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FUZZY SETS UNDER VARIOUS OPERATIONS (Part II)

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Abstract

This paper investigates the algebraic properties of fuzzy sets under new operations of "drastic product" and "drastic sum" introduced by Dubois in 1979, and the algebraic properties in the case where these new operations are combined with well-known operations for fuzzy sets. The properties of fuzzy relations are also shown by introducing new compositions of fuzzy relations which are defined by using drastic product.

1. Introduction

As the continuation of our study on "Fuzzy Sets under Various Operations" in BUSEFAL, No.4, 38-49, 1980, which shows the algebraic properties of fuzzy sets under the operations of "bounded-sum", "bounded-difference" and "bounded-product", and the results when these operations are combined with the well-known operations of intersection, union, algebraic product and algebraic sum, we shall investigate the algebraic properties of fuzzy sets under new operations "drastic product" and "drastic sum" introduced by Dubois (1979). The properties of fuzzy sets are also obtained in the case where

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these new operations are combined with the well-known operations of intersection, union; algebraic product, algebraic sum; and bounded-product, bounded-sum. Moreover, the properties of fuzzy relations are briefly discussed under new compositions which are defined by using the drastic product and bounded-product.

2. Fuzzy Sets and Their Operations

Let A and B be fuzzy sets in a universe of discourse U, then the operations on fuzzy sets A and B are listed as follows.

Intersection: $A \cap B \Leftrightarrow \mu_{A \cap B} = \mu_A \wedge \mu_B$ (1)

Union: $A \cup B \Leftrightarrow \mu_{A \cup B} = \mu_A \vee \mu_B$ (2)

Algebraic Product: $A \cdot B \Leftrightarrow \mu_{A \cdot B} = \mu_A \mu_B$ (3)

Algebraic Sum: $A \dot{+} B \Leftrightarrow \mu_{A \dot{+} B} = \mu_A + \mu_B - \mu_A \mu_B$ (4)

Bounded-Product: $A \odot B \Leftrightarrow \mu_{A \odot B} = 0 \vee (\mu_A + \mu_B - 1)$ (5)

Bounded-Sum: $A \oplus B \Leftrightarrow \mu_{A \oplus B} = 1 \wedge (\mu_A + \mu_B)$ (6)

Drastic Product:

$$A \mathbin{\hat{\cap}} B \Leftrightarrow \mu_{A \hat{\cap} B} = \begin{cases} \mu_A & \cdots \quad \mu_B = 1 \\ \mu_B & \cdots \quad \mu_A = 1 \\ 0 & \cdots \quad \mu_A, \mu_B < 1 \end{cases} \qquad (7)$$

Drastic Sum:

$$A \mathbin{\dot{\cup}} B \Leftrightarrow \mu_{A \dot{\cup} B} = \begin{cases} \mu_A & \cdots \quad \mu_B = 0 \\ \mu_B & \cdots \quad \mu_A = 0 \\ 1 & \cdots \quad \mu_A, \mu_B > 0 \end{cases} \qquad (8)$$

where the operations of $\wedge$, $\vee$, + and - represent min, max, arithmetic sum and arithmetic difference, respectively.

Clearly, the operations of intersection, algebraic product, bounded-product and drastic product are dual to those of union, algebraic sum, bounded-sum and drastic sum, respectively. Drastic product ($\cap\!\!\!\cdot$) and drastic sum ($\cup\!\!\!\cdot$) for fuzzy sets are corresponding to the operations Tw(x,y) and Tw*(x,y) by Dubois (1979), respectively. An interesting application of these operators to fuzzy numbers is discussed by Dubois and Prade (1981). In this paper we shall rewrite Tw(x,y) as $x \dot{\wedge} y$, and Tw*(x,y) as $x \dot{\vee} y$ for convenience. Thus,

$$x \dot{\wedge} y = Tw(x,y) = \begin{cases} x & \dots & y = 1 \\ y & \dots & x = 1 \\ 0 & \dots & x,\ y < 1 \end{cases} \qquad (9)$$

$$x \dot{\vee} y = Tw^{*}(x,y) = \begin{cases} x & \dots & y = 0 \\ y & \dots & x = 0 \\ 1 & \dots & x,\ y > 0 \end{cases} \qquad (10)$$

The following inequalities hold for these operators: For any $x,\ y \in [0,1]$,

$$x \dot{\wedge} y \leq x \odot y \leq x \cdot y \leq x \wedge y \qquad (11)$$

$$x \dot{\vee} y \geq x \oplus y \geq x \dotplus y \geq x \vee y \qquad (12)$$

where $\odot$, $\cdot$, $\wedge$, $\oplus$, $\dotplus$, $\vee$ stand for bounded-product, algebraic product, min, bounded-sum, algebraic sum and max, respectively, which correspond to the fuzzy set operations in (1)-(6).

From these inequalities it is found that $\dot{\wedge}$ is the most drastic operator, while $\odot$, $\cdot$ and $\wedge$ are less and less drastic (Dubois and Prade, 1981). The same holds for $\vee$, $\oplus$, $\dotplus$ and $\vee$. Therefore, in this paper we call the operator $\dot{\wedge}$ (as well, $\cap\!\!\!\cdot$) as "drastic product", and the operator $\dot{\vee}$ ($\cup\!\!\!\cdot$) as "drastic sum".

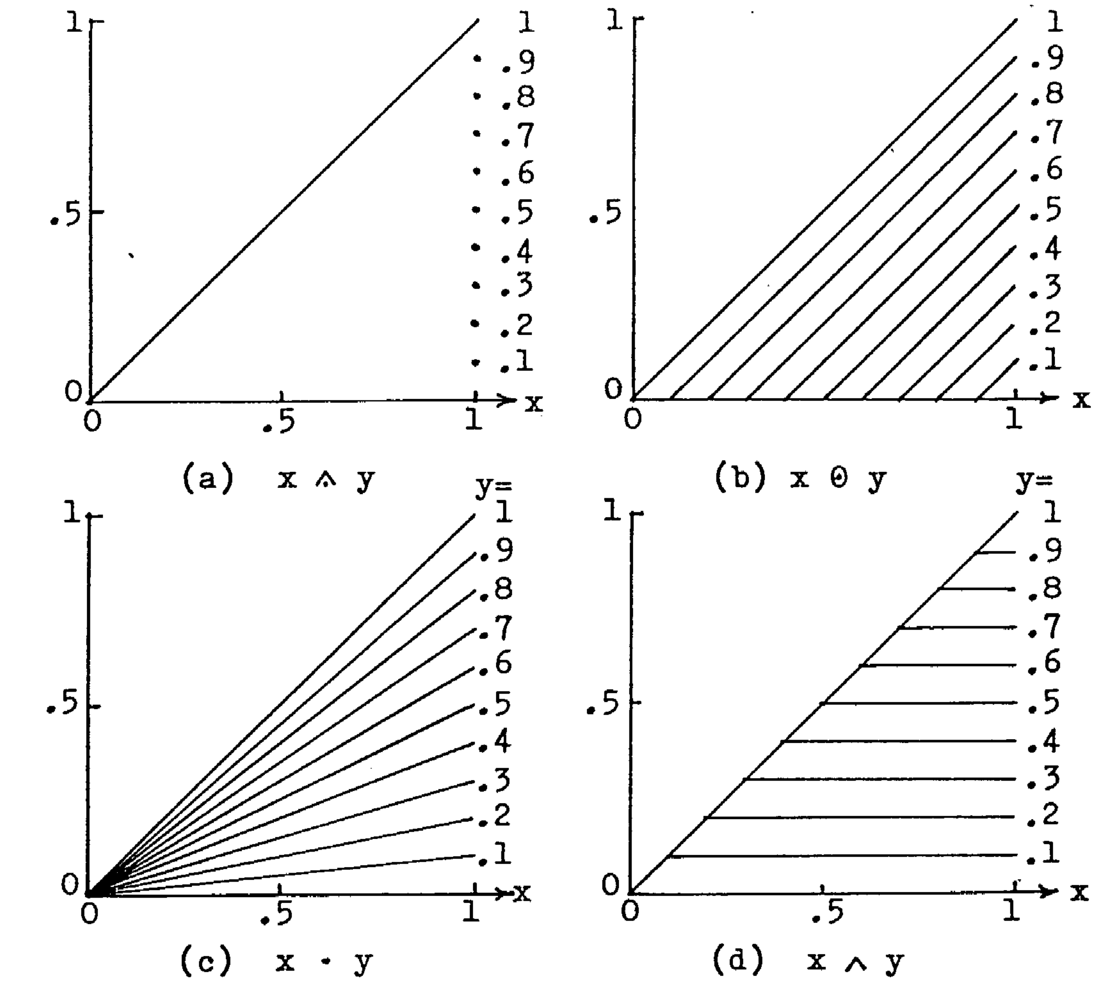


Fig. 1 Diagrams of $\dot{\wedge}$, $\ominus$, $\cdot$ and $\wedge$

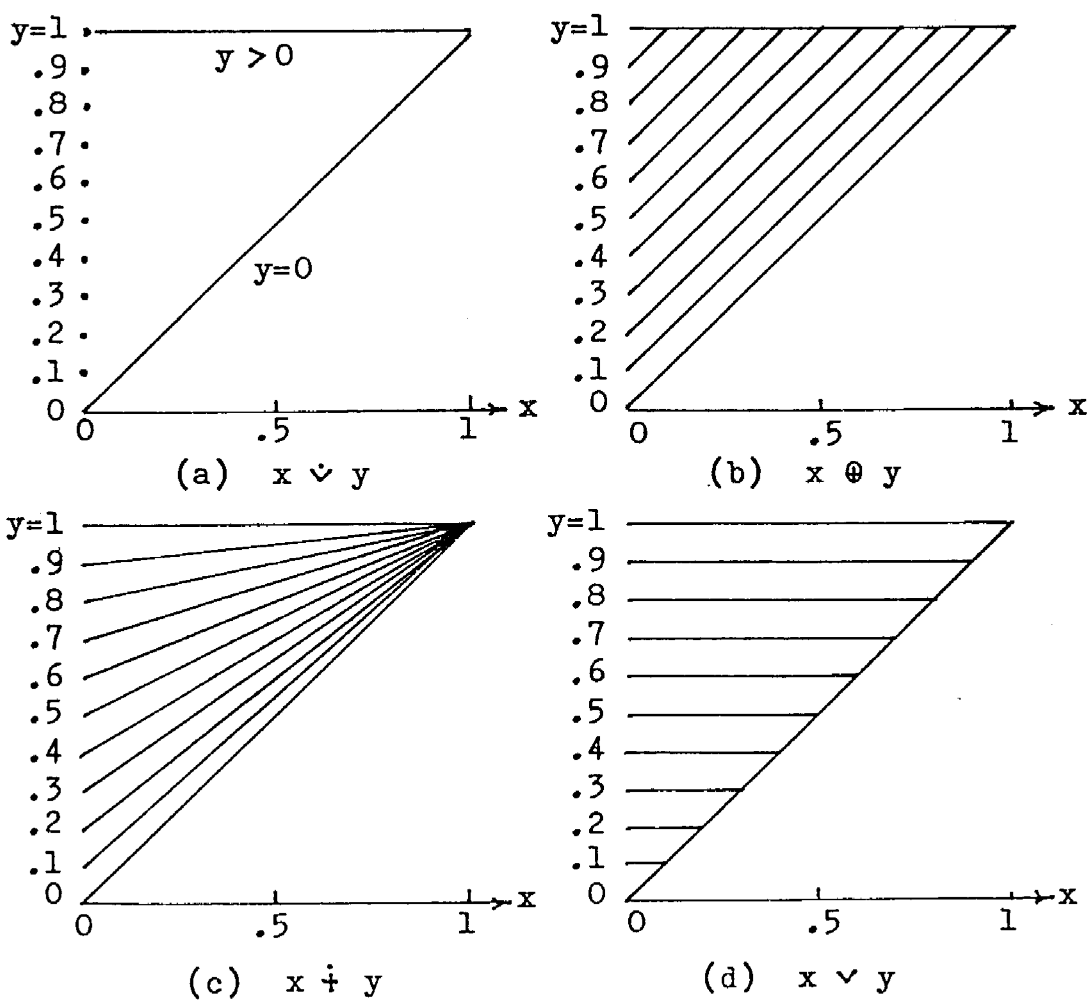
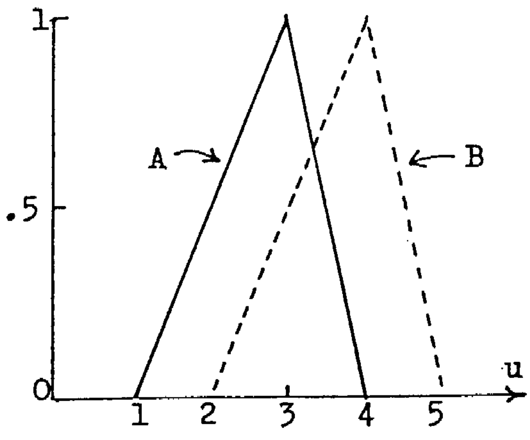
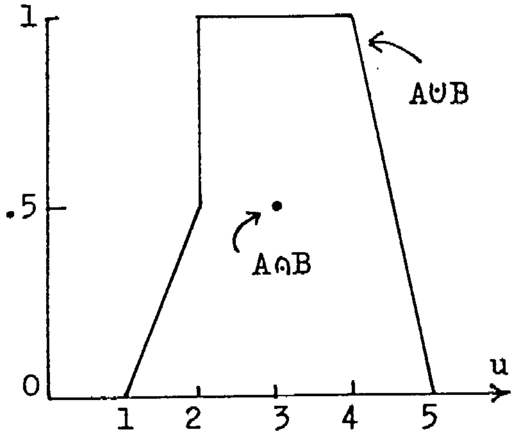


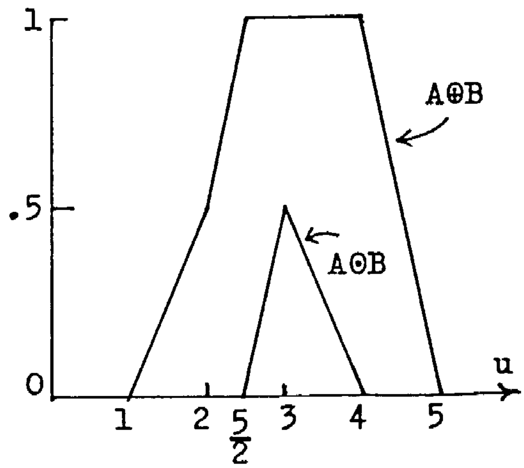
Fig. 2 Diagrams of $\dot{\vee}$, $\oplus$, $\dot{+}$ and $\vee$



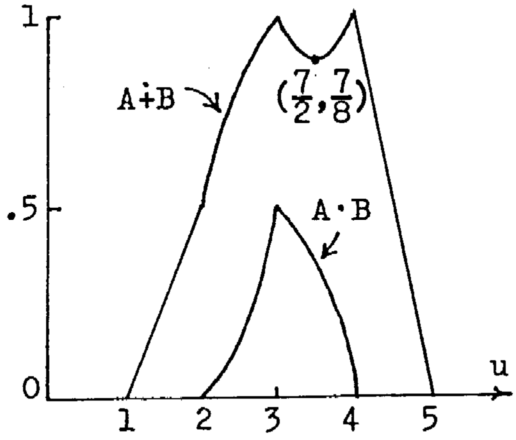
(a) Fuzzy sets A and B



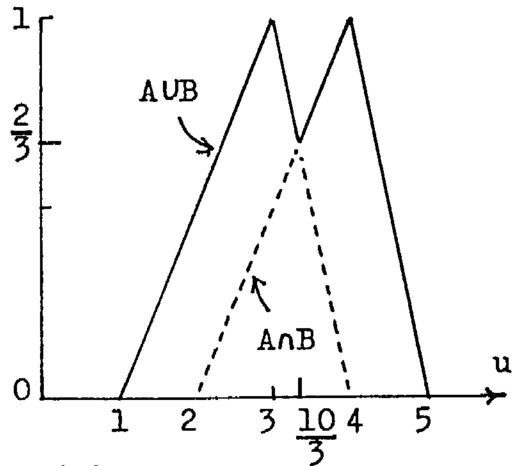
(b) Drastic product ⩀
and drastic sum ⩁



(c) Bounded-product ⊙
and bounded-sum ⊕



(d) Algebraic product ·
and algebraic sum ∔



(e) Intersection ∩
and union ∪

Fig. 3 Operation results of A and B

In Fig. 1-2 these operations are depicted by using a parameter y in order to see how drastic the operators $\dot{\wedge}$ and $\dot{\vee}$ are. Such tendencies can be also observed in the fuzzy set operations in (1)-(8) (See Fig.3).

3. Algebraic Properties of Fuzzy Sets

In this section we shall discuss the algebraic properties of fuzzy sets under the operations of drastic product ($\dot{\cap}$) and drastic sum ($\dot{\cup}$), and the properties of fuzzy sets when these operations are combined with the well-known operations in (1)-(6). The properties under other combinations of these fuzzy set operations are found in Mizumoto and Tanaka (1980, 1981).

(I) The Case of Drastic Product ($\dot{\cap}$) and Drastic Sum ($\dot{\cup}$):

Idempotency: $$A \dot{\cap} A \subseteq A, \qquad A \dot{\cup} A \supseteq A \tag{13}$$

Commutativity: $$A \dot{\cap} B = B \dot{\cap} A, \qquad A \dot{\cup} B = B \dot{\cup} A \tag{14}$$

Associativity: $$A \dot{\cap} (B \dot{\cap} C) = (A \dot{\cap} B) \dot{\cap} C, \qquad A \dot{\cup} (B \dot{\cup} C) = (A \dot{\cup} B) \dot{\cup} C \tag{15}$$

Absorption: $$A \dot{\cap} (A \dot{\cup} B) \subseteq A, \qquad A \dot{\cup} (A \dot{\cap} B) \supseteq A \tag{16}$$

Distributivity: $$A \dot{\cap} (B \dot{\cup} C) \neq (A \dot{\cap} B) \dot{\cup} (A \dot{\cap} C), \qquad A \dot{\cup} (B \dot{\cap} C) \neq (A \dot{\cup} B) \dot{\cap} (A \dot{\cup} C) \tag{17}$$

De Morgan's laws: $$\overline{A \dot{\cap} B} = \bar{A} \dot{\cup} \bar{B}, \qquad \overline{A \dot{\cup} B} = \bar{A} \dot{\cap} \bar{B} \tag{18}$$

Identities: $$A \dot{\cap} \emptyset = \emptyset, \quad A \dot{\cup} \Omega = \Omega, \quad A \dot{\cap} \Omega = A, \quad A \dot{\cup} \emptyset = A \tag{19}$$

Complementarity: $$A \dot{\cap} \bar{A} = \emptyset, \qquad A \dot{\cup} \bar{A} = \Omega \tag{20}$$

where $\emptyset$ is an empty set defined by $\mu_{\emptyset} = 0$, and Ω is a universe of discourse U and is defined by $\mu_\Omega = 1$.

<u>Theorem 1</u>: Fuzzy sets under $\dot{\cap}$ and $\dot{\cup}$ do not satisfy the absorption and distributive laws and hence they do not form such algebraic structures as a lattice and a semiring. Fuzzy sets under $\dot{\cap}$ form a commutative semigroup with unity ($=\Omega$) (that is, a commutative monoid). The duality holds for $\dot{\cup}$.

We shall next examine the absorption and distributive properties for fuzzy sets under the operations $\dot{\cap}$ and $\dot{\cup}$ which are combined with $\cap$ and $\cup$.

(II) <u>The Case of Drastic Product ($\dot{\cap}$) and Drastic Sum ($\dot{\cup}$) Combined with Intersection ($\cap$) and Union ($\cup$)</u>:

<u>Absorption</u>:

$$A \,\dot{\cap}\, (A \cap B) \subseteq A \qquad (21)$$

$$A \,\dot{\cap}\, (A \cup B) \subseteq A \qquad (22)$$

$$A \,\dot{\cup}\, (A \cap B) \supseteq A \qquad (23)$$

$$A \,\dot{\cup}\, (A \cup B) \supseteq A \qquad (24)$$

and

$$A \cap (A \,\dot{\cap}\, B) \subseteq A \qquad (25)$$

$$A \cap (A \,\dot{\cup}\, B) = A \qquad (26)$$

$$A \cup (A \,\dot{\cap}\, B) = A \qquad (27)$$

$$A \cup (A \,\dot{\cup}\, B) \supseteq A \qquad (28)$$

<u>Distributivity</u>:

$$A \,\dot{\cap}\, (B \cap C) = (A \,\dot{\cap}\, B) \cap (A \,\dot{\cap}\, C) \qquad (29)$$

$$A \,\dot{\cap}\, (B \cup C) = (A \,\dot{\cap}\, B) \cup (A \,\dot{\cap}\, C) \qquad (30)$$

$$A \,\dot{\cup}\, (B \cap C) = (A \,\dot{\cup}\, B) \cap (A \,\dot{\cup}\, C) \qquad (31)$$

$$A \,\dot{\cup}\, (B \cup C) = (A \,\dot{\cup}\, B) \cup (A \,\dot{\cup}\, C) \qquad (32)$$

and

$$A \cap (B \,\dot{\cap}\, C) \supseteq (A \cap B) \,\dot{\cap}\, (A \cap C) \qquad (33)$$

$$A \cap (B \,\dot{\cup}\, C) \subseteq (A \cap B) \,\dot{\cup}\, (A \cap C) \qquad (34)$$

$$A \cup (B \,\dot{\cap}\, C) \supseteq (A \cup B) \,\dot{\cap}\, (A \cup C) \qquad (35)$$

$$A \cup (B \,\dot{\cup}\, C) \subseteq (A \cup B) \,\dot{\cup}\, (A \cup C) \qquad (36)$$

Theorem 2: Fuzzy sets form a commutative semiring with unity ($=\Omega$) and zero ($=\emptyset$) under $\dot{\cap}$ (as multiplication) and $\cup$ (as addition). The duality holds for $\dot{\cup}$ and $\cap$. Moreover, fuzzy sets constitute a commutative semiring with unity ($=\emptyset$) under $\dot{\cup}$ (as multiplication) and $\cup$ (as addition). The duality holds for $\dot{\cap}$ and $\cap$. Fuzzy sets form a lattice ordered semigroup with unity ($=\Omega$) and zero ($=\emptyset$) under $\cap$, $\cup$ and $\dot{\cap}$, where $\dot{\cap}$ is a semigroup operation. The dualizy holds for $\cup$, $\cap$ and $\dot{\cup}$.

(III) The Case of Drastic Product ($\dot{\cap}$) and Drastic Sum ($\dot{\cup}$) Combined with Algebraic Product ($\cdot$) and Algebraic Sum ($\dotplus$):

Absorption:

$$A \,\dot{\cap}\, (A \cdot B) \subseteq A \qquad (37)$$

$$A \,\dot{\cap}\, (A \dotplus B) \subseteq A \qquad (38)$$

$$A \,\dot{\cup}\, (A \cdot B) \supseteq A \qquad (39)$$

$$A \,\dot{\cup}\, (A \dotplus B) \supseteq A \qquad (40)$$

and

$$A \cdot (A \,\dot{\cap}\, B) \subseteq A \qquad (41)$$

$$A \cdot (A \,\dot{\cup}\, B) \subseteq A \qquad (42)$$

$$A \dotplus (A \,\dot{\cap}\, B) \supseteq A \qquad (43)$$

$$A \dotplus (A \,\dot{\cup}\, B) \supseteq A \qquad (44)$$

Distributivity:

$$A \,\dot{\cap}\, (B \cdot C) \supseteq (A \,\dot{\cap}\, B) \cdot (A \,\dot{\cap}\, C) \qquad (45)$$

$$A \,\dot{\cap}\, (B \dotplus C) \subseteq (A \,\dot{\cap}\, B) \dotplus (A \,\dot{\cap}\, C) \qquad (46)$$

$$A \,\dot{\cup}\, (B \cdot C) \supseteq (A \,\dot{\cup}\, B) \cdot (A \,\dot{\cup}\, C) \qquad (47)$$

$$A \,\dot{\cup}\, (B \dotplus C) \subseteq (A \,\dot{\cup}\, B) \dotplus (A \,\dot{\cup}\, C) \qquad (48)$$

and

$$A \cdot (B \,\dot{\cap}\, C) \supseteq (A \cdot B) \,\dot{\cap}\, (A \cdot C) \qquad (49)$$

$$A \cdot (B \,\dot{\cup}\, C) \subseteq (A \cdot B) \,\dot{\cup}\, (A \cdot C) \qquad (50)$$

$$A \dotplus (B \,\dot{\cap}\, C) \supseteq (A \dotplus B) \,\dot{\cap}\, (A \dotplus C) \qquad (51)$$

$$A \dotplus (B \,\dot{\cup}\, C) \subseteq (A \dotplus B) \,\dot{\cup}\, (A \dotplus C) \qquad (52)$$

Theorem 3: Fuzzy sets do not form such algebraic structures as a lattice and a semigroup under $\dot{\cap}$ and $\dotplus$. The same is true of ($\dot{\cap}$, $\cdot$), ($\dot{\cup}$, $\cdot$) and ($\dot{\cup}$, $\dotplus$).

(IV) The Case of Drastic Product ($\dot\cap$) and Drastic Sum ($\dot\cup$) Combined with Bounded-Product ($\odot$) and Bounded-Sum ($\oplus$):

Absorption:

$$A \dot\cap (A \odot B) \subseteq A \tag{53}$$
$$A \dot\cap (A \oplus B) \subseteq A \tag{54}$$
$$A \dot\cup (A \odot B) \supseteq A \tag{55}$$
$$A \dot\cup (A \oplus B) \supseteq A \tag{56}$$

and

$$A \odot (A \dot\cap B) \subseteq A \tag{57}$$
$$A \odot (A \dot\cup B) \subseteq A \tag{58}$$
$$A \oplus (A \dot\cap B) \supseteq A \tag{59}$$
$$A \oplus (A \dot\cup B) \supseteq A \tag{60}$$

Distributivity:

$$A \dot\cap (A \odot C) \supseteq (A \dot\cap B) \odot (A \dot\cap C) \tag{61}$$
$$A \dot\cap (B \oplus C) \subseteq (A \dot\cap B) \oplus (A \dot\cap C) \tag{62}$$
$$A \dot\cup (B \odot C) \supseteq (A \dot\cup B) \odot (A \dot\cup C) \tag{63}$$
$$A \dot\cup (B \oplus C) \subseteq (A \dot\cup B) \oplus (A \dot\cup C) \tag{64}$$

and

$$A \odot (B \dot\cap C) \supseteq (A \odot B) \dot\cap (A \odot C) \tag{65}$$
$$A \odot (B \dot\cup C) \neq (A \odot B) \dot\cup (A \odot C) \tag{66}$$
$$A \oplus (B \dot\cap C) \neq (A \oplus B) \dot\cap (A \oplus C) \tag{67}$$
$$A \oplus (B \dot\cup C) \subseteq (A \oplus B) \dot\cup (A \oplus C) \tag{68}$$

Theorem 4: Fuzzy sets do not constitute such algebraic structures as a lattice and a semiring under $\dot\cap$ and $\oplus$. The same is true of ($\dot\cap$, $\odot$), ($\dot\cup$, $\odot$) and ($\dot\cup$, $\oplus$).

4. New Compositions of Fuzzy Relations

We shall briefly investigate new compositions of fuzzy relations by introducing bounded-product $\odot$ and drastic product $\dot\wedge$ into compositions of fuzzy relations. As is well-known, the max-min composition and max-product composition of fuzzy relations are defined as follows: Let R be a fuzzy relation U x V and S be a fuzzy relation in U x W, then we have

Max-Min Composition:

$$R \circ S \Leftrightarrow \mu_{R\circ S}(u,w) = \bigvee_{v} \{\mu_R(u,v) \wedge \mu_S(v,w)\} \qquad (69)$$

Max-Product Composition:

$$R \cdot S \Leftrightarrow \mu_{R\cdot S}(u,w) = \bigvee_{v} \{\mu_R(u,v) \cdot \mu_S(v,w)\} \qquad (70)$$

In the same way, we can easily define new compositions by using bounded-product $\odot$ and drastic product $\hat{\wedge}$ in (9).

Max-$\odot$ Composition:

$$R \boxdot S \Leftrightarrow \mu_{R\boxdot S}(u,w) = \bigvee_{v} \{\mu_R(u,v) \odot \mu_S(v,w)\} \qquad (71)$$

where

$$x \odot y = 0 \vee (x + y - 1)$$

Max-$\hat{\wedge}$ Compositions:

$$R \blacktriangle S \Leftrightarrow \mu_{R\blacktriangle S}(u,w) = \bigvee_{v} \{\mu_R(u,v) \hat{\wedge} \mu_S(v,w)\} \qquad (72)$$

Similarly, we can define a number of new compositions such as $\dot{+}$-min composition, $\oplus$-product composition, $\dot{\vee}$-$\odot$ composition and so on by combining $\vee$, $\dot{+}$, $\oplus$, $\dot{\vee}$, $\wedge$, $\cdot$, $\odot$ and $\hat{\wedge}$.

Example 1: Let R and S be fuzzy relations such as

$$R = \begin{bmatrix} .2 & .8 & 1 \\ .9 & .5 & .4 \\ .3 & .9 & .1 \end{bmatrix}, \qquad S = \begin{bmatrix} .8 & .9 & .1 \\ 1 & .7 & .8 \\ .1 & .4 & 1 \end{bmatrix}$$

then we have $R \circ S$, $R \cdot S$, $R \boxdot S$ and $R \blacktriangle S$ in the following.

$$R \circ S = \begin{bmatrix} .8 & .7 & 1 \\ .8 & .9 & .5 \\ .9 & .7 & .8 \end{bmatrix}, \qquad R \cdot S = \begin{bmatrix} .8 & .56 & 1 \\ .72 & .81 & .4 \\ .9 & .63 & .72 \end{bmatrix}$$

$$R \boxdot S = \begin{bmatrix} .8 & .5 & 1 \\ .7 & .8 & .4 \\ .9 & .6 & .7 \end{bmatrix}, \qquad R \blacktriangle S = \begin{bmatrix} .8 & .4 & 1 \\ .5 & 0 & .4 \\ .9 & 0 & .1 \end{bmatrix}$$

As was shown in this example, we obtain in general

$$R \blacktriangle S \subseteq R \,\square\, S \subseteq R \cdot S \subseteq R \circ S$$

by virtue of the property of (11) of $\wedge$, $\ominus$, $\cdot$ and $\wedge$.

<u>Example 2</u>: Let R be a fuzzy relation on a real line which represents "u is approximately equal to v", i.e., "u $\approx$ v":

$$\mu_R(u,v) = \max(0,\ 1 - |u - v|) \qquad (73)$$

Then we obtain

$$\mu_{R \circ R}(u,v) = \max(0,\ 1 - \frac{|u - v|}{2})$$

$$\mu_{R \cdot R}(u,v) = \begin{cases} (1 - \frac{|u - v|}{2})^2 & \dots\ |u - v| \leqq 2 \\ 0 & \dots\ |u - v| \geqq 2 \end{cases}$$

$$\mu_{R \square R}(u,v) = \max\ (0,\ 1 - |u - v|\ )$$

$$\mu_{R \blacktriangle R}(u,v) = \max\ (0,\ 1 - |u - v|\ )$$

Therefore,

$$R \circ R \supseteq R \cdot R \supseteq R, \qquad R \,\square\, R = R \blacktriangle R = R$$

From these results, we may say that the max-min composition R o R and max-product composition R • R fit our intuition in the case of R = $\approx$. However, it is noted that the max-$\ominus$ composition and max-$\wedge$ composition satisfy the transitive law and thus the fuzzy relation R which is reflexive and symmetric in nature becomes a fuzzy equivalence relation (Zadeh, 1971) under each of $\square$ and $\blacktriangle$.

As another example, let us consider a fuzzy relation S which also represents "u $\approx$ v" and is defined by

$$\mu_S(u,v) = \max\ (0,\ 1 - (u-v)^2) \qquad (74)$$

Then we have

$$\mu_{S\circ S}(u,v) = \max(0,\ 1 - \frac{(u-v)^2}{4}) \geqq \mu_S(u,v)$$

$$\mu_{S\cdot S}(u,v) = \begin{cases} (1 - \frac{(u-v)^2}{4})^2 & \ldots\ |u-v| \leqq 2 \\ 0 & \ldots\ |u-v| \geqq 2 \end{cases}$$

$$\geqq \mu_S(u,v)$$

$$\mu_{S\boxdot S}(u,v) = \max(0,\ 1 - \frac{(u-v)^2}{2}) \geqq \mu_S(u,v)$$

$$\mu_{S\blacktriangle S}(u,v) = \max(0,\ 1 - (u-v)^2) = \mu_S(u,v)$$

Namely,

$$S \circ S \supsetneqq S \cdot S \supsetneqq S \boxdot S \supsetneqq S \blacktriangle S \ (= S)$$

Thus, the fuzzy relation S also becomes a fuzzy equivalence relation under $\blacktriangle$.

As in the case of max-min composition "o", we can obtain the following properties under max-product composition "$\cdot$", max-$\odot$ composition "$\boxdot$" and max-$\wedge$ composition "$\blacktriangle$".

Let R, S and T be fuzzy relations on U, and let $* \in \{\circ, \cdot, \boxdot, \blacktriangle\}$, then

$$R * (S * T) = (R * S) * T \tag{75}$$

$$S \subseteq T \Rightarrow R * S \subseteq R * T \tag{76}$$

$$R * (S \cup T) = (R * S) \cup (R * T) \tag{77}$$

$$R * (S \cap T) \subseteq (R * S) \cap (R * T) \tag{78}$$

$$I * R = R * I = R, \quad 0 * R = R * 0 = 0 \tag{79}$$

$$(R * S)^c = S^c * R^c \tag{80}$$

where I and 0 are identity relation and null relation, respectively, and R^c stands for the converse of R.