cdots \quad \mu_B \leq \frac{3-\sqrt{5}}{2}\,, \\ \mu_B & \cdots \quad \mu_B \geq \frac{3-\sqrt{5}}{2}\,. \end{cases}$$

Namely,

$$\mu_{Db} = \mu_B \vee \frac{3-\sqrt{5}}{2}\,.$$

(iv) <u>The Case of Rgg' = $(A \times V \underset{g}{\Rightarrow} U \times B) \cap (\neg A \times V \underset{g}{\Rightarrow} U \times C)$</u>:
The membership function of Rgg' is given by

$$(\mu_A \underset{g}{\rightarrow} \mu_B) \wedge (1-\mu_A \underset{g}{\rightarrow} \mu_C).$$

The diagram with parameters $\mu_B$ and $\mu_C$ is in Fig.4 by taking $\wedge$ of the left and right figures of Fig.1(iv). The part ① in Fig.4 is a diagram at $\mu_C = 0.5$ and $\mu_B = 0.2$, and the part ② is at $\mu_C = 0.5$ and $\mu_B = 0.7$. The membership function $\mu_{Dgg}$ in (14) takes 0.2, 0.7, respectively, when $\mu_B = 0.2$, 0.7. The same holds for any $\mu_C$. Hence, in general

$$\mu_{Dgg} = \mu_B\,.$$

<u>Example</u>: We shall show a simple example using Table 1. When the fuzzy sets A and B, C are as in Fig.5(i)-(ii), the consequences by each method (Rm', Ra', Rb', Rgg') at A' = A, <u>very</u> A, <u>more or less</u> A, <u>not</u> A, <u>not very</u> A and <u>not more or less</u> A are shown in Fig.5(iii)-(viii).

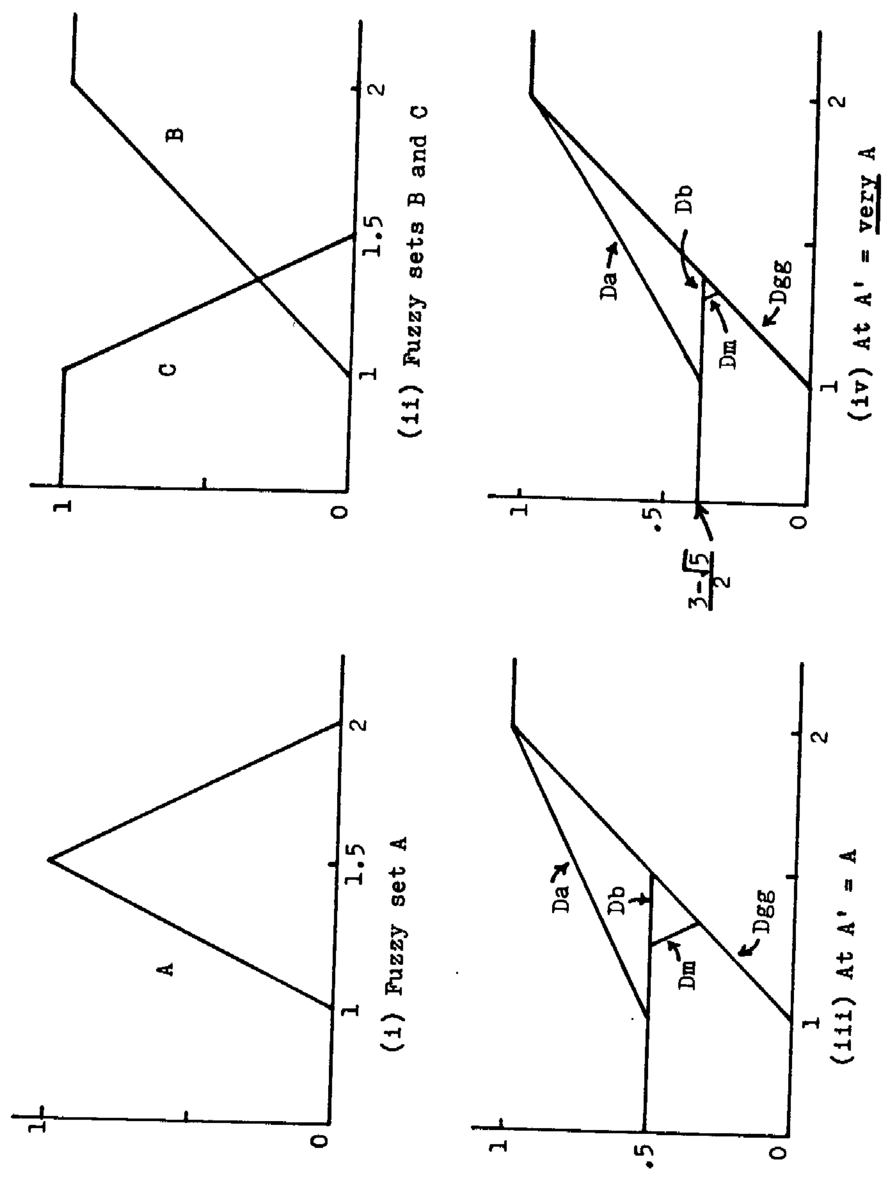


Fig. 5 Fuzzy sets A, B, C, and the consequences Dm, Da, Db, Dgg

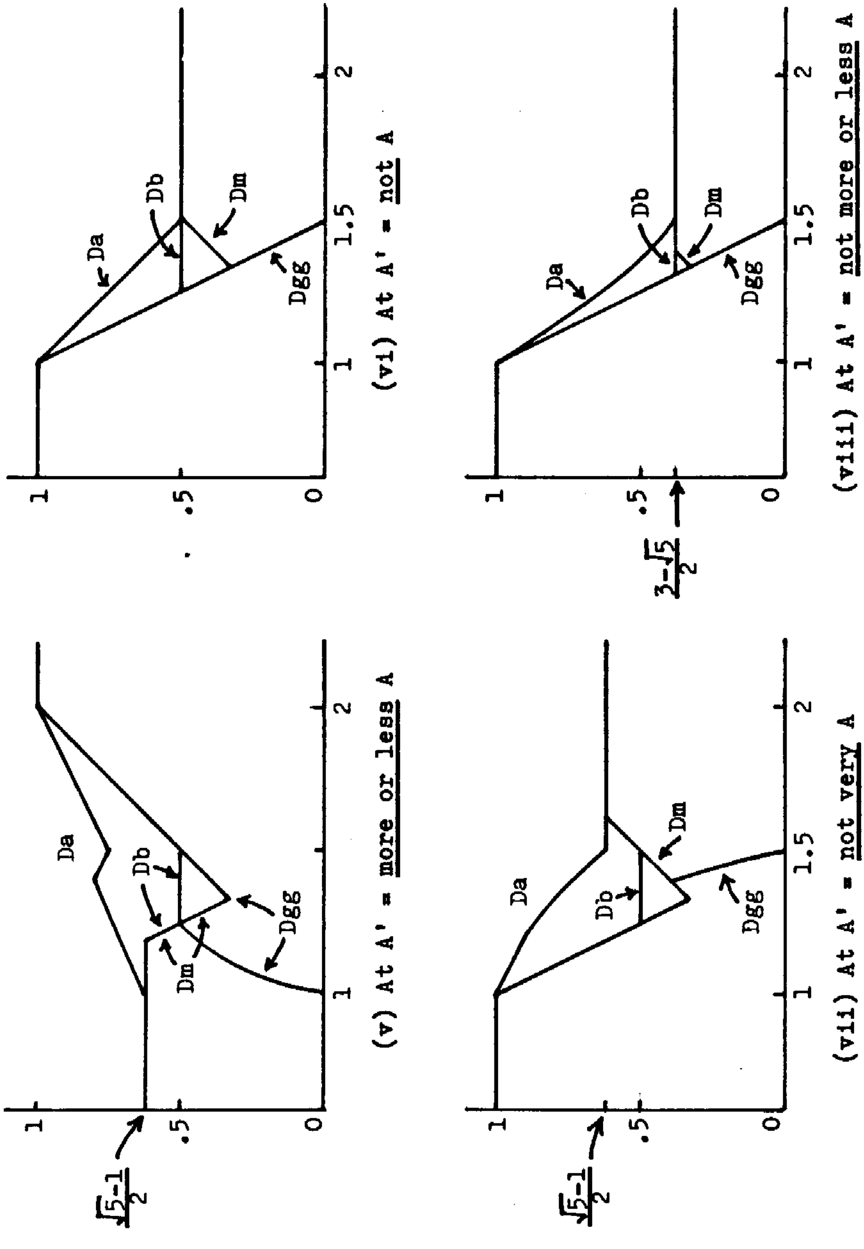


Fig. 5 (continued)

According to our intuitions, it seems that the consequence D in (2) should be B and C, respectively, when A' is A and not A under the conditional proposition "If x is A then y is B else y is C." Namely, the following inferences may be a quite natural demand.

| | | |
|---|---|---|
| Ant 1: | If x is A then y is B else y is C. | |
| Ant 2: | x is A. | (27) |
| Cons: | y is B. | |

| | | |
|---|---|---|
| Ant 1: | If x is A then y is B else y is C. | |
| Ant 2: | x is not A. | (28) |
| Cons: | y is C. | |

From Table 1 and Fig.5 it is found that the consequence Dgg is equal to B (i.e., $\mu_B$) at A' = A, and C (i.e., $\mu_C$) at A' = not A. Namely, the rule Rgg' satisfies the criteria (27) and (28). The other rules Rm', Ra' and Rb' do not satisfy such criteria. It is not yet known, however, what kinds of consequences are most suitable for the consequence D of (2), when A' is equal to very A, more or less A, not very A, and not more or less A. Therefore, we can not conclude that our new rule Rgg' is the best one for the fuzzy conditional inference with "If x is A then y is B else y is C," but we may say that Rgg' is a suitable method for the fuzzy conditional inference, since it satisfies the quite natural criteria (27) and (28).

## CONCLUSION

In this paper we pointed out that the methods Ra', Rm' and Rb' for the fuzzy conditional inference with "If x is A then y is B else y is C" do not give the consequences which fit our intuitions, and gave an improved method Rgg' which fits our intuition under quite natural criteria.

The formalization of inference methods for the more complicated form of inference, such as

If x is $A_1$ then y is $B_1$ else
If x is $A_2$ then y is $B_2$ else
⋮
If x is $A_n$ then y is $B_n$.
x is A'.
―――――――――
y is B'.

would be the future subjects.

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