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NOTE ON THE ARITHMETIC RULE BY ZADEH FOR FUZZY CONDITIONAL INFERENCE

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This paper shows that Zadeh's arithmetic rule for fuzzy conditional propositions "If x is A then y is B" and "If x is A then y is B else y is C" can infer quite reasonable consequences in a fuzzy conditional inference if new compositions of "max-$\odot$ composition" and "max-$\wedge$ composition" are used in the compositional rule of inference, though, as was pointed out before, this arithmetic rule cannot get suitable consequences in the compositional rule of inference which uses max-min composition. Moreover, it is shown that the arithmetic rule satisfies a syllogism under these two compositions.

INTRODUCTION

In our daily life we often make such an inference that its antecedents and consequences contain fuzzy concepts. Such an inference cannot be made sufficiently by the inference rules of classical two valued logic and many valued logic. In order to make such an inference with fuzzy concepts, Zadeh (1975) suggested an inference rule called "compositional rule of inference." In this compositional rule of inference, he proposed a translation rule named "arithmetic rule" for translating fuzzy conditional propositionals "If x is A then y is B" and "If x is A then y is B else y is C" into fuzzy relations. This arithmetic rule is based on the well-known implication in Lukasiewicz's L_{Aleph1} logic and has become the center of interest in the fuzzy reasoning problems (Baldwin, 1979a, b; Tsukamoto, 1979; Umano, 1978; Zadeh, 1979, 1980).

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In Mizumoto (1978, 1979a, b, c, 1980b, 1981a, c) and Fukami (1980), however, we have pointed out that the consequences inferred by the arithmetic rule do not fit our intuition and do not satisfy quite natural criteria such as modus ponens and modus tollens, and that the arithmetic rule does not satisfy a syllogism.

In this paper, on the contrary, we show that the arithmetic rule can infer quite reasonable consequences which fit our intuition if, instead of the max-min composition usually used in the compositional rule of inference, we use two kinds of compositions called "max-$\odot$ composition" and "max-$\triangle$ composition" in the compositional rule of inference, where $\odot$ is the operation of "bounded-product" which is dual to "bounded-sum" (Zadeh, 1975), and $\triangle$ is the operation of "drastic product" Tw(x,y) introduced by Dubois (1979). Moreover, we show that the syllogism holds under the arithmetic rule by using these two compositions, though the syllogism does not hold under the max-min composition (Mizumoto, 1979b).

ARITHMETIC RULE FOR FUZZY CONDITIONAL INFERENCE

We shall first consider the following form of inference in which a fuzzy conditional proposition "If x is A then y is B" is contained.

Ant 1:	If x is A then y is B.	
Ant 2:	x is A'.	(1)
Cons:	y is B'.	

where x and y are the names of objects, and A, A$'$, B and B$'$ are fuzzy concepts which are represented by fuzzy sets in universes of discourse U, U, V and V, respectively.

An example of this form of inference is:

If a demand is *large* then a price is *high*.
The demand of autos is *highly large*.
―――――――――――――――――――
The price of autos is *very high*.

The form of inference in (1) may be viewed as *generalized modus ponens* which reduces to the classical *modus ponens* when $A' = A$ and $B' = B$.

Furthermore, the following form of inference is also possible which also contains a fuzzy conditional proposition.

$$\begin{array}{ll} \text{Ant 1:} & \text{If x is A then y is B.} \\ \text{Ant 2:} & \text{y is B'.} \\ \hline \text{Cons:} & \text{x is A'.} \end{array} \tag{2}$$

This inference can be viewed as *generalized modus tollens* which reduces to the classical *modus tollens* when $B' = \textit{not}\ B$ and $A' = \textit{not}\ A$.

The Ant 1 of the form "If x is A then y is B" may represent a certain relationship between A and B. From this point of view, Zadeh (1975) proposed a translation rule called "arithmetic rule" for translating the fuzzy conditional proposition "If x is A then y is B" into a fuzzy relation in U × V.

Let A and B be fuzzy sets in U and V, respectively, which are written as

$$A = \int_U \mu_A(u)/u \; ; \qquad B = \int_V \mu_B(v)/v \; . \tag{3}$$

Then we have the *arithmetic rule* as

$$\begin{aligned} Ra &= (\neg A \times V) \oplus (U \times B) \\ &= \int_{U\times V} 1 \wedge (1-\mu_A(u)+\mu_B(v)) \;/\; (u,v) \end{aligned} \tag{4}$$

where ¬, x and ⊕ denote the complement, cartesian product and bounded-sum for fuzzy sets, respectively. It is noted that the arithmetic rule Ra is based on the implication rule in Lukasiewicz's logic $L_{\text{Aleph}1}$ (i.e., $p \to q = 1 \wedge (1 - p + q), p,q \in [0,1]$).

Figure 1 shows the diagram of Ra in which the symbols μ_A and μ_B are used instead of $\mu_A(u)$ and $\mu_B(v)$ for simplicity. The left figure depicted with parameter μ_B will be found to be useful to discuss the generalized modus ponens in (1), and the right figure with parameter μ_A is useful to analyze the generalized modus tollens in (2).

In the generalized modus ponens of (1), the consequence B′ in Cons can be deduced from Ant 1 and Ant 2 by using the *max-min composition* "o" of

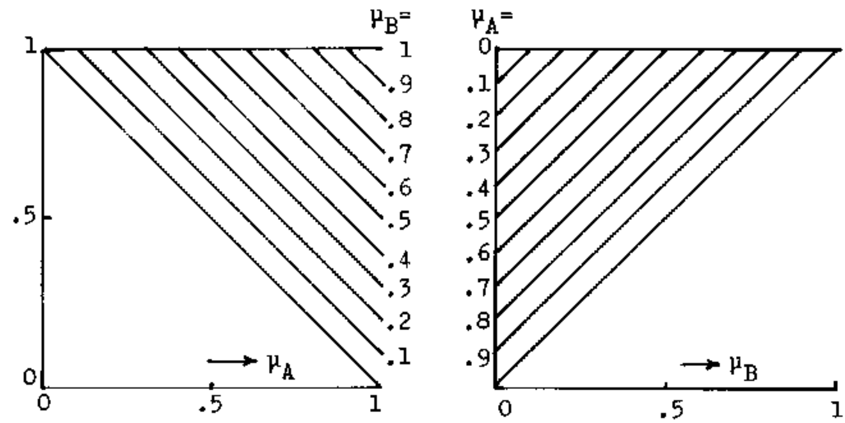

FIGURE 1. Ra: $1 \wedge (1 - \mu_A + \mu_B)$.

the fuzzy set A′ and the fuzzy relation Ra (the *compositional rule of inference*). That is to say,

$$\begin{aligned} B' &= A' \circ Ra \\ &= A' \circ [(\neg A \times V) \oplus (U \times B)] \end{aligned} \tag{5}$$

where the max-min composition o of A′ and Ra is defined as

$$\mu_{A' \circ Ra}(v) = \bigvee_u \{\mu_{A'}(u) \wedge \mu_{Ra}(u,v)\} \tag{6}$$

where $\vee$ and $\wedge$ stand for "max" and "min," respectively. Thus, the membership function of B′ of (5) is given by

$$\mu_{B'}(v) = \bigvee_u \{\mu_{A'}(u) \wedge [1 \wedge (1-\mu_A(u)+\mu_B(v))]\}. \tag{7}$$

Similarly, in the generalized modus tollens of (2), the consequence A′ in Cons can be inferred by using the max-min composition of Ra and B′. Namely,

$$A' = Ra \circ B' \tag{8}$$

$$= \left[(\neg A \times V) \oplus (U \times B)\right] \circ B'$$

$$= \int_U \bigvee_V \left\{ \left[1 \wedge (1-\mu_A(u)+\mu_B(v)\right] \wedge \mu_{B'}(v) \right\} / u \, . \tag{8 (Cont.)}$$

As was indicated in Mizumoto (1979b), for example, when $A' = A$ in the generalized modus ponens, the arithmetic rule infers such a consequence as

$$B' = A \circ \left[(\neg A \times V) \oplus (U \times B)\right]$$
$$= \int_V \frac{1+\mu_B(v)}{2} / v$$
$$\neq B.$$

Similarly, when $B' = not$ B (= $\neg B$) in the generalized modus tollens, we have

$$A' = \left[(\neg A \times V) \oplus (U \times B)\right] \circ \neg B$$
$$= \int_U 1 - \frac{\mu_A(u)}{2} / u$$
$$\neq \neg A.$$

These consequences B' and A' are found not to be equal to B and $\neg A$, respectively. In other words, the arithmetic rule cannot satisfy the following modus ponens and modus tollens which are quite reasonable demands in the fuzzy conditional inference.

If x is A then y is B.
x is A. (modus ponens) (9)

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y is B.

If x is A then y is B.
y is <u>not</u> B. (modus tollens) (10)

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x is <u>not</u> A.

As a generalization of the fuzzy conditional inference with a proposition "If x is A then y is B," Zadeh (1975) also proposed a fuzzy conditional inference of the form:

$$\begin{array}{l} \text{Ant 1: If x is A then y is B else y is C.} \\ \text{Ant 2: x is A'} \\ \hline \text{Cons: y is D.} \end{array} \tag{11}$$

where A, A′, B, C and D are fuzzy sets in U, U, V, V and V, respectively.

An example of such a form of inference which contains a fuzzy conditional proposition "If x is A then y is B else y is C" is the following.

If a demand is *large* then a price is *high* else a price is *fairly low*.
The demand of autos is *fairly large*.

The price of autos is *more or less high*.

For this form of inference with a fuzzy conditional proposition "If x is A then y is B else y is C," he gave a translation rule (arithmetic rule) for translating the proposition "If x is A then y is B else y is C" into a fuzzy relation to U × V.

Let A, B and C be fuzzy sets in U, V and V, respectively, which are represented as

$$A = \int_U \mu_A(u)/u; \; B = \int_V \mu_B(v)/v; \; C = \int_V \mu_C(v)/v \tag{12}$$

Then we have the arithmetic rule as

$$\begin{aligned} Ra' &= (\neg A \times V \oplus U \times B) \cap (A \times V \oplus U \times C) \\ &= \int_{U\times V} [1 \wedge (1-\mu_A(u)+\mu_B(v))] \wedge [1 \wedge (\mu_A(u)+\mu_C(v))] \;/\; (u,v) \\ &= \int_{U\times V} 1 \wedge (1-\mu_A(u)+\mu_B(v)) \wedge (\mu_A(u)+\mu_C(v)) \;/\; (u,v). \end{aligned} \tag{13}$$

Remark: If C is replaced by V (the universe of discourse of C) which is interpreted as "*unknown*," then the fuzzy conditional proposition "If x is A then y is B else y is C" becomes a proposition "If x is A then y is B else y is *unknown*," that is, "If x is A then y is B." Therefore, the arithmetic rule Ra′ in (13) reduces to the arithmetic rule Ra in (4) at C = V, i.e., $\mu_C = 1$.

In Fig. 2 the fuzzy relation Ra′ is illustrated by a diagram in which the symbols μ_A, μ_B and μ_C are used instead of $\mu_A(u)$, $\mu_B(v)$ and $\mu_C(v)$ for simplicity. The left figure shows $1 \wedge (1 - \mu_A + \mu_B)$ using a parameter μ_B, and the right figure shows $1 \wedge (\mu_A + \mu_C)$ with a parameter μ_C. Therefore, the expression $1 \wedge (1 - \mu_A + \mu_B) \wedge (\mu_A + \mu_C)$ with parameters μ_B and μ_C in (13) is obtained by taking min ($\wedge$) of the left and right figures.

The consequence D in Cons of (11) can be inferred from Ant 1 and Ant 2 using the max-min composition "o" of the fuzzy set A′ and the fuzzy relation Ra′. Namely,

$$D = A' \circ Ra' = A' \circ \left[(\neg A \times V \oplus U \times B) \cap (A \times V \oplus U \times C) \right]$$
$$= \int_V \bigvee_u \left\{ \mu_{A'}(u) \wedge \left[1 \wedge (1 - \mu_A(u) + \mu_B(v)) \wedge (\mu_A(u) + \mu_C(v)) \right] \right\} / v. \tag{14}$$

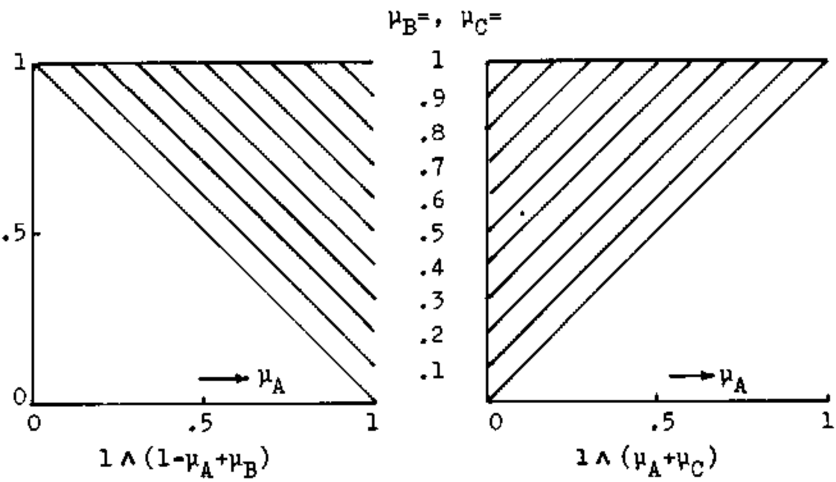


FIGURE 2. Ra′: $1 \wedge (1 - \mu_A + \mu_B) \wedge (\mu_A + \mu_C)$.

For example, when $A' = A$, and *not* A, the consequence D become as follows (Mizumoto, 1980b, 1981c).

$$D = A \circ \left[(\neg A \times V \oplus U \times B) \cap (A \times V \oplus U \times C) \right]$$

$$= \int_V \frac{1+\mu_B(v)}{2} / v$$

$$\neq B.$$

$$D = \neg A \circ \left[(\neg A \times V \oplus U \times B) \cap (A \times V \oplus U \times C) \right]$$

$$= \int_V \frac{1+\mu_C(v)}{2} / v$$

$$\neq C.$$

From these results, it is found that the consequence D is not equal to B at $A' = A$, and to C at $A' = not$ A. Namely, the arithmetic rule Ra' does not satisfy the following criteria which may be quite natural demands.

Ant 1: If x is A then y is B else y is C.
Ant 2: x is A. (15)

Cons: y is B.

Ant 1: If x is A then y is B else y is C.
Ant 2: x is <u>not</u> A. (16)

Cons: y is C.

From the above results it was found that the arithmetic rule does not satisfy the quite reasonable criteria (9), (10), (15) and (16). Therefore, it seems that the arithmetic rule is not a suitable method for the fuzzy conditional inference. But in the next sections we shall show that this arithmetic rule can satisfy these criteria and infer the consequence which fit our intuition in the case that we use new compositions different from the max-min composition in the compositional rule of inference.

MAX-$\odot$ COMPOSITIONS AND MAX-$\wedge$ COMPOSITIONS

We shall first review the properties of the operations of "bounded-product" $\odot$ and "drastic product" $\wedge$ in order to define new compositions of "max-$\odot$ composition" and "max-$\wedge$ composition" which will be used in the compositional rule of inference. The more detailed properties of these operations are found in Dubois (1979, 1980), Prade (1980) and Mizumoto (1980a, 1981b), and their interesting applications to fuzzy numbers are discussed by Dubois (1981).

The operation of "bounded-product" $\odot$ is defined as: For any x, $y \in [0, 1]$,

Bounded-Product

$$x \odot y = 0 \vee (x+y-1) \tag{17}$$

which is the dual operation of bounded-sum $\oplus$ introduced by Zadeh (1975).

The operation of "drastic product" $\wedge$ is the operation Tw(x,y) by Dubois (1979) and is defined by

Drastic Product

$$x \wedge y = \begin{cases} x & \ldots \quad y = 1 \\ y & \ldots \quad x = 1 \\ 0 & \ldots \quad x, y < 1 \end{cases} \tag{18}$$

The following inequality holds for these operations.

$$x \wedge y \leqq x \odot y \leqq x \cdot y \leqq x \wedge y, \qquad \forall\, x, y \in [0,1] \tag{19}$$

where $\cdot$ denotes algebraic product. From this inequality it is seen that $\wedge$ is the most drastic operator, while $\odot$, $\cdot$ and $\wedge$ are less and less drastic (Dubois, 1981). Therefore, we call the operator $\dot{\wedge}$ as "drastic product" in this paper. In Fig. 3 these operations are depicted with a parameter y in order to see how drastic the operator $\dot{\wedge}$ is.

The dual operations to $\odot$ and $\dot{\wedge}$ are defined as follows.

Bounded-Sum

$$x \oplus y = 1 \wedge (x + y) \tag{20}$$

Drastic Sum

$$x \dot{\vee} y = \begin{cases} x & \ldots \quad y = 0 \\ y & \ldots \quad x = 0 \\ 1 & \ldots \quad x, y > 0 \end{cases} \tag{21}$$

The following inequality holds:

$$x \dot{\vee} y \geqq x \oplus y \geqq x \dot{+} y \geqq x \vee y \tag{22}$$

where $\dot{+}$ is algebraic sum which is dual to algebraic product $(\cdot)$ and defined by $x \dot{+} y = x + y - x \cdot y$.

For the bounded-product $\odot$ and drastic product $\dot{\wedge}$, the following properties are obtained. The properties of the bounded-sum $\oplus$ and drastic sum $\dot{\vee}$ are omitted because they are dual to $\odot$ and $\dot{\wedge}$, respectively and they are not used in the discussion of the fuzzy conditional inference.

$$x \leqq y,\ z \leqq w \Rightarrow x \odot z \leqq y \odot w \qquad x \leqq y,\ z \leqq w \Rightarrow x \dot{\wedge} z \leqq y \dot{\wedge} w \tag{23}$$

$$x \odot x \leqq x \qquad x \dot{\wedge} x \leqq x \tag{24}$$

$$x \odot y = y \odot x \qquad x \dot{\wedge} y = y \dot{\wedge} x \tag{25}$$

$$x \odot (y \odot z) = (x \odot y) \odot z \qquad x \dot{\wedge} (y \dot{\wedge} z) = (x \dot{\wedge} y) \dot{\wedge} z \tag{26}$$

$$x \odot (y \oplus z) \neq (x \odot y) \oplus (x \odot z) \qquad x \dot{\wedge} (y \dot{\vee} z) \neq (x \dot{\wedge} y) \dot{\vee} (x \dot{\wedge} z) \tag{27}$$

$$1 - (x \odot y) = (1-x) \oplus (1-y) \qquad 1 - (x \dot{\wedge} y) = (1-x) \dot{\vee} (1-y) \tag{28}$$

$$x \odot 1 = x, \quad x \odot 0 = 0 \qquad x \dot{\wedge} 1 = x, \quad x \dot{\wedge} 0 = 0 \tag{29}$$

$$x \odot (1-x) = 0 \qquad x \dot{\wedge} (1-x) = 0 \tag{30}$$

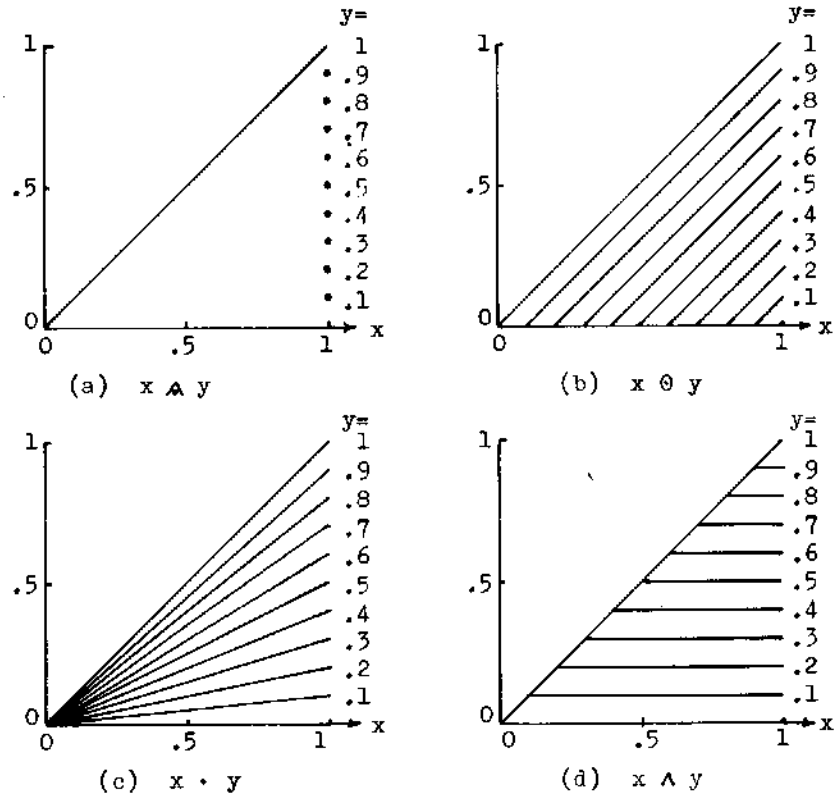


FIGURE 3. Diagrams of $\dot{\wedge}$, $\odot$, $\cdot$ and $\wedge$.

Moreover, the following properties are also given by combining $\odot$, $\dot{\wedge}$, with $\vee$, $\wedge$.

$$x \odot (y \vee z) = (x \odot y) \vee (x \odot z) \qquad x \,\dot{\wedge}\, (y \vee z) = (x \,\dot{\wedge}\, y) \vee (x \,\dot{\wedge}\, z) \qquad (31)$$

$$x \odot (y \wedge z) = (x \odot y) \wedge (x \odot z) \qquad x \,\dot{\wedge}\, (y \wedge z) = (x \,\dot{\wedge}\, y) \wedge (x \,\dot{\wedge}\, z) \qquad (32)$$

$$x \vee (y \odot z) \geqq (x \vee y) \odot (x \vee z) \qquad x \vee (y \,\dot{\wedge}\, z) \geqq (x \vee y) \,\dot{\wedge}\, (x \vee z) \qquad (33)$$

$$x \wedge (y \odot z) \geqq (x \wedge y) \odot (x \wedge z) \qquad x \wedge (y \,\dot{\wedge}\, z) \geqq (x \wedge y) \,\dot{\wedge}\, (x \wedge z) \qquad (34)$$

From these properties we can conclude that the systems $\langle [0,1], \odot \rangle$ and $\langle [0,1], \dot{\wedge} \rangle$ constitute cummutative semigroups with unity 1 (that is, commutative monoids) (Dubois, 1979). The systems $\langle [0,1], \odot, \oplus \rangle$ and $\langle [0,1],$

$\wedge, \vee\rangle$ do not form such algebraic structures as a lattice and a semigroup since they do not satisfy the idempotent laws (24) and distributive laws (27). But the systems $\langle[0,1], \wedge, \vee, \odot\rangle$ and $\langle[0,1], \wedge, \vee, \mathbin{\Lambda}\rangle$ form lattice ordered semigroups with unity 1 and zero 0 since they satisfy the distributive laws (31) and so on. Moreover, $\langle[0,1], \vee, \odot\rangle$ and $\langle[0,1], \vee, \mathbin{\Lambda}\rangle$ form commutative semirings with unity 1 and zero 0. See Mizumoto (1980a, 1981b).

Using the operations of bounded-product $\odot$ and drastic product $\mathbin{\Lambda}$, we can define the operations for fuzzy sets. Let A and B be fuzzy sets in U, then we have

Bounded-Product

$$A \odot B = \int_U \mu_A(u) \odot \mu_B(u) \,/\, u$$
$$= \int_U 0 \vee (\mu_A(u) + \mu_B(u) - 1) \,/\, u \tag{35}$$

Drastic Product

$$A \mathbin{\dot{\cap}} B = \int_U \mu_A(u) \mathbin{\Lambda} \mu_B(u) \,/\, u \tag{36}$$

where

$$\mu_A(u) \mathbin{\Lambda} \mu_B(u) = \begin{cases} \mu_A(u) \;\ldots\; \mu_B(u) = 1 \\ \mu_B(u) \;\ldots\; \mu_A(u) = 1 \\ 0 \qquad \ldots\; \mu_A(u),\ \mu_B(u) < 1 \end{cases}$$

As a simple illustration of using these operations for fuzzy sets, let us consider the fuzzy sets A and B in Fig. 4a which are represented as

$$A = \int_{-\infty}^{0.5} 1/u + \int_{0.5}^{2} \tfrac{2}{3}(2-u)/u$$

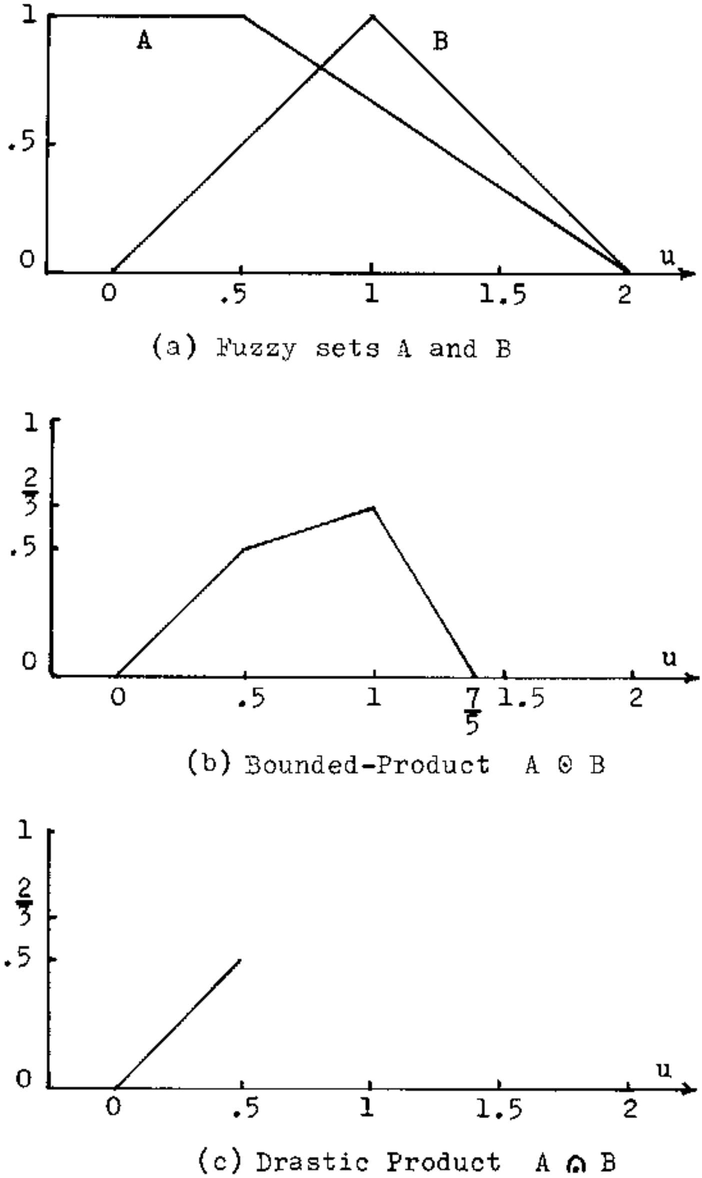


FIGURE 4. Fuzzy sets A and B, A ⊙ B, and A ⋒ B.

$$B = \int_0^1 u/u + \int_1^2 2-u/u$$

Then we have A ⊙ B and A ⋒ B as in Fig. 4b and c which are given by

$$A \odot B = \int_0^{0.5} u/u + \int_{0.5}^{1} \frac{1}{3}(u+1)/u + \int_1^{\frac{7}{5}} \frac{1}{3}(7-5u)/u$$

$$A \Cap B = \int_0^{0.5} u/u + \frac{2}{3}/1$$

We shall next introduce "max-⊙ composition" and "max-⋀ composition" using the bounded-product ⊙ and drastic product ⋀. These compositions are easily defined in the same way as the max-min composition "o."

Let R be a fuzzy relation in U × V and S be a fuzzy relation in V × W, then we can obtain max-⊙ composition "□" and max-⋀ composition "▲" of R and S by the following.

Max-⊙ Composition

$$\mu_{R \,\square\, S}(u,w) = \bigvee_{v} \left\{ \mu_R(u,v) \odot \mu_S(v,w) \right\} \tag{37}$$

Max-⋀ Composition

$$\mu_{R \,\blacktriangle\, S}(u,w) = \bigvee_{v} \left\{ \mu_R(u,v) \wedge\!\!\!\!\cdot\;\, \mu_S(v,w) \right\} \tag{38}$$

Example 1: Let R and S be fuzzy relations such as

$$R = \begin{bmatrix} .2 & .8 & 1 \\ .9 & .5 & .4 \\ .3 & .9 & .1 \end{bmatrix}, \qquad S = \begin{bmatrix} .8 & .9 & .1 \\ 1 & .7 & .8 \\ .1 & .4 & 1 \end{bmatrix}$$

then we have R o S, R · S, R □ S and R ▲ S in the following, where R · S means "max-product composition" (Kaufman, 1975) which is obtained from (37) by replacing Θ by algebraic product (·).

$$R \circ S = \begin{bmatrix} .8 & .7 & 1 \\ .8 & .9 & .5 \\ .9 & .7 & .8 \end{bmatrix}$$

$$R \cdot S = \begin{bmatrix} .8 & .56 & 1 \\ .72 & .81 & .4 \\ .9 & .63 & .72 \end{bmatrix}$$

$$R \blacksquare S = \begin{bmatrix} .8 & .5 & 1 \\ .7 & .8 & .4 \\ .9 & .6 & .7 \end{bmatrix}$$

$$R \blacktriangle S = \begin{bmatrix} .8 & .4 & 1 \\ .5 & 0 & .4 \\ .9 & 0 & .1 \end{bmatrix}$$

As was shown in this example, we have in general

$$R \blacktriangle S \subseteq R \blacksquare S \subseteq R \cdot S \subseteq R \circ S \tag{39}$$

by virtue of the property (19) of Λ, Θ, · and Λ.

Example 2: Let R be a fuzzy relation on the real line which represents "u is approximately equal to v," i.e., "u ≈ v":

$$\mu_R(u,v) = \max(0,\ 1-|u-v|)\quad .$$

then R □ R and R ▲ R become as follows.

$$R \,\square\, R = R \blacktriangle R = R$$

In this connection, using the max-min composition we have

$$\mu_{R\circ R}(u,v) = \max(0,\ 1-\frac{|u-v|}{2})$$

That is,

$$R \circ R \geq R$$

From these results, we may say that the max-min composition R o R fits our intuition. However, it is noted that □ and ▲ satisfy the transitive law and thus the fuzzy relation R which is reflexive and symmetric in nature becomes a fuzzy equivalence relation (Zadeh, 1971) under both □ and ▲.

As another example, let us consider a fuzzy relation S which also represents "u ≈ v" and is defined by

$$\mu_S(u,v) = \max(0,\ 1-(u-v)^2)$$

Then we obtain

$$\mu_{S\circ S}(u,v) = \max(0,\ 1-\frac{(u-v)^2}{4}) \geq \mu_S(u,v)$$

$$\mu_{S\square S}(u,v) = \max(0,\ 1-\frac{(u-v)^2}{2}) \geq \mu_S(u,v)$$

$$\mu_{S\blacktriangle S}(u,v) = \max(0,\ 1-(u-v)^2) = \mu_S(u,v)$$

The fuzzy relation S becomes a fuzzy equivalence relation under ▲.

As in the case of the max-min composition "o," the max-Θ composition "□" and max-A composition "▲" satisfy the following properties.

Let R, S and T be fuzzy relations on U, then we have

$$R \square (S \square T) = (R \square S) \square T \qquad R \blacktriangle (S \blacktriangle T) = (R \blacktriangle S) \blacktriangle T \tag{40}$$

$$S \subseteq T \Rightarrow R \square S \subseteq R \square T \qquad S \subseteq T \Rightarrow R \blacktriangle S \subseteq R \blacktriangle T \tag{41}$$

$$(R \cup S) \square T = (R \square T) \cup (S \square T) \qquad (R \cup S) \blacktriangle T = (R \blacktriangle T) \cup (S \blacktriangle T) \tag{42}$$

$$(R \cap S) \square T \subseteq (R \square T) \cap (S \square T) \qquad (R \cap S) \blacktriangle T \subseteq (R \blacktriangle T) \cap (S \blacktriangle T) \tag{43}$$

$$I \square R = R, \quad O \square R = O \qquad I \blacktriangle R = R, \quad O \blacktriangle R = O \tag{44}$$

$$(R \square S)^c = S^c \square R^c \qquad (R \blacktriangle S)^c = S^c \blacktriangle R^c \tag{45}$$

where I and O are identity relation and null relation, respectively, and R^c stands for the converse of R.

As a special case of the definitions of max-Θ composition (37) and max-$\wedge$ composition (38) of two fuzzy relations, let A be a fuzzy set in U and R be a fuzzy relation in U × V, then the max-Θ composition "□" and max-$\wedge$ composition "▲" of A and R are obtained as

$$\mu_{A \square R}(v) = \bigvee_u \left\{ \mu_A(u) \odot \mu_R(u,v) \right\} \tag{46}$$

$$\mu_{A \blacktriangle R}(v) = \bigvee_u \left\{ \mu_A(u) \wedge \mu_R(u,v) \right\} \tag{47}$$

From (42) and (43) we can have the following properties which will be useful to discuss the fuzzy conditional inference.

$$\begin{aligned} A \square (R_1 \cup R_2) &= (A \square R_1) \cup (A \square R_2), \\ A \blacktriangle (R_1 \cup R_2) &= (A \blacktriangle R_1) \cup (A \blacktriangle R_2) \end{aligned} \tag{48}$$

$$\begin{aligned} A \square (R_1 \cap R_2) &\subseteq (A \square R_1) \cap (A \square R_2), \\ A \blacktriangle (R_1 \cap R_2) &\subseteq (A \blacktriangle R_1) \cap (A \blacktriangle R_2) \end{aligned} \tag{49}$$

$$\begin{aligned} (A_1 \cup A_2) \square R &= (A_1 \square R) \cup (A_2 \square R), \\ (A_1 \cup A_2) \blacktriangle R &= (A_1 \blacktriangle R) \cup (A_2 \blacktriangle R) \end{aligned} \tag{50}$$

$$(A_1 \cap A_2) \boxdot R \subseteq (A_1 \boxdot R) \cap (A_2 \boxdot R),$$
$$(A_1 \cap A_2) \blacktriangle R \subseteq (A_1 \blacktriangle R) \cap (A_2 \blacktriangle R) \tag{51}$$

ARITHMETIC RULE UNDER MAX-Θ COMPOSITION

In this section we shall discuss what consequences can be inferred by the arithmetic rule when the max-Θ composition is used in the compositional rule of inference.

In the generalized modus ponens in (1), we shall show what the consequences B′ become when A′ is

$$A' = A = \int_U \mu_A(u)/u$$

$$A' = \underline{\text{very}}\ A = A^2 = \int_U \mu_A(u)^2/u$$

$$A' = \underline{\text{more or less}}\ A = A^{0.5} = \int_U \sqrt{\mu_A(u)}/u$$

$$A' = \underline{\text{not}}\ A = \neg A = \int_U 1-\mu_A(u)/u$$

which are typical examples of A′.

Similarly, in the generalized modus tollens in (2), we show what the consequences A′ is when B′ is

$$B' = \underline{\text{not}}\ B = \neg B = \int_V 1-\mu_B(v)/v$$

$$B' = \underline{\text{not very}}\ B = \neg B^2 = \int_V 1-\mu_B(v)^2/v$$

$$B' = \underline{\text{not more or less}}\ B = \neg B^{0.5} = \int_V 1-\sqrt{\mu_B(v)}/v$$

$$B' = B = \int_V \mu_B(v)/v$$

We shall begin with the generalized modus ponens in (1). In the same way as (5), the consequence B′ can be deduced from Ant 1 and Ant 2 by the following when we use the max-Θ composition "□" of A′ and Ra in the compositional rule of inference.

$$B' = A' \,\square\, Ra$$

$$= A' \,\square\, \left[(\neg A \times V) \oplus (U \times B)\right] \tag{52}$$

The membership function of B′ is

$$\mu_{B'}(v) = \bigvee_u \left\{ \mu_{A'}(u) \odot \mu_{Ra}(u,v) \right\}$$

$$= \bigvee_u \left\{ \mu_{A'}(u) \odot \left[1 \wedge (1-\mu_A(u)+\mu_B(v))\right] \right\} \tag{53}$$

by using (46). This expression can be simplified by omitting "(u)" and "(v)." Namely,

$$\mu_{B'} = \bigvee_u \left\{\mu_{A'} \odot \left[1 \wedge (1-\mu_A+\mu_B)\right]\right\} \tag{54}$$

Furthermore, this expression can be rewritten as (56) by letting

$$\mu_A = x, \quad \mu_{A'} = x', \quad \mu_B = b, \quad \mu_{B'} = b' \tag{55}$$

if $\mu_A(u)$ takes all values in [0,1] according to u varying all over U, that is, μ_A is a function onto [0,1], i.e., $x \in [0,1]$.

$$b' = \bigvee_x \left\{ x' \odot [1 \wedge (1-x+b)] \right\} \tag{56}$$

and let

$$f(x) = x' \odot [1 \wedge (1-x+b)] \tag{57}$$

Therefore, we shall assume in the generalized modus ponens that μ_A is a function onto [0,1]. Clearly, from this assumption, the fuzzy set A is a normal fuzzy set.

(i) *At* $A' = A$: When A′ is equal to A, i.e., $\mu_{A'} = \mu_A$, x′ becomes x. Thus, using the bounded-product ⊙ of (17), we can have (56) as†

$$\begin{aligned} b' &= \bigvee_x \left\{ x \odot [1 \wedge (1-x+b)] \right\} \\ &= \bigvee_x \left\{ 0 \vee [x + [1 \wedge (1-x+b)] - 1] \right\} \\ &= \bigvee_x \left\{ 0 \vee [(x+1-1) \wedge (x+1-x+b-1)] \right\} \\ &= \bigvee_x \left\{ 0 \vee (x \wedge b) \right\} = \bigvee_x \left\{ x \wedge b \right\} \\ &= b \cdots \text{at } x = 1 \end{aligned}$$

since x can take 1 from the assumption. Therefore, it is obtained that $b' = b$, i.e., $\mu_{B'} = \mu_B$ from (55). In other words, $B' = B$ at $A' = A$. Namely,

$$A \circ Ra = B \tag{58}$$

which shows that the modus ponens (9) is satisfied by the arithmetic rule Ra under the max-⊙ composition.

†For any real numbers x, y and z, we have in general

$$x + (y \wedge z) = (x + y) \wedge (x + z)$$
$$x + (y \vee z) = (x + y) \vee (x + z)$$

(ii) *At* $A' = very\ A$: When $A' = very\ A\ (= A^2)$, i.e., $\mu_{A'} = \mu_A^2$, x′ becomes x^2. Thus, (56) will be

$$
\begin{aligned}
b' &= \bigvee_x \{x^2 \circledcirc [1 \wedge (1-x+b)]\} \\
&= \bigvee_x \{0 \vee [x^2 + [1 \wedge (1-x+b)] - 1]\} \\
&= \bigvee_x \{0 \vee (x^2 \wedge (x^2 - x + b))\} \\
&= \bigvee_x \{x^2 \wedge [0 \vee (x^2 - x + b)]\} \\
&= \bigvee_x \ f(x). \qquad (59)
\end{aligned}
$$

Figure 5 shows the expressions x^2 and $0 \vee (x^2 - x + b)$ using a parameter b. When b is equal to, say, 0.2, f(x) of (59) is indicated by the broken line and hence $\bigvee_x f(x)$ at b = 0.2 becomes 0.2 by taking the maximum of this line. In the same way, at b = 0.7, f(x) is shown by the line "— · — ·" whose maximum value is 0.7. Thus, b′ = 0.7 at b = 0.7. In general we can have b′ = b at $x' = x^2$. That is to say, B′ = B at A′ = *very* A. Namely,

$$\underline{very}\ A \boxdot Ra = B \qquad (60)$$

(iii) *At* $A' =$ *more or less A*: Since x′ becomes $\sqrt{x}$, f(x) of (57) is

$$
\begin{aligned}
f(x) &= \sqrt{x} \circledcirc [1 \wedge (1-x+b)] \\
&= 0 \vee (\sqrt{x} \wedge (\sqrt{x}-x+b)) \\
&= \sqrt{x} \wedge (\sqrt{x}-x+b) \quad \ldots \quad \sqrt{x}-x+b \geqq 0 \qquad (61)
\end{aligned}
$$

The expressions $\sqrt{x}$ and $\sqrt{x} - x + b$ are depicted in Fig. 6 by using a parameter b. When b = 0.1, f(x) is shown by the line "· - - · -" whose maximum value equals to the maximum value of $\sqrt{x} - x + 0.1$. The expression $\sqrt{x} - x + b$ takes the maximum value (= b + 0.25) at x = 0.25. Thus, we

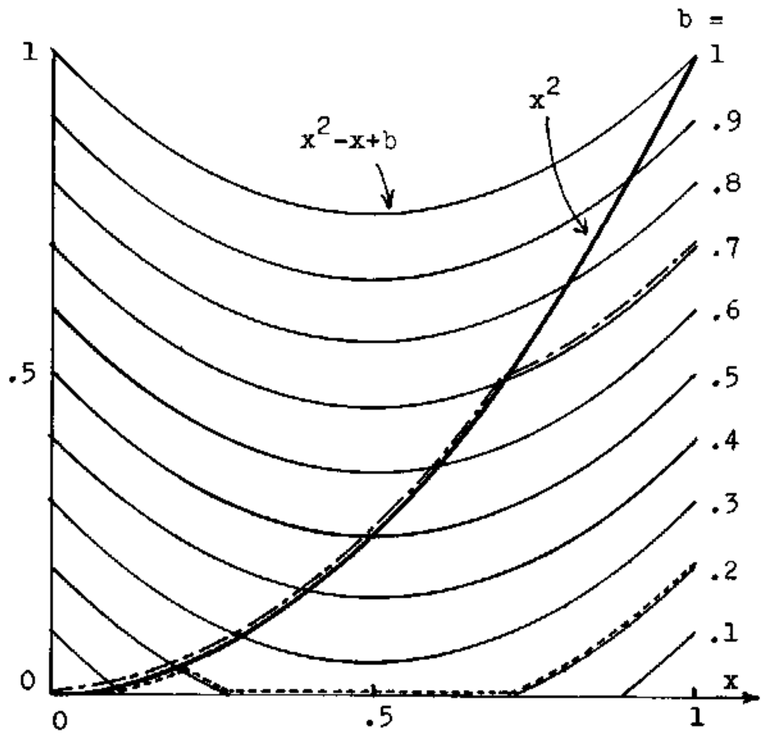


FIGURE 5. x^2 and $0 \vee (x^2 - x + b)$ in (59).

have $\vee_x\ f(x) = 0.1 + 0.25 = 0.35$ at $b = 0.1$. From this figure, it is found that $\vee_x\ f(x) = b + 0.25$ so long as $b \leqslant 0.25$. On the other hand, when $b = 0.7$ ($\geqslant 0.25$), f(x) is indicated by the line "— - — -". The maximum value of f(x) is equal to the height ($= \sqrt{b}$) of the cross point of $\sqrt{x}$ and $\sqrt{x} - x + b$. Thus, $\vee_x\ f(x) = \sqrt{0.7}$ at $b = 0.7$. In general, we can obtain $\vee_x\ f(x) = \sqrt{b}$ so long as $b \geqslant 0.25$. After all,

$$b' = \bigvee_x f(x) = \begin{cases} b + \frac{1}{4} & \cdots \quad b \leqq \frac{1}{4} \\ \sqrt{b} & \cdots \quad b \geqq \frac{1}{4} \end{cases}$$

Note that the black circles in the figure indicate the maximum value of f(x) for each parameter b. Therefore,

$\underline{\text{more or less}}$ A ∘ Ra = B'

where

$$\mu_{B'} = \begin{cases} \mu_B + \frac{1}{4} & \cdots \quad \mu_B \leq \frac{1}{4} \\ \sqrt{\mu_B} & \cdots \quad \mu_B \geq \frac{1}{4} \end{cases} \tag{62}$$

Since this fuzzy set B′ can be approximately represented by *almost more or less* **B**, we have

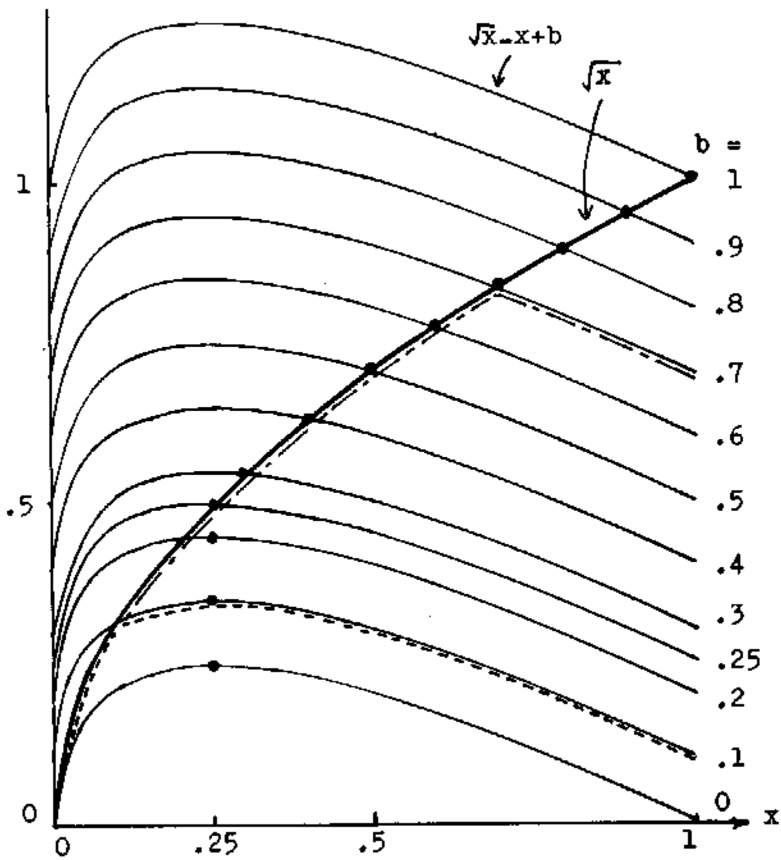


FIGURE 6. $\sqrt{x}$ and $\sqrt{x} - x + b$ in (61).

$$\underline{\text{more or less}}\ A \circ Ra = \underline{\text{almost more or less}}\ B \tag{63}$$

(iv) *At* $A' = \textit{not } A$: Since $x' = 1 - x$, we have (56) as

$$\begin{aligned} b' &= \bigvee_x \left\{ (1-x) \odot \left[1 \wedge (1-x+b) \right] \right\} \\ &= \bigvee_x \left\{ (1-x) \wedge \left[0 \vee (-2x+1+b) \right] \right\} \\ &= 1 \quad \ldots \quad \text{at } x = 0 \end{aligned}$$

because x can take O from the assumption. In the sequel,

$$\underline{\text{not}}\ A \circ Ra = \underline{\text{unknown}} \tag{64}$$

We shall next deal with the generalized modus tollens in (2). As in (8), the consequence A′ can be obtained by using the max-Θ composition of Ra and B′.

$$\begin{aligned} A' &= Ra \circ B' \\ &= \left[(\neg A \times V) \oplus (U \times B) \right] \circ B' \end{aligned} \tag{65}$$

and its membership function becomes

$$\mu_{A'} = \bigvee_v \left\{ \left[1 \wedge (1-\mu_A+\mu_B) \right] \odot \mu_{B'} \right\} \tag{66}$$

by omitting "(u)" and "(v)." This expression can be written as in (68) by letting

$$\mu_A = a, \quad \mu_{A'} = a', \quad \mu_B = x, \quad \mu_{B'} = x' \tag{67}$$

if μ_B is a function onto [0,1].

$$a' = \bigvee_x \left\{ \left[1 \wedge (1-a+x) \right] \odot x' \right\} \tag{68}$$

Therefore, we assume that μ_B is a function onto [0,1] in the generalized modus tollens.

(v) *At* $B' = not\ B$: When B′ is *not* B (= 7B), i.e., $\mu_{B'} = 1 - \mu_B$, x′ becomes 1 − x from (67). Then (68) is as follows.

$$a' = \bigvee_x \left\{ \left[1 \wedge (1-a+x) \right] \odot (1-x) \right\}$$

$$= \bigvee_x \left\{ 0 \vee \left[\left[1 \wedge (1-a+x) \right] + (1-x) - 1 \right] \right\}$$

$$= \bigvee_x \left\{ 0 \vee \left[(1-x) \wedge (1-a) \right] \right\}$$

$$= \bigvee_x \left\{ (1-x) \wedge (1-a) \right\}$$

$$= 1-a \quad \ldots \quad \text{at } x = 0$$

since x can take O from the assumption that μ_B (= x) is onto [0,1]. Thus, a′ = 1 − a, i.e., $\mu_{A'} = 1 - \mu_A$, which leads to A′ = *not* A at B′ = *not* B, that is,

$$Ra \ \square \ \underline{not}\ B = \underline{not}\ A \tag{69}$$

This result shows that the modus tollens (10) is satisfied by the arithmetic rule Ra under the max-Θ composition "□."

The consequences A′ at B′ = *not very* B, *not more or less* B, and B are obtained in the same way as the cases of A′ = A, *very* A, . . . , *not* A, and B′ = *not* B discussed above. We shall omit these methods because of limitations of space. The consequences are given by the following.

(vi) *At B′ = not very B*:

$$Ra \ \square \ \underline{not\ very}\ B = A'$$

where A′ is

$$\mu_{A'} = \begin{cases} 1 - \mu_A^2 & \cdots \quad \mu_A \leq \frac{1}{2} \\ \frac{1}{4} + (1-\mu_A) & \cdots \quad \mu_A \geq \frac{1}{2} \end{cases} \tag{70}$$

and is approximately represented by *almost not very* A. Thus,

$$Ra \circ \underline{\text{not very}}\ B = \underline{\text{almost not very}}\ A \tag{71}$$

(vii) *At B′ = not more or less B*:

$$Ra \circ \underline{\text{not more or less}}\ B = \underline{\text{not}}\ A \tag{72}$$

(viii) *At B′ = B*:

$$Ra \circ B = \underline{\text{unknown}} \tag{73}$$

Stated in English, these inferences obtained in (i)-(viii) can be expressed as follows.

If x is A then y is B.
x is A. (modus ponens) (74)

y is B.

If x is A then y is B.
x is very A. (75)

y is B.

If x is A then y is B.
x is more or less A. (76)

y is almost more or less B.

If x is A then y is B.
x is not A. (77)

y is unknown.

If x is A then y is B.
y is <u>not</u> B. (modus tollens) (78)

---

x is <u>not</u> A.

If x is A then y is B.
y is <u>not very</u> B. (79)

---

x is <u>almost not very</u> A.

If x is A then y is B.
y is <u>not more or less</u> B. (80)

---

x is <u>not</u> A.

If x is A then y is B.
y is B. (81)

---

x is <u>unknown</u>.

From these results it is found that the consequences inferred by the arithmetic rule under the max-Θ composition are quite reasonable consequences and fit our intuition.

Finally, we shall consider the fuzzy conditional inference of (11) with a fuzzy conditional proposition "If x is A then y is B else y is C." The consequence D in Cons of (11) can be obtained from Ant 1 and Ant 2 by using the max-Θ composition "□" of the fuzzy set A′ and the fuzzy relation Ra′ (13).

$$D = A' \boxdot Ra' = A' \boxdot \left[(\neg A \times V \oplus U \times B) \cap (A \times V \oplus U \times C) \right] \tag{82}$$

The membership function of D is given by

$$\mu_D(v) = \bigvee_u \left\{ \mu_{A'}(u) \odot \left[1 \wedge (1-\mu_A(u)+\mu_B(v)) \wedge (\mu_A(u)+\mu_C(v)) \right] \right\} \tag{83}$$

If $\mu_A(u)$ takes all values in [0,1] according to u varying all over U, that is, μ_A is a function onto [0,1], the expression (83) can be rewritten as (85) by letting

$$\mu_A = x,\ \mu_{A'} = x',\ \mu_B = b,\ \mu_C = c,\ \mu_D = d \quad (84)$$

$$d = \bigvee_x \left\{ x' \odot \left[1 \wedge (1-x+b) \wedge (x+c) \right] \right\} \quad (85)$$

and let

$$f(x) = x' \odot \left[1 \wedge (1-x+b) \wedge (x+c) \right] \quad (86)$$

Now we shall show what the consequences D, i.e., d, will be when A′ is

A′ = A

A′ = <u>very</u> A (= A^2)

A′ = <u>more or less</u> A (= $A^{0.5}$)

A′ = <u>not</u> A (= $\neg A$)

A′ = <u>not very</u> A (= $\neg A^2$)

A′ = <u>not more or less</u> A (= $\neg A^{0.5}$)

(i) *At A′ = A*: When A′ = A, x′ becomes x from (84). Thus, (85) will be

$$\begin{aligned} d &= \bigvee_x \left\{ x \odot \left[1 \wedge (1-x+b) \wedge (x+c) \right] \right\} \\ &= \bigvee_x \left\{ 0 \vee \left[x + \left[1 \wedge (1-x+b) \wedge (x+c) \right] - 1 \right] \right\} \\ &= \bigvee_x \left\{ 0 \vee \left[x \wedge b \wedge (2x-1+c) \right] \right\} \\ &= \bigvee_x \left\{ x \wedge b \wedge \left[0 \vee (2x-1+c) \right] \right\} = \bigvee_x f(x) \end{aligned} \quad (87)$$

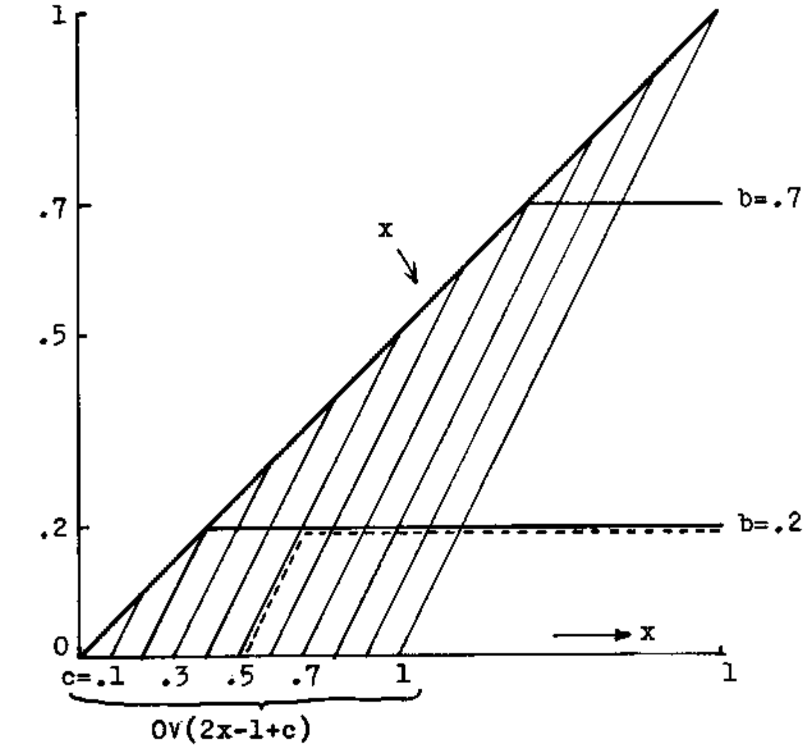


FIGURE 7. x and $0 \vee (2x - 1 + c)$.

In Fig. 7 the expressions x and $0 \vee (2x - 1 + c)$ are depicted partly by using a parameter c. For example, when $b = 0.2$ and $c = 0.5$, f(x) of (87) is indicated by the line "- - - -" and its maximum value is 0.2. For any parameter c we can have 0.2 as the maximum value. Thus $d = \vee_x f(x) = 0.2$ at $b = 0.2$. Similarly, when $b = 0.7$ we obtain $d = \vee_x f(x) = 0.7$ for any c. Therefore, in general, we have $d = b$ at $x' = x$, i.e., $D = B$ at $A' = A$. Stated alternatively,

$$A \,\Box\, Ra' = B \tag{88}$$

It is found from this result that the criterion (15) is satisfied by the arithmetic rule Ra′ under the max-⊙ composition "□."

(ii) *At A′ = very A*: When $A' = very\ A\ (= A^2)$, x′ is x^2. Then f(x) of (86) is given by

$$f(x) = x^2 \odot [1 \wedge (1-x+b) \wedge (x+c)] \tag{89}$$

$$= 0 \vee \{x^2 + [1 \wedge (1-x+b) \wedge (x+c)] - 1\}$$

$$= 0 \vee \{x^2 \wedge (x^2-x+b) \wedge (x^2+x+c-1)\}$$

$$= x^2 \wedge [0 \vee (x^2-x+b)] \wedge [0 \vee (x^2+x+c-1)] \quad \text{(89) (Cont.)}$$

Figure 8 shows the expressions x^2 and $0 \vee (x^2 + x + c - 1)$ with a parameter c. When $b = 0.7$ and $c = 0.5$, f(x) of (89) is shown by the line "- - - -". The maximum value of this line is 0.7. In the same way, we have 0.7 for any parameter c. Thus, $d = \vee_x \; f(x) = 0.7$ at $b = 0.7$. Similarly, when $b = 0.2$, we have $d = \vee_x \; f(x) = 0.2$ for any c. Therefore, in general, for any b, we obtain $d = b$ at $x' = x^2$, i.e., $D = B$ at $A = A^2$. Hence,

$$\underline{\text{very}}\ A \;\square\; Ra' = B \quad (90)$$

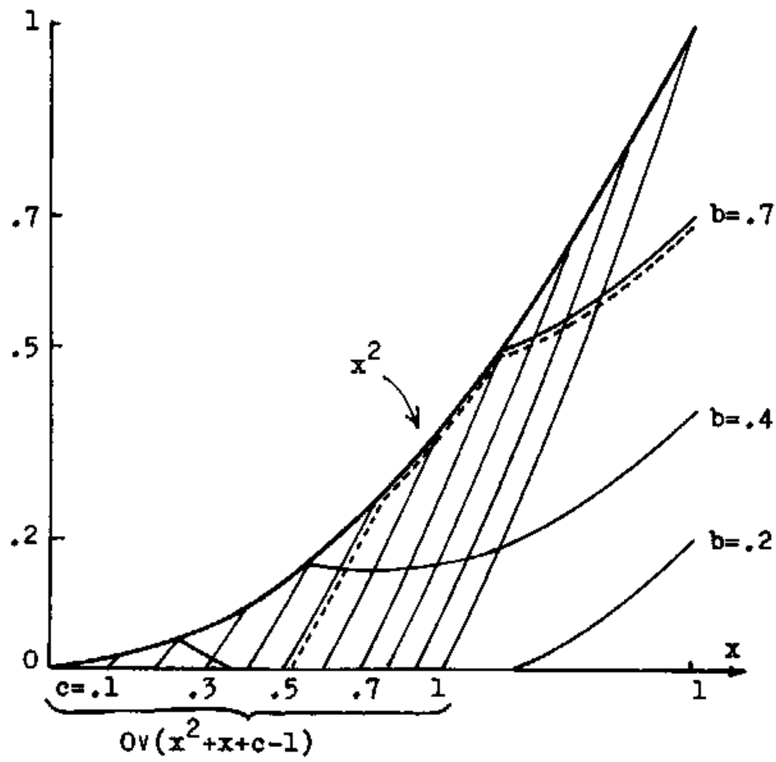


FIGURE 8. x^2, $0 \vee (x^2 + x + c - 1)$ and $0 \vee (x^2 - x + b)$.

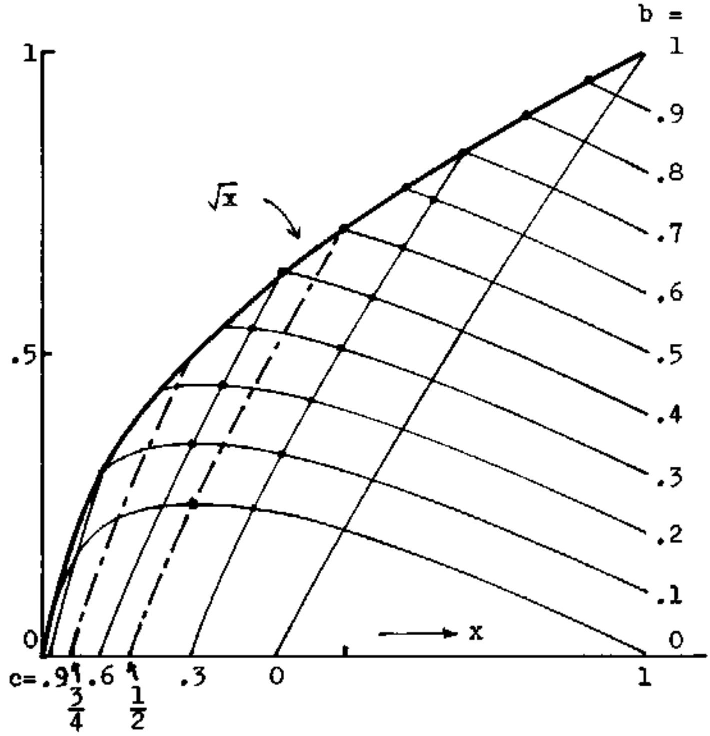


FIGURE 9. $\sqrt{x}$, $\sqrt{x} - x + b$, and $0 \vee (\sqrt{x} + x - 1 + c)$ of (91).

(iii) *At A' = more or less A*: Since $x' = \sqrt{x}$, f(x) of (86) is

$$\begin{aligned} f(x) &= \sqrt{x} \circledcirc [1 \wedge (1-x+b) \wedge (x+c)] \\ &= 0 \vee \{\sqrt{x} \wedge (\sqrt{x}-x+b) \wedge (\sqrt{x}+x-1+c)\} \\ &= \sqrt{x} \wedge (\sqrt{x}-x+b) \wedge [0 \vee (\sqrt{x}+x-1+c)] \end{aligned} \tag{91}$$

In Fig. 9 the expressions $\sqrt{x}$, $\sqrt{x} - x + b$ and $0 \vee (\sqrt{x} + x - 1 + c)$ are drawn partly using parameters b and c, respectively.

(a) *Case of* $c \leqslant \frac{1}{2}$: For example, when c = 0.3 and $b \leqslant 0.7$ (= 1 − c), the maximum value of $f(x) = \sqrt{x} \wedge (\sqrt{x} - x + b) \wedge [0 \vee (\sqrt{x} + x - 1 + 0.3)]$ is given as the height of the cross point of $\sqrt{x} - x + b$ and $\sqrt{x} + x - 1 + 0.3$. Thus, in general, when $c \leqslant 0.5$ and $b \leqslant 1 - c$, the maximum value of f(x) is equal to the height of the cross point of $\sqrt{x} - x + b$ and $\sqrt{x} + x - 1 + c$. The height is given by $\sqrt{(b+1-c)/2} + (b+c-1)/2$. Therefore,

$$d = \bigvee_x f(x) = \sqrt{\frac{b+1-c}{2}} + \frac{b+c-1}{2} \quad \cdots \quad c \leq 0.5,\ b \leq 1-c \tag{92}$$

On the other hand, when $b \geqslant 1 - c$, the maximum value of f(x) is given as the height ($= \sqrt{b}$) of the cross point of $\sqrt{x}$ and $\sqrt{x} - x + b$. Thus,

$$d = \bigvee_x f(x) = \sqrt{b} \quad \cdots \quad c \leq 0.5, \quad b \geq 1-c \tag{93}$$

(b) *Case of* $\frac{1}{2} \leqslant c \leqslant \frac{3}{4}$: Let us consider the case of $c = 0.6$. When $b \leqslant 0.1$ ($= c - \frac{1}{2}$), the maximum value of f(x) is equal to the maximum value ($= b + \frac{1}{4}$) of $\sqrt{x} - x + b$. When $0.1 \leqslant b \leqslant 0.4$ ($= 1 - c$), the maximum value of f(x) is given as the height $[= \sqrt{(b+1-c)/2} + (b+c-1)/2]$ of the cross point of $\sqrt{x} + x - 1 + c$ and $\sqrt{x} - x + b$. When $b \geqslant 0.4$, the maximum value of f(x) equals to the height ($= \sqrt{b}$) of the cross point of $\sqrt{x}$ and $\sqrt{x} - x + b$. Therefore, for $\frac{1}{2} \leqslant c \leqslant \frac{3}{4}$, we can have in general

$$d = \bigvee_x f(x) = \begin{cases} b + \frac{1}{4} & \cdots \quad b \leq c - \frac{1}{2} \\ \sqrt{\frac{b+1-c}{2}} + \frac{b+c-1}{2} & \cdots \quad c - \frac{1}{2} \leq b \leq 1-c \\ \sqrt{b} & \cdots \quad b \geq 1-c \end{cases} \tag{94}$$

(c) *Case of* $c \geqslant \frac{3}{4}$: When $b \leqslant \frac{1}{4}$ the maximum value of f(x) is equal to the maximum value ($= b + \frac{1}{4}$) of $\sqrt{x} - x + b$. When $b \geqslant \frac{1}{4}$, the maximum value is the height ($= \sqrt{b}$) of the cross point of $\sqrt{x}$ and $\sqrt{x} - x + b$. Therefore, in general,

$$d = \bigvee_x f(x) = \begin{cases} b + \frac{1}{4} & \cdots \quad b \leq \frac{1}{4} \\ \sqrt{b} & \cdots \quad b \geq \frac{1}{4} \end{cases} \tag{95}$$

In the sequel, d, i.e., μ_D is given from three cases of (a), (b) and (c) by the following.

$$\mu_D = \begin{cases} \left.\begin{array}{ll} \sqrt{\dfrac{\mu_B+1-\mu_C}{2}} + \dfrac{\mu_B+\mu_C-1}{2} & \cdots\ \mu_B \leqq 1-\mu_C \\ \sqrt{\mu_B} & \cdots\ \mu_B \geqq 1-\mu_C \end{array}\right] , & \mu_C \leqq \frac{1}{2} \\ \left.\begin{array}{ll} \mu_B + \frac{1}{4} & \cdots\ \mu_B \leqq \mu_C - \frac{1}{2} \\ \sqrt{\dfrac{\mu_B+1-\mu_C}{2}} + \dfrac{\mu_B+\mu_C-1}{2} & \cdots\ \mu_C - \frac{1}{2} \leqq \mu_B \leqq 1-\mu_C \\ \sqrt{\mu_B} & \cdots\ \mu_B \geqq 1-\mu_C \end{array}\right] , & \frac{1}{2} \leqq \mu_C \leqq \frac{3}{4} \\ \left.\begin{array}{ll} \mu_B + \frac{1}{4} & \cdots\ \mu_B \leqq \frac{1}{4} \\ \sqrt{\mu_B} & \cdots\ \mu_B \geqq \frac{1}{4} \end{array}\right] , & \mu_C \geqq \frac{3}{4} \end{cases} \tag{96}$$

This membership function μ_D of the consequence D is very complicated and so we shall show in Fig. 10 the diagram of μ_D using a parameter μ_C.

From the figure it is found that μ_D is approximately equal to $\sqrt{\mu_B}$. Therefore, we may represent the consequence D as *almost more or less* B, which leads to

$$\underline{\text{more or less}}\ A \,\square\, Ra' = \underline{\text{almost more or less}}\ B \tag{97}$$

As the consequences D at A′ = *not* A, *not very* A, and *not more or less* A can be obtained in the same way, we shall list these consequences in the following.

(iv) *At A′ = not A*:

$$\underline{\text{not}}\ A \,\square\, Ra' = C \tag{98}$$

This inference result shows that the criterion (16) is satisfied by the arithmetic rule under □.

(v) *At A′ = not very A*: The membership function of the consequence D is given by

$$\mu_D = \begin{cases} \left.\begin{array}{ll} \mu_C + \frac{1}{4} & \cdots\ \mu_C \leq \mu_B \\ -\left(\frac{1+\mu_B-\mu_C}{2}\right)^2 + \frac{1+\mu_B+\mu_C}{2} & \cdots\ \mu_B \leq \mu_C \leq 1-\mu_B \\ 1-(1-\mu_C)^2 & \cdots\ \mu_C \geq 1-\mu_B \end{array}\right] , & \mu_B \leq \frac{1}{2} \\ \left.\begin{array}{ll} \mu_C + \frac{1}{4} & \cdots\ \mu_C \leq 1-\mu_B \\ 1-(1-\mu_C)^2 & \cdots\ \mu_C \geq 1-\mu_B \end{array}\right] , & \mu_B \geq \frac{1}{2} \end{cases} \tag{99}$$

μ_D is depicted in Fig. 11 by using a parameter μ_B. From the figure, μ_D is

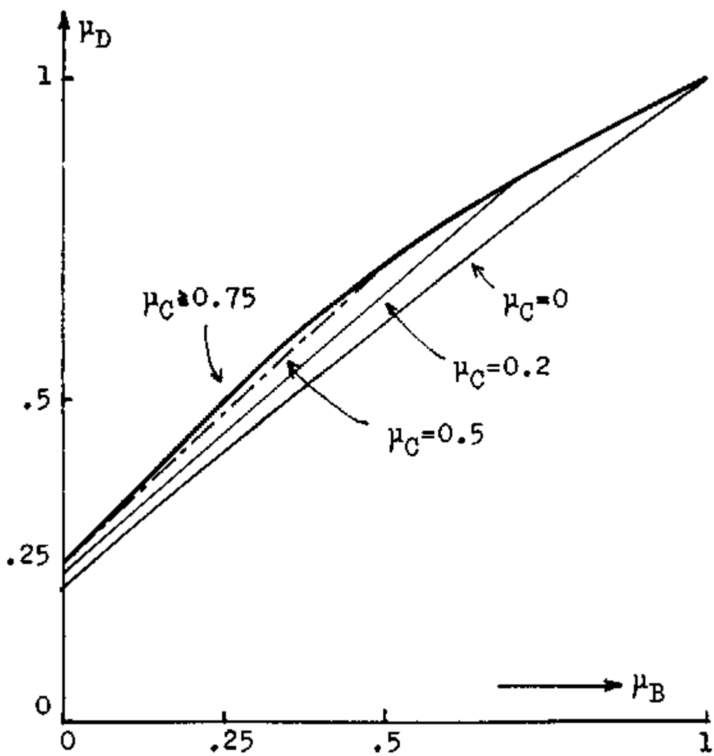


FIGURE 10. μ_D of (96).

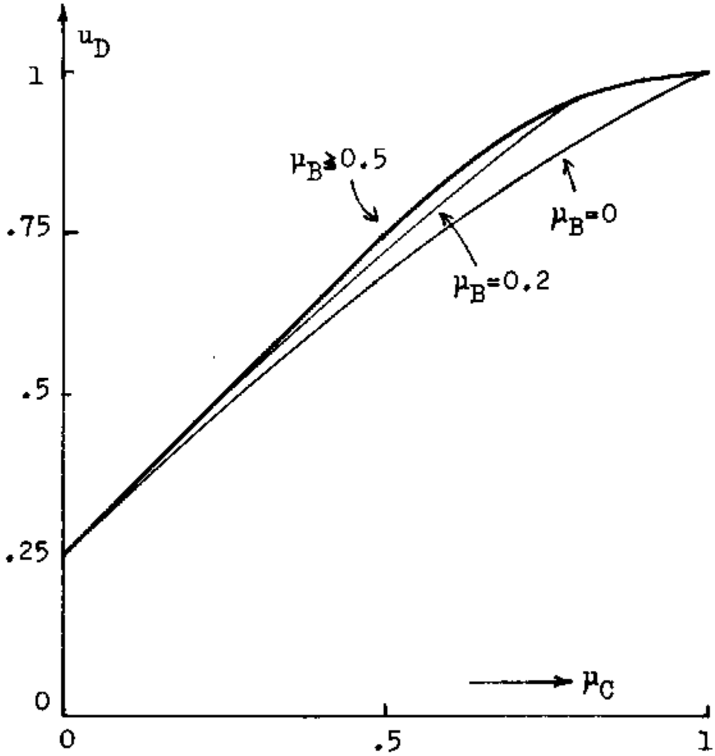


FIGURE 11. μ_D of (99).

approximately equal to $1-(1-\mu_C)^2$ and thus the consequence D is represented as *almost not very not* C, that is,

$$\underline{\text{not very}}\ A \circ Ra' = \underline{\text{almost not very not}}\ C \tag{100}$$

Note: *very not* C is not grammatical. But if, say, C = *happy*, and *not happy* is replaced by the single term *unhappy*, then *very unhappy* becomes meaningful.

(vi) *At A′ = not more or less A*:

$$\underline{\text{not more or less}}\ A \circ Ra' = C \tag{101}$$

Stated in English, the inference results in (i)-(vi) by the arithmetic rule under the max-Θ composition are as follows.

If x is A then y is B else y is C.
x is A. (102)

---

y is B.

If x is A then y is B else y is C.
x is <u>very</u> A. (103)

---

y is B.

If x is A then y is B else y is C.
x is <u>more or less</u> A. (104)

---

y is <u>almost more or less</u> B.

If x is A then y is B else y is C.
x is <u>not</u> A. (105)

---

y is C.

If x is A then y is B else y is C.
x is <u>not very</u> A. (106)

---

y is <u>almost not very not</u> A.

If x is A then y is B else y is C.
x is <u>not more or less</u> A. (107)

---

y is C.

From these results we can conclude that the consequences inferred by the arithmetic rule under the max-Θ composition are quite reasonable and fit our intuition.

ARITHMETIC RULE UNDER MAX-$\wedge$ COMPOSITION

In this section we shall observe what consequences can be obtained by the arithmetic rule when the max-$\wedge$ composition is used in the compositional rule of inference.

We shall first discuss the generalized modus ponens in (1). The consequence B′ is given as follows by using the max-∧ composition "▲" (47) of A′ and Ra.

$$B' = A' \blacktriangle Ra$$
$$= A' \blacktriangle \left[(\neg A \times V) \oplus (U \times B)\right].$$

The membership function of B′ becomes

$$\mu_{B'} = \bigvee_u \{\mu_{A'} \wedge [1 \wedge (1-\mu_A+\mu_B)]\} \tag{108}$$

by omitting "(u)" and "(v)." Let us assume as in the preceding section that μ_A is a function onto [0,1]. Then we have (108) as

$$b' = \bigvee_x \{x' \wedge [1 \wedge (1-x+b)]\} \tag{109}$$

and let

$$f(x) = x' \wedge [1 \wedge (1-x+b)] \tag{110}$$

where

$$x = \mu_A, \quad x' = \mu_{A'}, \quad b = \mu_B, \quad b' = \mu_{B'}, \tag{111}$$

We shall indicate what consequences B′ can be inferred when A′ is equal to A, *very* A, *more or less* A, *and not* A.

(i) *At A′ = A*: When A′ = A, x′ becomes x. Thus, f(x) of (110) will be

$$f(x) = x \wedge [1 \wedge (1-x+b)] \tag{112}$$

In Fig. 12(i), the expressions $1 \wedge (1 - x + b)$ and x are depicted. In this figure, f(x) of (112) is shown by the solid line and the black circle. That is to say,

$$f(x) = \begin{cases} x & \ldots \quad 0 \leqq x \leqq b \\ b & \ldots \quad x = 1 \\ 0 & \ldots \quad \text{otherwise} \end{cases}$$

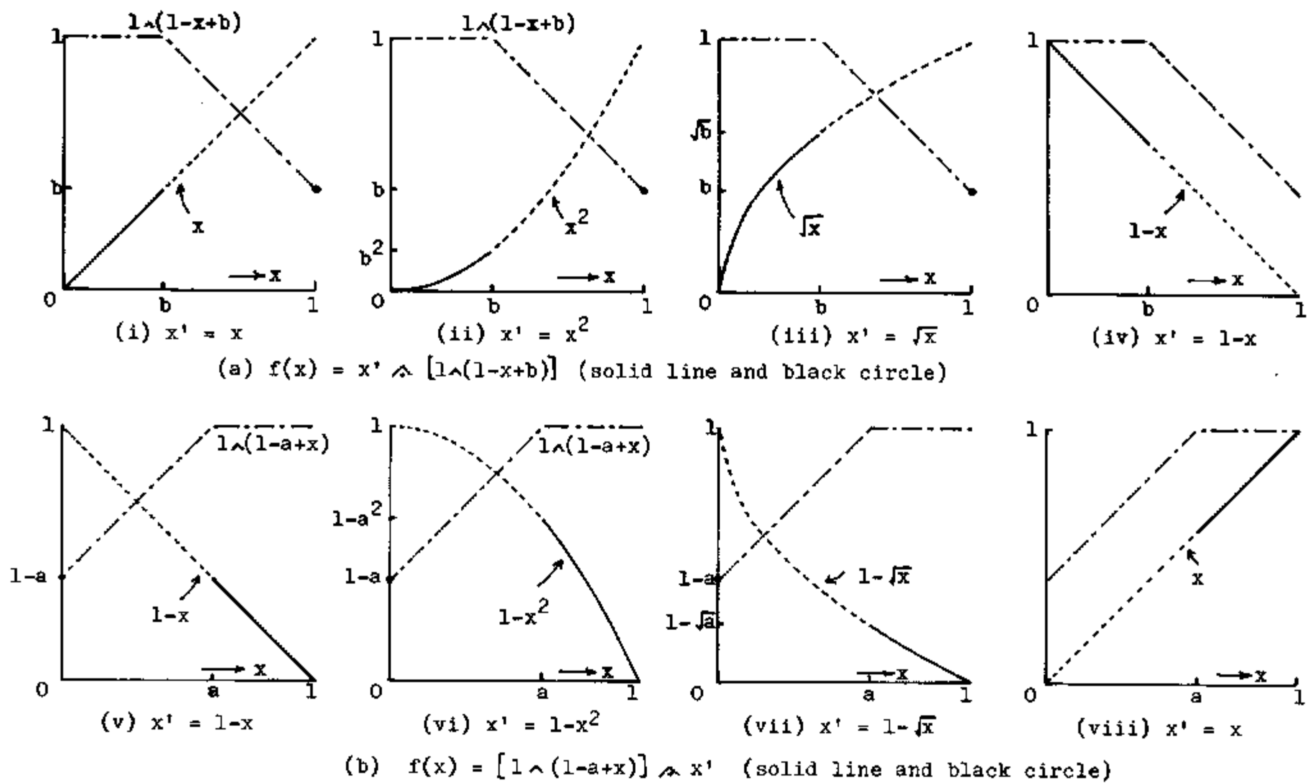


FIGURE 12. The method of obtaining $\bigvee_x f(x)$ of (109) and (117).

Thus,

$$b' = \bigvee_{x} f(x) = \left(\bigvee_{x \in [0,b]} x \right) \vee b$$

$$= b \vee b$$

$$= b$$

Therefore, we have $b' = b$, i.e., $B' = B$ at $A' = A$. Namely,

$$A \blacktriangle Ra = B \tag{113}$$

(ii) *At* $A' = $ *very* A: From Fig. 12(ii), $f(x) = x^2 \wedge [1 \wedge (1 - x + b)]$ becomes

$$f(x) = \begin{cases} x^2 & \ldots \quad 0 \leq x \leq b \\ b & \ldots \quad x = 1 \\ 0 & \ldots \quad \text{otherwise} \end{cases}$$

Thus,

$$b' = \bigvee_{x} f(x) = \left(\bigvee_{x \in [0,b]} x^2 \right) \vee b$$

$$= b^2 \vee b$$

$$= b$$

Namely, we have $b' = b$ at $x' = x^2$. Therefore,

$$\underline{\text{very}}\ A \blacktriangle Ra = B \tag{114}$$

(iii) *At* $A' = $ *more or less* A: From Fig. 12(iii) we have

$$b' = \bigvee_{x} f(x) = \left(\bigvee_{x \in [0,b]} \sqrt{x} \right) \vee b$$

$$= \sqrt{b} \vee b = \sqrt{b}$$

Thus,

$$\underline{\text{more or less}}\ A \blacktriangle Ra = \underline{\text{more or less}}\ B \tag{115}$$

(iv) *At A′ = not A*:

$$b' = (\bigvee_{x \in [0,b]} 1-x) = 1$$

Thus,

$$\underline{\text{not}}\ A \blacktriangle Ra = \underline{\text{unknown}} \tag{116}$$

We shall next consider the case of the generalized modus tollens of (2). The consequence A′ is obtained by taking the max-$\wedge$ composition "▲" of Ra and B′.

$$A' = Ra \blacktriangle B'$$

$$= [(\neg A \times V) \oplus (U \times B)] \blacktriangle B'$$

Then we can have

$$a' = \bigvee_{x} \{[1 \wedge (1-a+x)] \hat{\wedge} x'\} \tag{117}$$

$$f(x) = [1 \wedge (1-a+b)] \hat{\wedge} x' \tag{118}$$

where

$$a = \mu_A, \quad a' = \mu_{A'}, \quad x = \mu_B, \quad x' = \mu_{B'} \tag{119}$$

We shall obtain the consequences a′, i.e., A′ at B′ = *not* B, *not very* B, *not more or less* B, and B.

(v) *At B′ = not B*: When B′ = *not* B, we have x′ = 1 − x from (119). Then f(x) of (118) becomes

$$f(x) = [1 \wedge (1-a+x)] \hat{\wedge} (1-x)$$

and from Fig. 12(v) f(x) is obtained as

$$f(x) = \begin{cases} 1-a & \ldots \quad x = 0 \\ 1-x & \ldots \quad a \leq x \leq 1 \\ 0 & \ldots \quad \text{otherwise} \end{cases}$$

Therefore, we have a′ as

$$a' = \bigvee_x f(x) = (1-a) \vee \Big(\bigvee_{x \in [a,1]} 1-x \Big)$$

$$= (1-a) \vee (1-a) = 1-a$$

Hence,

$$Ra \blacktriangle \underline{\text{not}}\ B = \underline{\text{not}}\ A \tag{120}$$

In the similar way, we can obtain the consequences A′ at B′ = *not very* B, *not more or less* B and B. The obtained results are as follows.

(vi) *At B′ = not very B*;

$$Ra \blacktriangle \underline{\text{not very}}\ B = \underline{\text{not very}}\ A \tag{121}$$

(vii) *At B′ = not more or less B*:

$$Ra \blacktriangle \underline{\text{not more or less}}\ B = \underline{\text{not}}\ A \tag{122}$$

(viii) *At B′ = B*:

$$Ra \blacktriangle B = \underline{\text{unknown}} \tag{123}$$

In the sequel, the inference results in (i)-(viii) by the arithmetic rule under the max-∧ composition are stated in English as follows.

If x is A then y is B.
x is A. (modus ponens)

---

y is B. (124)

If x is A then y is B.
x is <u>very</u> A. (125)
---
y is B.

If x is A then y is B.
x is <u>more or less</u> A. (126)
---
y is <u>more or less</u> B.

If x is A then y is B.
x is <u>not</u> A. (127)
---
y is <u>unknown</u>.

If x is A then y is B.
y is <u>not</u> B. (modus tollens) (128)
---
x is <u>not</u> A.

If x is A then y is B.
y is <u>not very</u> B. (129)
---
x is <u>not very</u> A.

If x is A then y is B.
y is <u>not more or less</u> B. (130)
---
x is <u>not</u> A.

If x is A then y is B.
y is B. (131)
---
x is <u>unknown</u>.

As was founded in these inference results, the arithmetic rule can infer quite reasonable consequences under the max-$\wedge$ composition as well as the max-Θ composition discussed previously.

Lastly, we shall discuss the fuzzy conditional inference of (11) under the max-$\wedge$ composition. The consequence D is given by

$$D = A' \blacktriangle Ra'$$
$$= A' \blacktriangle \left[(\neg A \times V \oplus U \times B) \cap (A \times V \oplus U \times C)\right]$$

and the membership function of D is

$$\mu_D = \bigvee_u \left\{\mu_{A'} \hat{\wedge} \left[1 \wedge (1-\mu_A+\mu_B) \wedge (\mu_A+\mu_C)\right]\right\}$$

Furthermore, u_D is rewritten as

$$d = \bigvee_x \left\{x' \hat{\wedge} \left[1 \wedge (1-x+b) \wedge (x+c)\right]\right\} \tag{132}$$

and let

$$f(x) = x' \hat{\wedge} \left[1 \wedge (1-x+b) \wedge (x+c)\right] \tag{133}$$

where

$$x = \mu_A, \quad x' = \mu_{A'}, \quad b = \mu_B, \quad c = \mu_C, \quad d = \mu_D$$

We shall now obtain the consequence d, i.e., D at A$'$ = A, *very* A, *more or less* A, *not* A, *not very* A, and *not more or less* A. From Figure Fig. 2, we can draw the expression $1 \wedge (1 - x + b) \wedge (x + c)$ with parameters b and c in (133) by the solid lines in Fig. 13. The left figure is at $b + c < 1$, and the right figure is at $b + c \geqslant 1$.

(i) *At A$'$ = A*: When A$'$ = A, x$'$ becomes x. Thus, f(x) of (133) is

$$f(x) = x \hat{\wedge} \left[1 \wedge (1-x+b) \wedge (x+c)\right]$$

When $b + c < 1$ (the left figure of Fig. 14(i)), f(x) is given by

$$f(x) = \begin{cases} b & \dots \quad x = 1 \\ 0 & \dots \quad \text{otherwise} \end{cases}$$

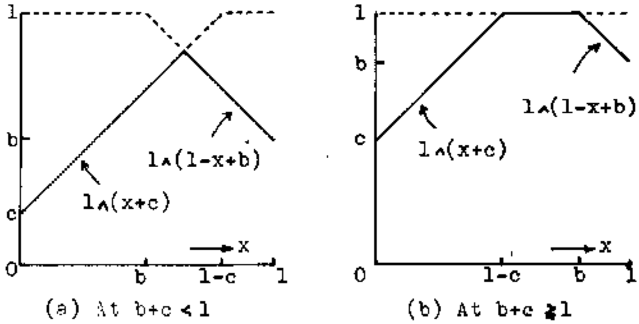


FIGURE 13. $1 \wedge (1 - x + b) \wedge (x + c)$ (solid line).

and

$$d = \bigvee_x f(x) = b \quad \ldots \quad \text{at } b+c < 1 \tag{134}$$

On the other hand, when $b + c \geqslant 1$ (the right figure of Fig. 14(i)), f(x) becomes

$$f(x) = \begin{cases} x & \ldots \quad 1-c \leqq x \leqq b \\ b & \ldots \quad x = 1 \\ 0 & \ldots \quad \text{otherwise} \end{cases}$$

Then,

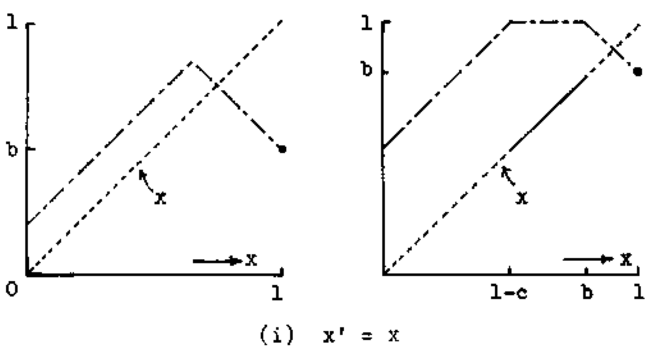
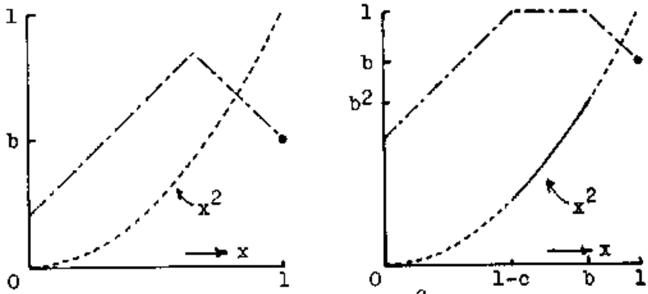
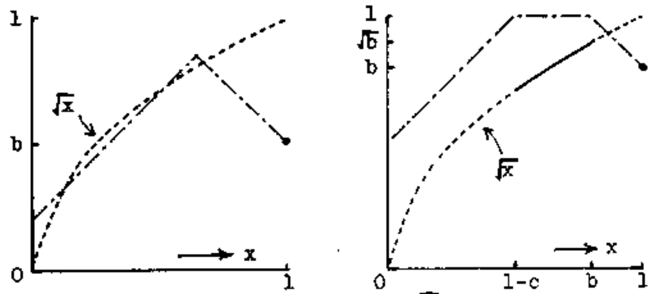


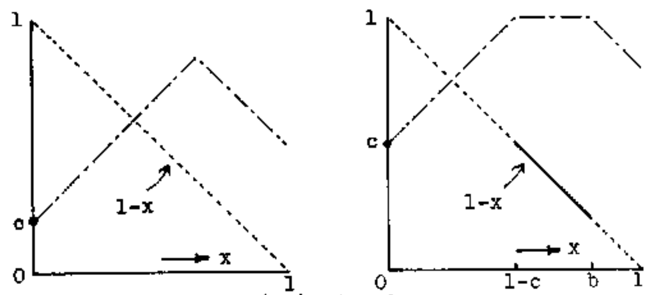
FIGURE 14. $f(x) = x' \wedge [1 \wedge (1 - x + b) \wedge (x + c)]$ (solid line and black circle).



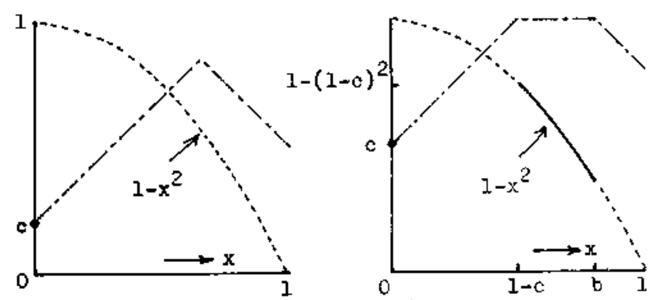
(ii) $x' = x^2$



(iii) $x' = \sqrt{x}$

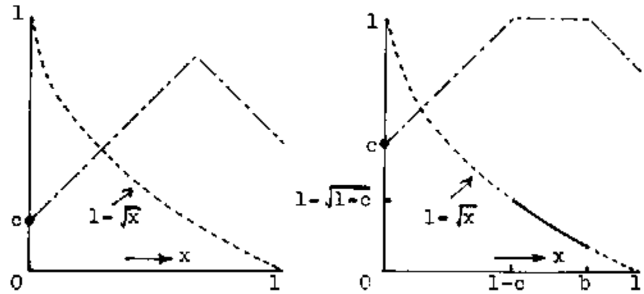


(iv) $x' = 1-x$



(v) $x' = 1-x^2$

FIGURE 14. *(Continued)*



(vi) $x' = 1-\sqrt{x}$

FIGURE 14. *(Continued)*

$$d = \bigvee_x f(x) = \Big(\bigvee_{x \in [1-c,\, b]} x\Big) \vee b$$

$$= b \vee b$$

$$= b \quad \ldots \quad \text{at } b+c \geqq 1 \tag{135}$$

From (134) and (135) we can have d = b for any b and c. Hence

$$A \blacktriangle Ra' = B \tag{136}$$

(ii) *At A′ = very A*: Since $x' = x^2$, f(x) of (133) becomes

$$f(x) = x^2 \mathbin{\blacktriangle} [1 \wedge (1-x+b) \wedge (x+c)]$$

From the left figure (at $b + c < 1$) of Fig. 14(ii), f(x) is given by

$$f(x) = \begin{cases} b & \ldots \quad x = 1 \\ 0 & \ldots \quad \text{otherwise} \end{cases}$$

and

$$d = \bigvee_x f(x) = b \quad \ldots \quad \text{at } b+c < 1$$

When $b + c \geqslant 1$, f(x) becomes

$$f(x) = \begin{cases} x^2 & \ldots \quad 1-c \leqq x \leqq b \\ b & \ldots \quad x = 1 \\ 0 & \ldots \quad \text{otherwise} \end{cases}$$

and thus

$$d = \bigvee_x f(x) = \Big(\bigvee_{x \in [1-c,b]} x^2 \Big) \vee b$$

$$= b^2 \vee b$$

$$= b$$

Therefore, we obtain d = b for any b and c. Namely,

$$\underline{\text{very}}\ A \blacktriangle Ra' = B \tag{137}$$

(iii) *At A′ = more or less A*: From Fig. 14(iii), we have

$$d = \bigvee_x f(x) = b \quad \ldots \quad b+c < 1$$

$$d = \bigvee_x f(x) = \Big(\bigvee_{x \in [1-c,b]} \sqrt{x} \Big) \vee b$$

$$= \sqrt{b} \vee b$$

$$= \sqrt{b} \quad \ldots \quad b+c \geqq 1$$

Therefore,

$$\underline{\text{more or less}}\ A \blacktriangle Ra' = D$$

where

$$\mu_D = \begin{cases} \mu_B & \ldots \quad \mu_B + \mu_C < 1 \\ \sqrt{\mu_B} & \ldots \quad \mu_B + \mu_C \geqq 1 \end{cases} \tag{138}$$

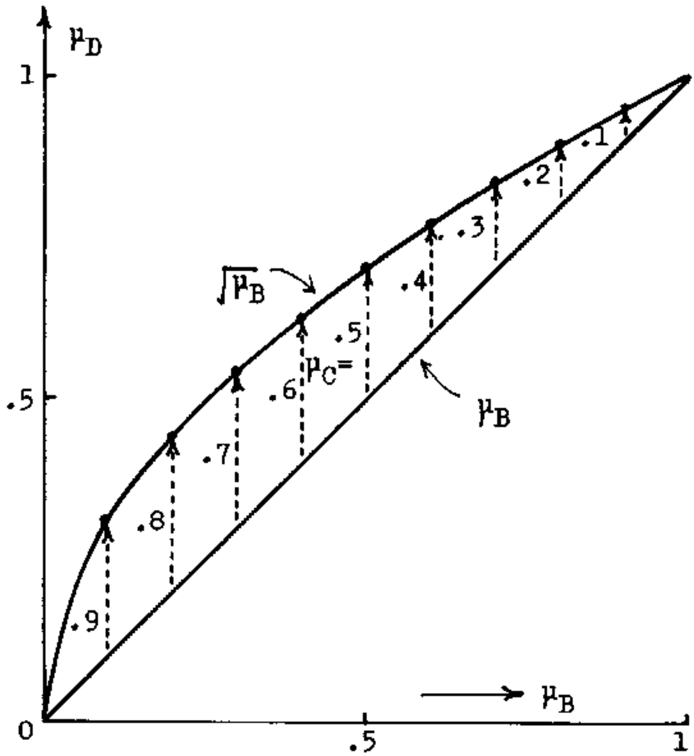


FIGURE 15. μ_D of (138).

In Fig. 15, μ_D is depicted by using a parameter μ_C. From this figure, the D can be roughly represented as *almost* (*more or less* B or B). Hence,

$$\underline{\text{more or less}}\ A \blacktriangle Ra' = \underline{\text{almost}}\ (\underline{\text{more or less}}\ B \text{ or } B) \tag{139}$$

(iv) *At* $A' = not\ A$: From Fig. 14(iv), d is given as

$$d = c \quad \ldots \quad \text{at } b+c < 1$$

$$d = \Big(\bigvee_{x \in [1-c,\, b]} 1-x \Big) \vee c$$

$$= c \vee c$$

$$= c \quad \ldots \quad \text{at } b+c \geq 1$$

Therefore, we have

$$\underline{\text{not}}\ A \blacktriangle Ra' = C \tag{140}$$

(v) *At A′ = not very A*:

$$\underline{\text{not very}}\ A \blacktriangle Ra' = D$$

where

$$\mu_D = \begin{cases} \mu_C & \cdots \quad \mu_B + \mu_C < 1 \\ 1-(1-\mu_C)^2 & \cdots \quad \mu_B + \mu_C \geqq 1 \end{cases} \tag{141}$$

Figure 16 shows μ_D with a parameter μ_B, and D can be approximated as *almost* (*not very not* C or C), which leads to

$$\underline{\text{not very}}\ A \blacktriangle Ra' = \underline{\text{almost}}\ (\underline{\text{not very not}}\ C \text{ or } C) \tag{142}$$

(vi) *At A′ = not more or less A*:

$$\underline{\text{not more or less}}\ A \blacktriangle Ra' = C \tag{143}$$

After all, the inference results obtained in (i)-(vi) by the arithmetic rule under the max-∧ composition are stated in English as:

If x is A then y is B else y is C.
x is A. (144)

---

y is B.

If x is A then y is B else y is C.
x is <u>very</u> A. (145)

---

y is B.

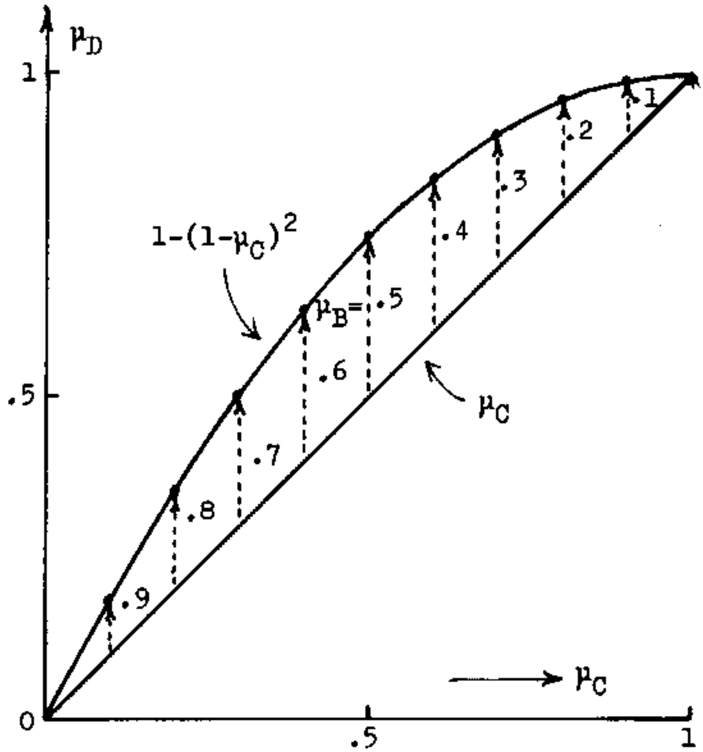


FIGURE 16. μ_D of (141).

$$\begin{array}{l} \text{If x is A then y is B else y is C.} \\ \text{x is } \underline{\text{more or less}} \text{ A.} \\ \hline \text{y is } \underline{\text{almost}} \text{ (}\underline{\text{more or less}} \text{ B or B).} \end{array} \tag{146}$$

$$\begin{array}{l} \text{If x is A then y is B else y is C.} \\ \text{x is } \underline{\text{not}} \text{ A.} \\ \hline \text{y is C.} \end{array} \tag{147}$$

$$\begin{array}{l} \text{If x is A then y is B else y is C.} \\ \text{x is } \underline{\text{not very}} \text{ A.} \\ \hline \text{y is } \underline{\text{almost}} \text{ (}\underline{\text{not very not}} \text{ C or C).} \end{array} \tag{148}$$

If x is A then y is B else y is C.
x is <u>not more or less</u> A.
——————————————————— (149)
y is C.

It is found from the above results that the arithmetic rule under the max-Λ composition also satisfies the criteria (15) and (16) and can get quite reasonable consequences which fit our intuition.

SYLLOGISM BY THE ARITHMETIC RULE

In this section we shall investigate an interesting concept of "syllogism" and show that the syllogism holds for the arithmetic rule under the max-Θ composition and the max-Λ composition, though the syllogism does not hold under the max-min composition.

Let P_1, P_2 and P_3 by fuzzy conditional propositions such as

P_1: If x is A then y is B

P_2: If y is B then z is C

P_3: If x is A then z is C

where A, B and C are fuzzy sets in U, V and W, respectively. If the proposition P_3 is deduced from the propositions P_1 and P_2, that is, the following holds:

P_1: If x is A then y is B.
P_2: If y is B then z is C.
——————————————————— (150)
P_3: If x is A then z is C.

then it is said that the syllogism holds.

Let Ra(A,B), Ra(B,C) and Ra(A,C) be fuzzy relations in U × V, V × W and U × W, respectively, which are obtained from the propositions P_1, P_2 and P_3 by using the arithmetic rule (4). If Ra(A,C) can be obtained from Ra(A,B) and Ra(B,C) by taking the composition of Ra(A,B) and

Ra(B,C), then we can say that the syllogism holds under the composition.

We shall first discuss the syllogism under the max-min composition "o." The fuzzy relations Ra(A,B) and Ra(B,C) are obtained from the propositions P_1 and P_2 by using (4):

$$Ra(A,B) = (\neg A \times V) \oplus (U \times B) \tag{151}$$

$$Ra(B,C) = (\neg B \times W) \oplus (V \times C) \tag{152}$$

Then the max-min composition "o" of Ra(A,B) and Ra(B,C) will be

$$Ra(A,B) \circ Ra(B,C)$$
$$= \left[(\neg A \times V) \oplus (U \times B)\right] \circ \left[(\neg B \times W) \oplus (V \times C)\right] \tag{153}$$

and its membership function becomes as follows.

$$\mu_{Ra(A,B)\circ Ra(B,C)}(u,w)$$
$$= \bigvee_{v} \left\{ [1 \wedge (1-\mu_A(u)+\mu_B(v))] \wedge [1 \wedge (1-\mu_B(v)+\mu_C(w))] \right\} \tag{154}$$

Moreover, this expression can be rewritten as

$$d = \bigvee_{x} \left\{ [1 \wedge (1-a+x)] \wedge [1 \wedge (1-x+c)] \right\} \tag{155}$$

under the assumption that μ_B is a function onto [0,1], where

$$a = \mu_A, \qquad x = \mu_B, \qquad c = \mu_C \tag{156}$$

The expression $1 \wedge (1-a+x)$ can be depicted by using a parameter a as in Fig. 17(a), and the expression $1 \wedge (1-x+c)$ is shown by using a parameter c as in Fig. 17(b). These figures are from Fig. 1. From these figures the expression $[1 \wedge (1-a+x)] \wedge [1 \wedge (1-x+c)]$ in (155) with parameters

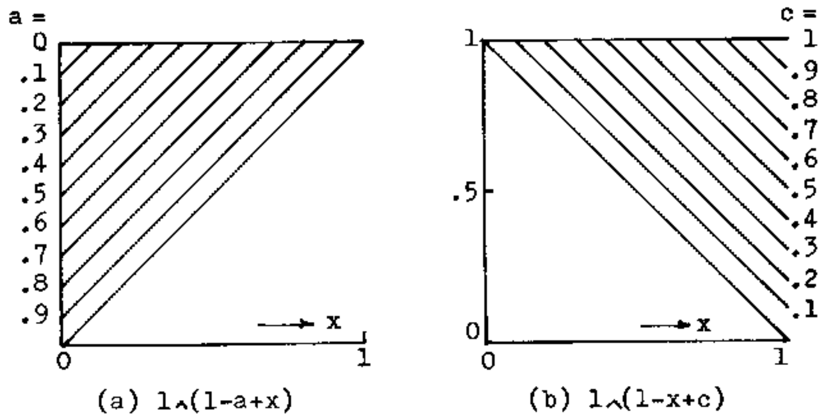


FIGURE 17. $1 \wedge (1-a+x)$ and $1 \wedge (1-x+c)$.

$a = a'$ and $c = c'$ is shown by the line "· · · ·" in Fig. 18 and its maximum value (by virtue of (155)) is equal to the height $(=0.5 + (1 - a' + c')/2)$ of the cross point of $1 - a' + x$ and $1 - x + c'$. On the other hand, if the parameter a is taken to be a″ as in Fig. 18, the maximum value of its line "– - – -" becomes 1. Therefore, in general, for any parameters a and c, the maximum value of $[1 \wedge (1-a+x)] \wedge [1 \wedge (1-x+c)]$ is shown to be $1 \wedge (0.5 + (1-a+c)/2)$, that is

$$d = 1 \wedge \left(0.5 + \frac{1-a+c}{2}\right) \tag{157}$$

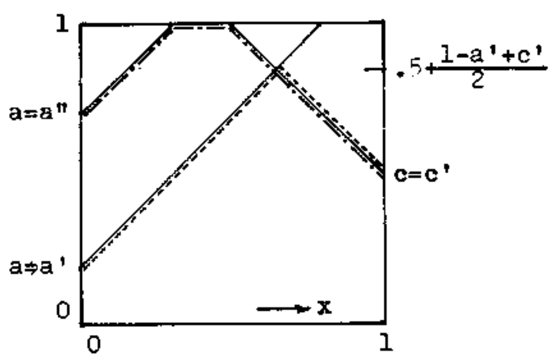


FIGURE 18. $[1 \wedge (1-a+x)] \wedge [1 \wedge (1-x+c)]$ in (155).

Therefore, the membership function $\mu_{Ra(A,B)oRa(B,C)}(u,w)$ of (154) becomes as follows.

$$\mu_{Ra(A,B)oRa(B,C)}(u,w) = 1 \wedge (0.5 + \frac{1-\mu_A(u)+\mu_C(w)}{2}) \tag{158}$$

From this result, we can have

$$\begin{aligned} & Ra(A,B) \text{ o } Ra(B,C) \\ & = \int_{U\times W} 1 \wedge (0.5 + \frac{1-\mu_A(u)+\mu_C(w)}{2}) \;/\; (u,w) \\ & \neq Ra(A,C) \; (= \int_{U\times W} 1 \wedge (1-\mu_A(u)+\mu_C(w)) \;/\; (u,w)) \end{aligned} \tag{159}$$

Hence, we can conclude that the arithmetic rule does not satisfy the syllogism under the max-min composition.

We shall next discuss the syllogism under the max-Θ composition "□." The max-Θ composition (37) of Ra(A,B) and Ra(B,C) is given by

$$\begin{aligned} & Ra(A,B) \,\square\, Ra(B,C) \\ & = [(\neg A \times V) \oplus (U \times B)] \,\square\, [(\neg B \times W) \oplus (V \times C)] \end{aligned}$$

and its membership function is

$$\begin{aligned} & \mu_{Ra(A,B)\,\square\,Ra(B,C)}(u,w) \\ & = \bigvee_{v} \{ [1 \wedge (1-\mu_A(u)+\mu_B(v))] \\ & \quad \odot [1 \wedge (1-\mu_B(v)+\mu_C(w))] \} \end{aligned}$$

As in the case of the max-min composition, this expression can be given as

$$d' = \bigvee_x \left\{ [1 \wedge (1-a+x)] \odot [1 \wedge (1-x+c)] \right\} \tag{160}$$

and let

$$f(x) = [1 \wedge (1-a+x)] \odot [1 \wedge (1-x+c)]$$

Then f(x) becomes as follows by using the bounded product $\odot$ in (17).

$$\begin{aligned}
&f(x) \\
&= 0 \vee \left\{ [1 \wedge (1-a+x)] + [1 \wedge (1-x+c)] - 1 \right\} \\
&= 0 \vee \left\{ [1 \wedge (1-a+x)] + [0 \wedge (-x+c)] \right\} \\
&= 0 \vee \left\{ 1 \wedge (1-x+c) \wedge (1-a+x) \wedge (1-a+x-x+c) \right\} \\
&= 0 \vee \left\{ 1 \wedge (1-x+c) \wedge (1-a+x) \wedge (1-a+c) \right\} \\
&= 1 \wedge (1-x+c) \wedge (1-a+x) \wedge (1-a+c) \\
&= 1 \wedge (1-a+x) \wedge (1-x+c) \wedge 1 \wedge (1-a+c)
\end{aligned}$$

Thus we have d′ of (160) as

$$\begin{aligned}
d' &= \bigvee_x f(x) \\
&= \bigvee_x \left\{ 1 \wedge (1-a+x) \wedge (1-x+c) \right\} \wedge [1 \wedge (1-a+c)] \\
&= \left[1 \wedge \left(0.5 + \frac{1-a+c}{2}\right) \right] \wedge [1 \wedge (1-a+c)] \quad \ldots \text{ from (155) and (157)} \\
&= 1 \wedge (1-a+c) \quad \ldots \quad 0.5 + \frac{1-a+c}{2} \geq 1-a+c
\end{aligned}$$

Therefore,

$$\mathrm{Ra}(A,B) \,\square\, \mathrm{Ra}(B,C)$$

$$= \int_{U \times W} 1 \wedge (1-\mu_A(u)+\mu_C(w)) \,/\, (u,w)$$

$$= \mathrm{Ra}(A,C) \tag{161}$$

which leads to the satisfaction of the syllogism under the max-Θ composition.

Finally, we investigate the case of the max-$\triangle$ composition. The membership function of the max-$\triangle$ composition of Ra(A,B) and Ra(B,C) is given in (162) in the same way as the cases of the max-min composition and the max-Θ composition.

$$d'' = \bigvee_x \left\{ [1 \wedge (1-a+x)] \,\hat{\wedge}\, [1 \wedge (1-x+c)] \right\} \tag{162}$$

$$f(x) = [1 \wedge (1-a+x)] \,\hat{\wedge}\, [1 \wedge (1-x+c)] \tag{163}$$

where $\hat{\wedge}$ is the drastic product of (18).

Using Fig. 17(a) and (b), the function f(x) of (163) is depicted by the solid line as in Fig. 19. When $c \leqslant a$, f(x) is obtained from Fig. 19(a) as

$$f(x) = \begin{cases} 1-a+x & \dots \quad 0 \leqq x \leqq c \\ 1-x+c & \dots \quad a \leqq x \leqq 1 \\ 0 & \dots \quad \text{otherwise} \end{cases}$$

Thus, from (162)

$$d'' = \bigvee_x f(x)$$

$$= \Big(\bigvee_{x \in [0,c]} 1-a+x \Big) \vee \Big(\bigvee_{x \in [a,1]} 1-x+c \Big)$$

$$= (1-a+c) \vee (1-a+c)$$

$$= 1-a+c \quad \dots \quad \text{at } c \leqq a$$

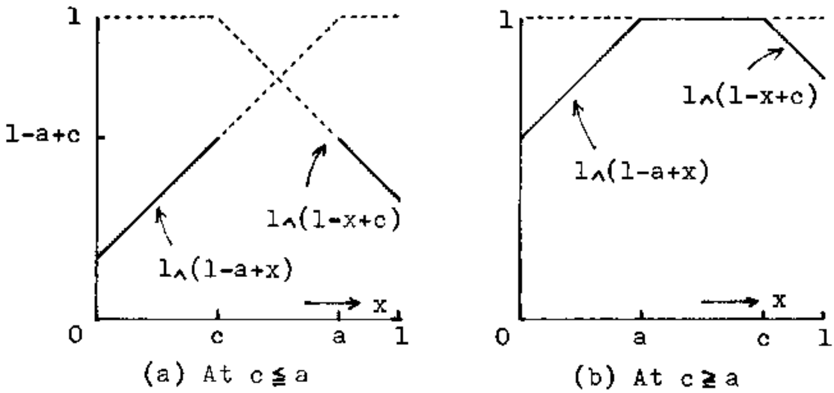


FIGURE 19. $f(x) = [1 \wedge (1-a+x)] \wedge [1 \wedge (1-x+c)]$ of (163) (solid line).

When $c \geqslant a$ [Fig. 19(b)],

$$d'' = \bigvee_x f(x) = 1 \quad \ldots \quad \text{at } c \geqq a$$

Thus, for any parameters a and c, we have

$$d'' = 1 \wedge (1-a+c)$$

Therefore, the max-∧ composition "▲" of Ra(A,B) and Ra(B,C) becomes

$$Ra(A,B) \blacktriangle Ra(B,C)$$

$$= \int_{U\times W} 1 \wedge (1-\mu_A(u)+\mu_C(w)) \;/\; (u,w)$$

$$= Ra(A,C) \tag{164}$$

Therefore, the syllogism also holds under the max-∧ composition.

CONCLUDING REMARKS

We have shown that the arithmetic rule can get quite reasonable inference results in the fuzzy conditional inference with "If . . . then . . ." and "If . . .

then . . . else . . ." when the max-Θ composition and the max-A composition are used in the compositional rule of inference. Moreover, the arithmetic rule satisfies the syllogism under these compositions.

In this connection, it is possible to introduce the max-product composition "·" (39) in the compositional rule of inference. For example, we can have such inference results as

$$A \cdot Ra = \int_V \left(\frac{1+\mu_B(v)}{2}\right)^2 / v$$

$$Ra \cdot \underline{\text{not}}\ B = \int_U \left(1 - \frac{\mu_A(u)}{2}\right)^2 / u$$

$$A \cdot Ra' = \int_V \left(\frac{1+\mu_B(v)}{2}\right)^2 / v$$

$$\underline{\text{not}}\ A \cdot Ra' = \int_V \left(\frac{1+\mu_C(v)}{2}\right)^2 / v$$

$$Ra(A,B) \cdot Ra(B,C) = \int_{U \times W} 1 \wedge (0.5 + \frac{1-\mu_A(u)+\mu_C(w)}{2})^2 / (u,w)$$

It is found from these results that the inference results under the max-product composition are better than those under the max-min composition, but they do not satisfy the reasonable criteria (9), (10), (15) and (16).

It will be of interest to apply the max-Θ composition and max-A composition to other fuzzy inference rules such as "maximum rule" and "fuzzified binary rule." These results will be presented in subsequent papers.

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