Reprinted from INFORMATION AND CONTROL

Vol. 50, No. 2, August 1981
Printed in Belgium

Fuzzy Sets and Their Operations, II

MASAHARU MIZUMOTO

Department of Management Engineering,
Osaka Electro-Communication University,
Neyagawa, Osaka 572, Japan

This paper investigates the algebraic properties of fuzzy sets under the new operations "drastic product" and "drastic sum" introduced by Dubois in 1979, and the algebraic properties in the case where these new operations are combined with the well-known operations for fuzzy sets. The properties of fuzzy relations are also shown under a new composition of fuzzy relations which is defined by using the drastic product.

1. INTRODUCTION

As the continuation of our study on "Fuzzy Sets and Their Operations" (Mizumoto and Tanaka, 1981a) which shows the algebraic properties of fuzzy sets under the operations of "bounded-sum," "bounded-difference" and "bounded-product," and the algebraic properties of fuzzy sets when these operations are combined with the well-known operations of intersection, union, algebraic product and algebraic sum, this paper investigates the algebraic properties of fuzzy sets under the new operations "drastic product" and "drastic sum" introduced by Dubois (1979). The properties of fuzzy sets are also obtained in the case where these new operations are combined with the well-known operations of intersection, union; algebraic product, algebraic sum; and bounded-product, bounded-sum. Moreover, the properties of fuzzy relations are briefly discussed under new compositions which are defined by using the drastic product and bounded-product.

The operations of drastic product and drastic sum have found very interesting applications to interactive fuzzy numbers (Dubois and Prade, 1981) and the fuzzy reasoning problem (Mizumoto, 1981b, 1982). Therefore, it will be valuable to investigate the properties of these operations for further applications.

0019-9958/81/080160-15$02.00/0

2. Fuzzy Sets and Their Operations

Let A and B be fuzzy sets in a universe of discourse U, and μ_A and μ_B be the membership functions of A and B, respectively, then the operations over fuzzy sets A and B are listed as follows:

$$\textit{Intersection}: \quad A \cap B \Leftrightarrow \mu_{A \cap B} = \mu_A \wedge \mu_B; \tag{1}$$

$$\textit{Union}: \quad A \cup B \Leftrightarrow \mu_{A \cup B} = \mu_A \vee \mu_B; \tag{2}$$

$$\textit{Algebraic Product}: \quad A \cdot B \Leftrightarrow \mu_{A \cdot B} = \mu_A \mu_B; \tag{3}$$

$$\textit{Algebraic Sum}: \quad A \dot{+} B \Leftrightarrow \mu_{A \dot{+} B} = \mu_A + \mu_B - \mu_A \mu_B; \tag{4}$$

$$\textit{Bounded-Product}: \quad A \odot B \Leftrightarrow \mu_{A \odot B} = 0 \vee (\mu_A + \mu_B - 1); \tag{5}$$

$$\textit{Bounded-Sum}: \quad A \oplus B \Leftrightarrow \mu_{A \oplus B} = 1 \wedge (\mu_A + \mu_B); \tag{6}$$

$$\textit{Drastic Product}: \quad A \mathbin{\dot\cap} B \Leftrightarrow \mu_{A \dot\cap B} = \begin{cases} \mu_A \cdots \mu_B = 1 \\ \mu_B \cdots \mu_A = 1 \\ 0 \cdots \mu_A, \mu_B < 1; \end{cases} \tag{7}$$

$$\textit{Drastic Sum}: \quad A \mathbin{\dot\cup} B \Leftrightarrow \mu_{A \dot\cup B} = \begin{cases} \mu_A \cdots \mu_B = 0 \\ \mu_B \cdots \mu_A = 0 \\ 1 \cdots \mu_A, \mu_B > 0; \end{cases} \tag{8}$$

where the operations of $\wedge$, $\vee$, $+$ and $-$ represent min, max, arithmetic sum and arithmetic difference, respectively.

It is easily checked that the operations of intersection, algebraic product, bounded-product and drastic product are dual to those of union, algebraic sum, bounded-sum and drastic sum, respectively. Drastic product ($\dot\cap$) and drastic sum ($\dot\cup$) for fuzzy sets are corresponding to the operations $Tw(x, y)$ and $Tw^*(x, y)$ which were originally studied by Schweizer and Sklar (1963) as a semigroup operation and then introduced by Dubois (1979) into fuzzy set theory. In this paper we shall rewrite $Tw(x, y)$ as $x \mathbin{\dot\wedge} y$, and $Tw^*(x, y)$ as $x \mathbin{\dot\vee} y$ for convenience. Thus, for $x, y \in [0, 1]$,

$$x \mathbin{\dot\wedge} y = Tw(x, y) = \begin{cases} x \cdots y = 1 \\ y \cdots x = 1 \\ 0 \cdots x, y < 1, \end{cases} \tag{9}$$

$$x \mathbin{\dot\vee} y = Tw^*(x, y) = \begin{cases} x \cdots y = 0 \\ y \cdots x = 0 \\ 1 \cdots x, y > 0. \end{cases} \tag{10}$$

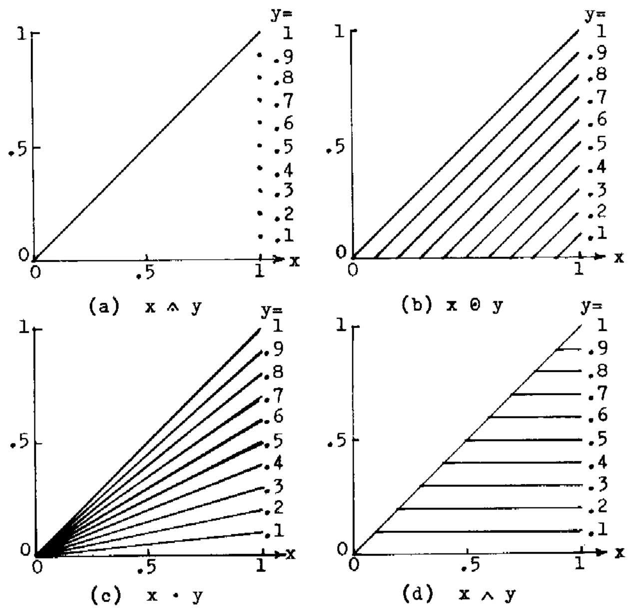

FIG. 1. Diagrams of $\dot{\wedge}$, $\odot$, $\cdot$ and $\wedge$.

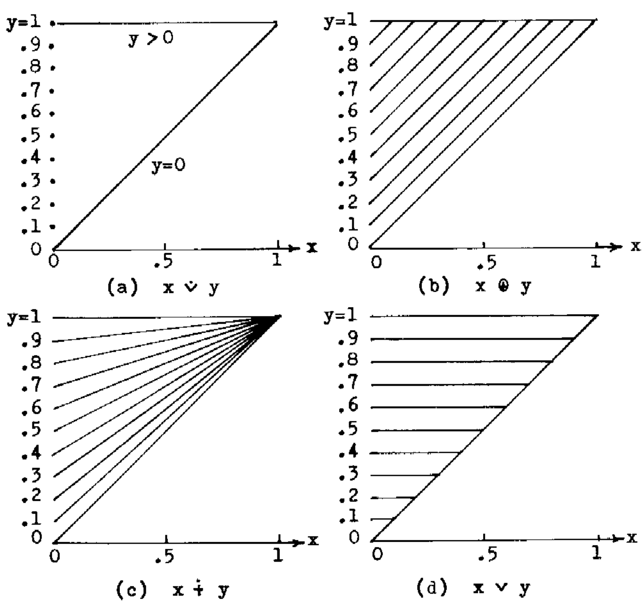

FIG. 2. Diagrams of $\dot{\vee}$, $\oplus$, $\dot{+}$ and $\vee$.

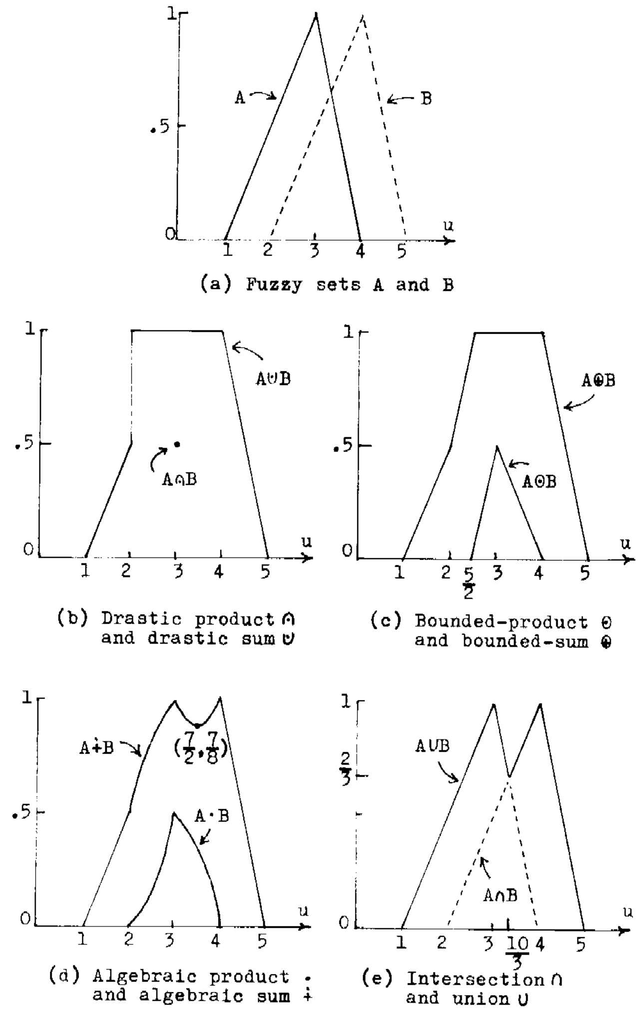

FIG. 3. Operation results of A and B.

The following inequalities hold for these operations: For $x, y \in [0, 1]$,

$$x \mathbin{\dot{\wedge}} y \leqq x \odot y \leqq x \cdot y \leqq x \wedge y, \tag{11}$$

$$x \mathbin{\dot{\vee}} y \geqq x \oplus y \geqq x \dot{+} y \geqq x \vee y, \tag{12}$$

where $\odot$, $\cdot$, $\wedge$, $\oplus$, $\dot{+}$ and $\vee$ stand for bounded-product, algebraic product, min, bounded-sum, algebraic sum and max, respectively, which correspond to the fuzzy set operations in (1)–(6).

It follows from these inequalities that $\dot{\wedge}$ is the most drastic operator, while $\odot$, $\cdot$ and $\wedge$ are less drastic (Dubois and Prade, 1981). In fact, the drastic product $\dot{\wedge}$ is not only the smallest among $\{\dot{\wedge}, \odot, \cdot, \wedge\}$, but also among any semigroup operation $*$ of $[0, 1]$ with identity 1, commutativity and non-decreasingness, while the min $\wedge$ is the greatest among these operations. That is, $x \mathbin{\dot{\wedge}} y \leqq x*y \leqq x \wedge y$. Hence $\dot{\cap}$ is the smallest possible fuzzy set intersection and $\cap$ is the greatest one. The dual property holds for $\dot{\vee}(\dot{\cup})$ and $\vee(\cup)$. See Schweizer and Sklar (1963) and Dubois (1979). Therefore, in this paper we call the operator $\dot{\wedge}$ (as well, $\dot{\cap}$) as "drastic product," and the operator $\dot{\vee}(\dot{\cup})$ as "drastic sum." In Figs. 1–2 these operations are depicted by using a parameter y in order to see how drastic the operations $\dot{\wedge}$ and $\dot{\vee}$ are. Such tendencies can be also observed for the fuzzy set operations in (1)–(8) (see Fig. 3).

From the inequalities in (11) and (12), we can have the following ordering for fuzzy set operations.

$$\begin{aligned} &A \mathbin{\dot{\cap}} B \subseteq A \odot B \subseteq A \cdot B \subseteq A \cap B \\ &\subseteq A \cup B \subseteq A \mathbin{\dot{+}} B \subseteq A \oplus B \subseteq A \mathbin{\dot{\cup}} B. \end{aligned} \tag{13}$$

3. Algebraic Properties of Fuzzy Sets

In this section we shall discuss the algebraic properties of fuzzy sets under the operations of drastic product ($\dot{\cap}$) and drastic sum ($\dot{\cup}$), and the properties of fuzzy sets when these operations are combined with the well-known operations in (1)–(6). Moreover, as a summary of this paper and the previous paper (Mizumoto and Tanaka, 1981a), this section lists the algebraic properties and algebraic structures under all the fuzzy set operations (1)–(8).

I. *The Case of Drastic Product* ($\dot{\cap}$) *and Drastic Sum* ($\dot{\cup}$)

Let A, B and C be fuzzy sets in a universe of discourse U, then we have

Idempotency:
$$\begin{aligned} &A \mathbin{\dot{\cap}} A \subseteq A, \\ &A \mathbin{\dot{\cup}} A \supseteq A; \end{aligned} \tag{14}$$

Commutativity:
$$\begin{aligned} &A \mathbin{\dot{\cap}} B = B \mathbin{\dot{\cap}} A, \\ &A \mathbin{\dot{\cup}} B = B \mathbin{\dot{\cup}} A; \end{aligned} \tag{15}$$

Associativity:
$$\begin{aligned} &A \mathbin{\dot{\cap}} (B \mathbin{\dot{\cap}} C) = (A \mathbin{\dot{\cap}} B) \mathbin{\dot{\cap}} C, \\ &A \mathbin{\dot{\cup}} (B \mathbin{\dot{\cup}} C) = (A \mathbin{\dot{\cup}} B) \mathbin{\dot{\cup}} C; \end{aligned} \tag{16}$$

$$\textit{Absorption:}\quad A \mathbin{\dot{\cap}} (A \mathbin{\dot{\cup}} B) \subseteq A, \qquad A \mathbin{\dot{\cup}} (A \mathbin{\dot{\cap}} B) \supseteq A; \tag{17}$$

$$\textit{Distributivity:}\quad A \mathbin{\dot{\cap}} (B \mathbin{\dot{\cup}} C) \neq (A \mathbin{\dot{\cap}} B) \mathbin{\dot{\cup}} (A \mathbin{\dot{\cap}} C), \qquad A \mathbin{\dot{\cup}} (B \mathbin{\dot{\cap}} C) \neq (A \mathbin{\dot{\cup}} B) \mathbin{\dot{\cap}} (A \mathbin{\dot{\cup}} C); \tag{18}$$

$$\textit{De Morgan's laws:}\quad \overline{A \mathbin{\dot{\cap}} B} = \bar{A} \mathbin{\dot{\cup}} \bar{B}, \qquad \overline{A \mathbin{\dot{\cup}} B} = \bar{A} \mathbin{\dot{\cap}} \bar{B}; \tag{19}$$

$$\textit{Identity:}\quad A \mathbin{\dot{\cap}} U = A, \qquad A \mathbin{\dot{\cup}} \phi = A; \tag{20}$$

$$\textit{Nullity:}\quad A \mathbin{\dot{\cap}} \phi = \phi, \qquad A \mathbin{\dot{\cup}} U = U; \tag{21}$$

$$\textit{Complementarity:}\quad A \mathbin{\dot{\cap}} \bar{A} = \phi, \qquad A \mathbin{\dot{\cup}} \bar{A} = U; \tag{22}$$

where ϕ is an empty set defined by $\mu_\phi = 0$.

THEOREM 1. *Fuzzy sets under* $\dot{\cap}$ *form a commutative semigroup with unity* $(= U)$.[1] *The duality holds for* $\dot{\cup}$. *Fuzzy sets under* $\dot{\cap}$ *and* $\dot{\cup}$ *do not satisfy the absorption and distributive laws and hence they do not form such algebraic structures as a lattice*[2] *and a semiring.*

We shall next examine the absorption and distributive properties for fuzzy sets under the operations $\dot{\cap}$ and $\dot{\cup}$ which are combined with $\cap$ and $\cup$.

II. *The Case of Drastic Product* ($\dot{\cap}$) *and Drastic Sum* ($\dot{\cup}$) *Combined with Intersection* ($\cap$) *and Union* ($\cup$)

$$\textit{Absorption:}\quad A \mathbin{\dot{\cap}} (A \cap B) \subseteq A, \tag{23}$$

$$A \mathbin{\dot{\cap}} (A \cup B) \subseteq A, \tag{24}$$

$$A \mathbin{\dot{\cup}} (A \cap B) \supseteq A; \tag{25}$$

[1] A semigroup $\langle S, * \rangle$ is a set S together with an operation $*$ such that $*$ is associative. A *commutative semigroup with unity* 1 (or a *commutative monoid*), $\langle S, *, 1 \rangle$, is a semigroup such that $*$ is commutative and has a unity 1 such as $a * 1 = 1 * a = a$.

[2] A set L with two operations $\wedge$ and $\vee$ satisfying idempotent laws, commutative laws, associative laws and absorption laws is said to be a lattice and denoted as $\langle L, \wedge, \vee \rangle$. A *pseudo-complemented distributive lattice* $\langle L, \wedge, \vee, I, 0 \rangle$ is a lattice which satisfies distributive laws and has a pseudo-complement for each element of L. I and 0 are the greatest and least elements, such as $a \vee I = I$ and $a \wedge 0 = 0$.

$$A \dot{\cup} (A \cup B) \supseteq A; \tag{26}$$

and $$A \cap (A \dot{\cap} B) \subseteq A, \tag{27}$$

$$A \cap (A \dot{\cup} B) = A, \tag{28}$$

$$A \cup (A \dot{\cap} B) = A, \tag{29}$$

$$A \cup (A \dot{\cup} B) \supseteq A; \tag{30}$$

Distributivity: $$A \dot{\cap} (B \cap C) = (A \dot{\cap} B) \cap (A \dot{\cap} C), \tag{31}$$

$$A \dot{\cap} (B \cup C) = (A \dot{\cap} B) \cup (A \dot{\cap} C), \tag{32}$$

$$A \dot{\cup} (B \cap C) = (A \dot{\cup} B) \cap (A \dot{\cup} C), \tag{33}$$

$$A \dot{\cup} (B \cup C) = (A \dot{\cup} B) \cup (A \dot{\cup} C); \tag{34}$$

and $$A \cap (B \dot{\cap} C) \supseteq (A \cap B) \dot{\cap} (A \cap C), \tag{35}$$

$$A \cap (B \dot{\cup} C) \subseteq (A \cap B) \dot{\cup} (A \cap C), \tag{36}$$

$$A \cup (B \dot{\cap} C) \supseteq (A \cup B) \dot{\cap} (A \cup C), \tag{37}$$

$$A \cup (B \dot{\cup} C) \subseteq (A \cup B) \dot{\cup} (A \cup C). \tag{38}$$

THEOREM 2. *Fuzzy sets form a commutative semiring with unity* $(= U)$ *and zero* $(= \phi)$[3] *under* $\dot{\cap}$ *(as multiplication) and* $\cup$ *(as addition). The duality holds for* $\dot{\cup}$ *and* $\cap$. *Moreover, fuzzy sets constitute a commuative semiring with unity* $(= \phi)$ *under* $\dot{\cup}$ *(as multiplication) and* $\cup$ *(as addition). The duality holds for* $\dot{\cap}$ *and* $\cap$. *Fuzzy sets form a lattice ordered semigroup with unity* $(= U)$ *and zero* $(= \phi)$[4] *under* $\cap$, $\cup$ *and* $\dot{\cap}$, *where* $\dot{\cap}$ *is a semigroup operation. The duality holds for* $\cup$, $\cap$ *and* $\dot{\cup}$.

[3] A semiring $\langle R, \times, + \rangle$ is a set R with two operations of $+$ (addition) and $\times$ (multiplication) such that $+$ is associative and commutative, and $\times$ is associative and distributive over $+$, i.e., $a \times (b + c) = (a \times b) + (a \times c)$. A *commutative semiring with unity* 1, $\langle R, \times, +, 1 \rangle$, is a semiring such that $\times$ is commutative and has a unity 1. Moreover, if $+$ has a unity 0 and satisfies $0 \times a = a \times 0 = 0$, it is said to be a *commutative semiring with unity* 1 *and zero* 0 and written as $\langle R, \times, +, 1, 0 \rangle$.

[4] A lattice $\langle L, \wedge, \vee \rangle$ which is a semigroup under $*$ and also satisfies the following distributive law is called a lattice ordered semigroup $\langle L, \wedge, \vee, * \rangle$. $a * (b \vee c) = (a * b) \vee (a * c)$. A *lattice ordered semigroup with unity I and zero* 0, $\langle L, \wedge, \vee, *, I, 0 \rangle$, is a lattice ordered semigroup satisfying

$$a \vee I = I, \qquad a * I = I * a = a,$$

$$a \vee 0 = a, \qquad a * 0 = 0 * a = 0.$$

III. *The Case of Drastic Product* ($\dot{\cap}$) *and Drastic Sum* ($\dot{\cup}$) *Combined with Algebraic Product* ($\cdot$) *and Algebraic Sum* ($\dot{+}$)

Absorption: $$A \mathbin{\dot{\cap}} (A \cdot B) \subseteq A, \tag{39}$$

$$A \mathbin{\dot{\cap}} (A \mathbin{\dot{+}} B) \subseteq A, \tag{40}$$

$$A \mathbin{\dot{\cup}} (A \cdot B) \supseteq A, \tag{41}$$

$$A \mathbin{\dot{\cup}} (A \mathbin{\dot{+}} B) \supseteq A; \tag{42}$$

and $$A \cdot (A \mathbin{\dot{\cap}} B) \subseteq A, \tag{43}$$

$$A \cdot (A \mathbin{\dot{\cup}} B) \subseteq A, \tag{44}$$

$$A \mathbin{\dot{+}} (A \mathbin{\dot{\cap}} B) \supseteq A, \tag{45}$$

$$A \mathbin{\dot{+}} (A \mathbin{\dot{\cup}} B) \supseteq A; \tag{46}$$

Distributivity: $$A \mathbin{\dot{\cap}} (B \cdot C) \supseteq (A \mathbin{\dot{\cap}} B) \cdot (A \mathbin{\dot{\cap}} C), \tag{47}$$

$$A \mathbin{\dot{\cap}} (B \mathbin{\dot{+}} C) \subseteq (A \mathbin{\dot{\cap}} B) \mathbin{\dot{+}} (A \mathbin{\dot{\cap}} C), \tag{48}$$

$$A \mathbin{\dot{\cup}} (B \cdot C) \supseteq (A \mathbin{\dot{\cup}} B) \cdot (A \mathbin{\dot{\cup}} C), \tag{49}$$

$$A \mathbin{\dot{\cup}} (B \mathbin{\dot{+}} C) \subseteq (A \mathbin{\dot{\cup}} B) \mathbin{\dot{+}} (A \mathbin{\dot{\cup}} C); \tag{50}$$

and $$A \cdot (B \mathbin{\dot{\cap}} C) \supseteq (A \cdot B) \mathbin{\dot{\cap}} (A \cdot C), \tag{51}$$

$$A \cdot (B \mathbin{\dot{\cup}} C) \subseteq (A \cdot B) \mathbin{\dot{\cup}} (A \cdot C), \tag{52}$$

$$A \mathbin{\dot{+}} (B \mathbin{\dot{\cap}} C) \supseteq (A \mathbin{\dot{+}} B) \mathbin{\dot{\cap}} (A \mathbin{\dot{+}} C), \tag{53}$$

$$A \mathbin{\dot{+}} (B \mathbin{\dot{\cup}} C) \subseteq (A \mathbin{\dot{+}} B) \mathbin{\dot{\cup}} (A \mathbin{\dot{+}} C). \tag{54}$$

THEOREM 3. *Fuzzy sets do not form such algebraic structures as a lattice and a semiring under* $\dot{\cap}$ *and* $\dot{+}$. *The same is true of* ($\dot{\cap}, \cdot$), ($\dot{\cup}, \cdot$) *and* ($\dot{\cup}, \dot{+}$).

IV. *The Case of Drastic Product* ($\dot{\cap}$) *and Drastic Sum* ($\dot{\cup}$) *Combined with Bounded-Product* ($\odot$) *and Bounded-Sum* ($\oplus$)

Absorption: $$A \mathbin{\dot{\cap}} (A \odot B) \subseteq A, \tag{55}$$

$$A \mathbin{\dot{\cap}} (A \oplus B) \subseteq A, \tag{56}$$

$$A \mathbin{\dot{\cup}} (A \odot B) \supseteq A, \tag{57}$$

$$A \mathbin{\dot{\cup}} (A \oplus B) \supseteq A; \tag{58}$$

and $$A \odot (A \mathbin{\dot{\cap}} B) \subseteq A, \tag{59}$$

$$A \odot (A \mathbin{\dot{\cup}} B) \subseteq A, \tag{60}$$

$$A \oplus (A \dot{\cap} B) \supseteq A, \tag{61}$$

$$A \oplus (A \dot{\cup} B) \supseteq A; \tag{62}$$

Distributivity:

$$A \dot{\cap} (B \odot C) \supseteq (A \dot{\cap} B) \odot (A \dot{\cap} C), \tag{63}$$

$$A \dot{\cap} (B \oplus C) \subseteq (A \dot{\cap} B) \oplus (A \dot{\cap} C), \tag{64}$$

$$A \dot{\cup} (B \odot C) \supseteq (A \dot{\cup} B) \odot (A \dot{\cup} C), \tag{65}$$

$$A \dot{\cup} (B \oplus C) \subseteq (A \dot{\cup} B) \oplus (A \dot{\cup} C); \tag{66}$$

and

$$A \odot (B \dot{\cap} C) \supseteq (A \odot B) \dot{\cap} (A \odot C), \tag{67}$$

$$A \odot (B \dot{\cup} C) \neq (A \odot B) \dot{\cup} (A \odot C), \tag{68}$$

$$A \oplus (B \dot{\cap} C) \neq (A \oplus B) \dot{\cap} (A \oplus C), \tag{69}$$

$$A \oplus (B \dot{\cup} C) \subseteq (A \oplus B) \dot{\cup} (A \oplus C). \tag{70}$$

THEOREM 4. *Fuzzy sets do not constitute such algebraic structures as a lattice and a semiring under $\dot{\cap}$ and $\oplus$. The same is true of* $(\dot{\cap}, \odot)$, $(\dot{\cup}, \odot)$ *and* $(\dot{\cup}, \oplus)$.

As a summary of this paper and our previous paper (Mizumoto and Tanaka, 1981a), we shall list the algebraic properties of fuzzy sets under all the operations in (1)–(8). Moreover, we shall summarize the algebraic structures which fuzzy sets form under these operations.

Table I lists the algebraic properties of idempotency, commutativity,..., complementarity under the fuzzy set operations. The symbol "=" represents that such a property is satisfied. The symbol "$\subseteq$" means, say, $A \dot{\cap} A \subseteq A$ in the case of $\dot{\cap}$, and "$\supseteq$" means, say, $A \dot{\cup} A \supseteq A$ in the case of $\dot{\cup}$. The

TABLE I

Algebraic Properties under Fuzzy Set Operations

	$\dot{\cap}$	$\dot{\cup}$	$\odot$	$\oplus$	$\cdot$	$\dot{+}$	$\cap$	$\cup$
Idempotency	$\subseteq$	$\supseteq$	$\subseteq$	$\supseteq$	$\subseteq$	$\supseteq$	=	=
Commutativity	=	=	=	=	=	=	=	=
Associativity	=	=	=	=	=	=	=	=
De Morgan's laws	=		=		=		=	
Identity	=	=	=	=	=	=	=	=
Nullity	=	=	=	=	=	=	=	=
Complementarity	=	=	=	=	$\neq$	$\neq$	$\neq$	$\neq$

complementarity under $\cdot$ and $\dotplus$, and $\cap$ and $\cup$ is not satisfied, but we certainly have

$$\phi \subseteq A \cdot \bar{A} \subseteq 0.25U; \qquad 0.75U \subseteq A \dotplus \bar{A} \subseteq U, \tag{71}$$

$$\phi \subseteq A \cap \bar{A} \subseteq 0.5U; \qquad 0.5U \subseteq A \cup \bar{A} \subseteq U, \tag{72}$$

where, for example, $0.25U$ is a fuzzy set defined by $\mu_{0.25U} = 0.25\mu_U = 0.25 \times 1 = 0.25$.

Table II shows the absorption property under each pair of fuzzy set operations. The operations are ordered as in the case of (13). Each symbol of the table means the following. For example,

$$= : \quad A \cap (A \cup B) = A \text{ for the ordered pair } \langle \cap, \cup \rangle,$$

$$\subseteq : \quad A ⩀ (A \odot B) \subseteq A \text{ for } \langle ⩀, \odot \rangle,$$

$$\supseteq : \quad A \cup (A \dotplus B) \supseteq A \text{ for } \langle \cup, \dotplus \rangle.$$

Table III indicates the distributivity under each pair of fuzzy sets operations, where the symbols represent that, for example,

$$= : \quad A ⩀ (B \cup C) = (A ⩀ B) \cup (A ⩀ C) \text{ for the pair } \langle ⩀, \cup \rangle,$$

$$\subseteq : \quad A ⩀ (B \dotplus C) \subseteq (A ⩀ B) \dotplus (A ⩀ C) \text{ for } \langle ⩀, \dotplus \rangle,$$

$$\supseteq : \quad A ⩀ (B \odot C) \supseteq (A ⩀ B) \odot (A ⩀ C) \text{ for } \langle ⩀, \odot \rangle,$$

$$\neq : \quad A \odot (B \cdot C) \neq (A \odot B) \cdot (A \odot C) \text{ for } \langle \odot, \cdot \rangle.$$

TABLE II

Absorption Property under Fuzzy Set Operations

	$⩀$	$\odot$	$\cdot$	$\cap$	$\cup$	$\dotplus$	$\oplus$	$⊍$
$⩀$		$\subseteq$	$\subseteq$	$\subseteq$	$\subseteq$	$\subseteq$	$\subseteq$	$\subseteq$
$\odot$	$\subseteq$		$\subseteq$	$\subseteq$	$\subseteq$	$\subseteq$	$\subseteq$	$\subseteq$
$\cdot$	$\subseteq$	$\subseteq$		$\subseteq$	$\subseteq$	$\subseteq$	$\subseteq$	$\subseteq$
$\cap$	$\subseteq$	$\subseteq$	$\subseteq$		$=$	$=$	$=$	$=$
$\cup$	$=$	$=$	$=$	$=$		$\supseteq$	$\supseteq$	$\supseteq$
$\dotplus$	$\supseteq$	$\supseteq$	$\supseteq$	$\supseteq$	$\supseteq$		$\supseteq$	$\supseteq$
$\oplus$	$\supseteq$	$\supseteq$	$\supseteq$	$\supseteq$	$\supseteq$	$\supseteq$		$\supseteq$
$⊍$	$\supseteq$	$\supseteq$	$\supseteq$	$\supseteq$	$\supseteq$	$\supseteq$	$\supseteq$	

TABLE III

Distributivity under Fuzzy Set Operations

	⩀	$\odot$	$\cdot$	$\cap$	$\cup$	$\dot{+}$	$\oplus$	⊍
⩀		$\supseteq$	$\supseteq$	$=$	$=$	$\subseteq$	$\subseteq$	$\neq$
$\odot$	$\supseteq$		$\neq$	$=$	$=$	$\neq$	$\neq$	$\neq$
$\cdot$	$\supseteq$	$\supseteq$		$=$	$=$	$\subseteq$	$\subseteq$	$\subseteq$
$\cap$	$\supseteq$	$\supseteq$	$\supseteq$		$=$	$\subseteq$	$\subseteq$	$\subseteq$
$\cup$	$\supseteq$	$\supseteq$	$\supseteq$	$=$		$\subseteq$	$\subseteq$	$\subseteq$
$\dot{+}$	$\supseteq$	$\supseteq$	$\supseteq$	$=$	$=$		$\subseteq$	$\subseteq$
$\oplus$	$\neq$	$\neq$	$\neq$	$=$	$=$	$\neq$		$\subseteq$
⊍	$\neq$	$\supseteq$	$\supseteq$	$=$	$=$	$\subseteq$	$\subseteq$	

TABLE IV

Algebraic Structures under Fuzzy Set Operations

Commutative semigroup with unity 1 $\langle S, *, 1\rangle$	$\langle \mathscr{F}$, ⩀, $U\rangle$, $\langle \mathscr{F}, \odot, U\rangle$, $\langle \mathscr{F}, \cdot, U\rangle$, $\langle \mathscr{F}, \cap, U\rangle$ $\langle \mathscr{F}$, ⊍, $\phi\rangle$, $\langle \mathscr{F}, \oplus, \phi\rangle$, $\langle \mathscr{F}, \dot{+}, \phi\rangle$, $\langle \mathscr{F}, \cup, \phi\rangle$
Pseudo-complemented distributive lattice $\langle L, \wedge, \vee, I, 0\rangle$	$\langle \mathscr{F}, \cap, \cup, U, \phi\rangle$ $\langle \mathscr{F}, \cup, \cap, \phi, U\rangle$
Commutative semiring with unity 1 and zero 0 $\langle R, \times, +, 1, 0\rangle$	$\langle \mathscr{F}$, ⩀, $\cup, U, \phi\rangle$, $\langle \mathscr{F}$, ⊍, $\cap, \phi, U\rangle$, $\langle \mathscr{F}, \odot, \cup, U, \phi\rangle$, $\langle \mathscr{F}, \oplus, \cap, \phi, U\rangle$, $\langle \mathscr{F}, \cdot, \cup, U, \phi\rangle$, $\langle \mathscr{F}, \dot{+}, \cap, \phi, U\rangle$
Commutative semiring with unity 1 $\langle R, \times, +, 1\rangle$	$\langle \mathscr{F}$, ⩀, $\cap, U\rangle$, $\langle \mathscr{F}, \odot, \cap, U\rangle$, $\langle \mathscr{F}, \cdot, \cap, U\rangle$ $\langle \mathscr{F}$, ⊍, $\cup, \phi\rangle$, $\langle \mathscr{F}, \oplus, \cup, \phi\rangle$, $\langle \mathscr{F}, \dot{+}, \cup, \phi\rangle$
Lattice ordered semigroup with unity I and zero 0 $\langle L, \wedge, \vee, *, I, 0\rangle$	$\langle \mathscr{F}, \cap, \cup$, ⩀, $U, \phi\rangle$, $\langle \mathscr{F}, \cup, \cap$, ⊍, $\phi, U\rangle$, $\langle \mathscr{F}, \cap, \cup, \odot, U, \phi\rangle$, $\langle \mathscr{F}, \cup, \cap, \oplus, \phi, U\rangle$, $\langle \mathscr{F}, \cap, \cup, \cdot, U, \phi\rangle$, $\langle \mathscr{F}, \cup, \cap, \dot{+}, \phi, U\rangle$

Table IV shows the algebraic structures under fuzzy set operations, where $\mathscr{F}$ is the family of fuzzy sets in U. The definition of each algebraic structure is found in Footnotes 1–4.

4. New Compositions of Fuzzy Relations

We shall briefly investigate new compositions of fuzzy relations obtained by introducing bounded-product $\odot$ and drastic product ⩀. The new composition called max-$\odot$ composition and max-$\wedge\!\!\!\cdot$ composition have been shown by Mizumoto (1981b, 1982) to be very useful to fuzzy reasoning problem: Quite reasonable inference results can be obtained in the fuzzy conditional inference if these new compositions are used in the compositional rule of inference, though good inference results are not obtained in general when a max-min composition is used.

As is well-known, the max-min composition and max-product composition of fuzzy relations are defined as follows:

Let R be a fuzzy relation in $U \times V$ and S be a fuzzy relation in $V \times W$, then we have

Max-min Composition:

$$R \circ S \Leftrightarrow \mu_{R \circ S}(u, w) = \bigvee_{v} \{\mu_R(u, v) \wedge \mu_S(v, w)\}. \tag{73}$$

Max-Product Composition:

$$R \cdot S \Leftrightarrow \mu_{R \cdot S}(u, w) = \bigvee_{v} \{\mu_R(u, v) \cdot \mu_S(v, w)\}. \tag{74}$$

In the same way, we can easily propose new compositions by using bounded-product $\odot$ and drastic product $\wedge\!\!\!\cdot$.

Max-$\odot$ Composition:

$$R \square S \Leftrightarrow \mu_{R \square S}(u, w) = \bigvee_{v} \{\mu_R(u, v) \odot \mu_S(v, w)\}, \tag{75}$$

where

$$x \odot y = 0 \vee (x + y - 1).$$

Max-$\wedge\!\!\!\cdot$ Composition:

$$R \blacktriangle S \Leftrightarrow \mu_{R \blacktriangle S}(u, w) = \bigvee_{v} \{\mu_R(u, v) \wedge\!\!\!\cdot\, \mu_S(v, w)\}, \tag{76}$$

where

$$x \wedge\!\!\!\cdot\, y = \begin{cases} x \cdots y = 1 \\ y \cdots x = 1 \\ 0 \cdots x, y < 1. \end{cases}$$

Similarly, we could define a number of new compositions such as $\dot{+}$-min composition, $\oplus$-product composition and $\vee\!\!\!\cdot - \odot$ composition if $\vee$, $\dot{+}$, $\oplus$, $\vee\!\!\!\cdot$, $\wedge$, $\cdot$, $\odot$ and $\wedge\!\!\!\cdot$ were combined with each other.

EXAMPLE 1. Let R and S be fuzzy relations such as

$$R = \begin{bmatrix} 0.2 & 0.8 & 1 \\ 0.9 & 0.5 & 0.4 \\ 0.3 & 0.9 & 0.1 \end{bmatrix}, \qquad S = \begin{bmatrix} 0.8 & 0.9 & 0.1 \\ 1 & 0.7 & 0.8 \\ 0.1 & 0.4 & 1 \end{bmatrix},$$

then we have $R \circ S$, $R \cdot S$, $R \,\Box\, S$ and $R \blacktriangle S$ in the following.

$$R \circ S = \begin{bmatrix} 0.8 & 0.7 & 1 \\ 0.8 & 0.9 & 0.5 \\ 0.9 & 0.7 & 0.8 \end{bmatrix}, \qquad R \cdot S = \begin{bmatrix} 0.8 & 0.56 & 1 \\ 0.72 & 0.81 & 0.4 \\ 0.9 & 0.63 & 0.72 \end{bmatrix},$$

$$R \,\Box\, S = \begin{bmatrix} 0.8 & 0.5 & 1 \\ 0.7 & 0.8 & 0.4 \\ 0.9 & 0.6 & 0.7 \end{bmatrix}, \qquad R \blacktriangle S = \begin{bmatrix} 0.8 & 0.4 & 1 \\ 0.5 & 0 & 0.4 \\ 0.9 & 0 & 0.1 \end{bmatrix}.$$

As was shown in this example, we can obtain in general

$$R \blacktriangle S \subseteq R \,\Box\, S \subseteq R \cdot S \subseteq R \circ S \tag{77}$$

by virtue of the property of (11) for $\wedge$, $\odot$, $\cdot$ and $\wedge$.

EXAMPLE 2. Let R be a fuzzy relation on the real line which represents "u is approximately equal to v," i.e., "$u \approx v$":

$$\mu_R(u, v) = \max\{0, 1 - |u - v|\}. \tag{78}$$

Then we obtain

$$\mu_{R \circ R}(u, v) = \max\left\{0, 1 - \frac{|u - v|}{2}\right\},$$

$$\mu_{R \cdot R}(u, v) = \begin{cases} \left(1 - \dfrac{|u - v|}{2}\right)^2 & \cdots |u - v| \leqq 2 \\ 0 & \cdots |u - v| \geqq 2, \end{cases}$$

$$\mu_{R \Box R}(u, v) = \max\{0, 1 - |u - v|\},$$

$$\mu_{R \blacktriangle R}(u, v) = \max\{0, 1 - |u - v|\}.$$

Therefore,

$$R \circ R \supseteq R \cdot R \supseteq R, \qquad R \,\Box\, R = R \blacktriangle R = R.$$

From these results, we may say that the max-min composition $R \circ R$ and max-product composition $R \cdot R$ fit our intuition in the case of $R = \approx$. However, it is noted that the max-$\odot$ composition $R \,\Box\, R$ and max-$\wedge$

composition $R \blacktriangle R$ satisfy the transitive law and thus the fuzzy relation R which is reflexive and symmetric in nature becomes a fuzzy equivalence relation (Zadeh, 1971) under each of $\square$ and $\blacktriangle$.

As another example, let us consider a fuzzy relation S which also represents "$u \approx v$" and is defined by

$$\mu_S(u, v) = \max\{0, 1 - (u - v)^2\}. \tag{79}$$

Then we have

$$\mu_{S \circ S}(u, v) = \max \left\{0, 1 - \frac{(u-v)^2}{4}\right\} \geqq \mu_S(u, v),$$

$$\mu_{S \cdot S}(u, v) = \begin{cases} \left(1 - \dfrac{(u-v)^2}{4}\right)^2 & \cdots |u - v| \leqq 2 \\ 0 & \cdots |u - v| \geqq 2 \end{cases}$$

$$\geqq \mu_S(u, v),$$

$$\mu_{S \square S}(u, v) = \max \left\{0, 1 - \frac{(u-v)^2}{2}\right\} \geqq \mu_S(u, v),$$

$$\mu_{S \blacktriangle S}(u, v) = \max\{0, 1 - (u - v)^2\} = \mu_S(u, v).$$

Namely,

$$S \circ S \supseteq S \cdot S \supseteq S \square S \supseteq S \blacktriangle S \ (= S).$$

Thus, the fuzzy relation S also becomes a fuzzy equivalence relation under $\blacktriangle$.

As in the case of max-min composition "$\circ$", we can obtain the following properties under max-product composition "$\cdot$", max-$\odot$ composition "$\square$" and max-$\wedge$ composition "$\blacktriangle$".

Let R, S and T be fuzzy relations on U, and let $* \in \{\circ, \cdot, \square, \blacktriangle\}$, then

$$R \blacktriangle S \subseteq R \square S \subseteq R \cdot S \subseteq R \circ S, \tag{80}$$

$$R * (S * T) = (R * S) * T, \tag{81}$$

$$S \subseteq T \Rightarrow R * S \subseteq R * T, \tag{82}$$

$$R * (S \cup T) = (R * S) \cup (R * T), \tag{83}$$

$$R * (S \cap T) \subseteq (R * S) \cap (R * T), \tag{84}$$

$$I * R = R * I = R, \tag{85}$$

$$0 * R = R * 0 = 0, \tag{86}$$

$$(R * S)^c = S^c * R^c, \tag{87}$$

where I and 0 are identity relation and null relation, respectively, and R^c stands for the converse of R.

5. Conclusion

As was shown by Dubois and Prade (1981) and Mizumoto (1981b, 1982), the new operations of drastic product and drastic sum have found very interesting applications to the problems of fuzzy number and fuzzy reasoning. Therefore, it is hoped that these operations will develop further interesting application fields. For example, it will benefit such problems as fuzzy classification (Tamura *et al.*, 1971; Ozawa, 1978) to discuss a general form of reflexive and symmetric fuzzy relation R which satisfies $R * R = R$ under $* = \blacktriangle, \square, \cdot, \circ$ and to obtain a fuzzy equivalence class and fuzzy partition of R under $*$.

Acknowledgments

This work was attained during the author's stay (Nov. 1980–Aug. 1981) at RWTH Aachen, West Germany, with the assistance of the Alexander von Humboldt Foundation. He acknowledges the invaluable help of Prof. Dr. H.-J. Zimmermann and the members of fuzzy research group at RWTH Aachen.

RECEIVED: June 11, 1981; REVISED: January 8, 1982

References

DUBOIS, D. (1979), Quelques classes d'opérateurs remarquables pour combiner des ensembles flous, *Busefal*, Automne, 29–35.

DUBOIS, D. AND PRADE, H. (1981), Additions of interactive fuzzy numbers, *IEEE Trans. Auto. Contr.* AC-26, 926–936.

MIZUMOTO, M. AND TANAKA, K. (1981a), Fuzzy sets and their operations, I, *Inform. Contr.* **48**, 30–48.

MIZUMOTO, M. (1981b), Note on the arithmetic rule by Zadeh for fuzzy conditional inference, *Cybernet. Systems* **12**, 247–306.

MIZUMOTO, M. (1982), Fuzzy inferences using max-$\wedge$ composition in the compositional rule of inference, *in* "Fuzzy Information and Decision Processes" (Gupta and Sanchez, Eds.), North-Holland, Amsterdam, in press.

OZAWA, K. (1978), Classification of the keyhole shaped tombs by template matching methods, *IEEE Trans. Comput.* **C-27**, 461–467.

SCHWEIZER, B. AND SKLAR, A. (1963), Associative functions and abstract semi-groups, *Publ. Math. Debrecen* **10**, 69–81.

TAMURA, S., HIGUCHI, S., AND TANAKA, K. (1971), Pattern classification based on fuzzy relations, *IEEE Trans. Systems Man Cybernet.* **SMC-1**, 61–66.

ZADEH, L. A. (1971). Similarity relations and fuzzy orderings, *Inform. Sci.* **3**, 177–200.

Printed by the St. Catherine Press Ltd., Tempelhof 41, Bruges, Belgium