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FUZZY SETS OF TYPE 2 UNDER ALGEBRAIC PRODUCT AND ALGEBRAIC SUM

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The concept of fuzzy sets of type 2 has been proposed by L.A. Zadeh as an extension of ordinary fuzzy sets. A fuzzy set of type 2 can be defined by a fuzzy membership function, the grade (or fuzzy grade) of which is taken to be a fuzzy set in the unit interval [0, 1] rather than a point in [0, 1].

This paper investigates the algebraic properties of fuzzy grades (that is, fuzzy sets of type 2) under the operations of algebraic product and algebraic sum which can be defined by using the concept of the extension principle and shows that fuzzy grades under these operations do not form such algebraic structures as a lattice and a semiring. Moreover, the properties of fuzzy grades are also discussed in the case where algebraic product and algebraic sum are combined with the well-known operations of join and meet for fuzzy grades and it is shown that normal convex fuzzy grades form a lattice ordered semigroup under join, meet and algebraic product.

Keywords: Fuzzy sets of type 2, Fuzzy grades, Algebraic product, Algebraic sum, Negation, Join, Meet.

1. Introduction

Recently, L.A. Zadeh [8] has formulated the interesting concept of the extension principle by which a binary operation defined on a set X may be extended to fuzzy sets in X and defined the operations for fuzzy sets of type 2, fuzzy numbers and fuzzy linguistic logic.

In Mizumoto and Tanaka [2] we discussed what kinds of algebraic structures the grades (or fuzzy grades) of fuzzy sets of type 2 form under join ($\sqcup$), meet ($\sqcap$) and negation ($\neg$), and showed that normal convex fuzzy grades form a distributive lattice and convex fuzzy grades form a commutative semiring under join and meet.

In this paper we investigate the algebraic properties of fuzzy grades (or fuzzy sets of type 2) under the operations of algebraic product and algebraic sum which are defined by using the extension principle [8] and show that, unlike the

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operations of join and meet, fuzzy grades under algebraic product and algebraic sum do not constitute such algebraic structures as a lattice and a semiring. Furthermore, the properties of fuzzy grades are also discussed in the case where algebraic product and algebraic sum are combined with join and meet, and it is shown that normal convex fuzzy grades form a lattice ordered semigroup under join, meet and algebraic product.

2. Fuzzy grades

We shall briefly describe the concept of fuzzy sets of type 2 and their operations of algebraic product and algebraic sum obtained using the extension principle.

Fuzzy sets of type 2: A fuzzy set of type 2, A, in a universe of discourse X is characterized by a *fuzzy membership function* μ_A as

$$\mu_A : X \to [0, 1]^{[0,1]}, \tag{1}$$

where the value $\mu_A(x)$ is called a *fuzzy grade* and is a fuzzy set in the unit interval $[0, 1]$. A fuzzy grade $\mu_A(x)$ is represented by

$$\mu_A(x) = \int f(u)/u, \quad u \in [0, 1], \tag{2}$$

where f is a membership function for the fuzzy grade $\mu_A(x)$ and is defined as

$$f : [0, 1] \to [0, 1]. \tag{3}$$

Example 1. Suppose that $X = \{\text{Susie, Betty, Helen, Ruth, Pat}\}$ is a set of women and that A is a fuzzy set of type 2 of *beautiful* women in X. Then we may have

A = BEAUTIFUL = **high**/Susie + **middle**/Betty + **low**/Helen
+ **not low**/Ruth + **very high**/Pat,

where the fuzzy grades labeled **high, middle**, . . . , **very high** may be depicted as in Fig. 1.

We shall next define the operations of algebraic product[1] and algebraic sum for fuzzy grades using the concept of the extension principle.[2]

[1] Algebraic product and algebraic sum performed on ordinary fuzzy sets A and B are defined as follows [6]:

Algebraic product: $AB \Leftrightarrow \mu_{AB}(x) = \mu_A(x) \cdot \mu_B(x)$,
Algebraic sum: $A \dotplus B \Leftrightarrow \mu_{A \dotplus B}(x) = \mu_A(x) + \mu_B(x) - \mu_A(x) \cdot \mu_B(x)$,

where the symbols $\cdot$, $+$, $-$ represent arithmetic product, arithmetic sum, and arithmetic difference, respectively, and $\mu_A(x)$ and $\mu_B(x)$ are both in $[0, 1]$.

[2] Let $A = \int \mu_A(x)/x$ and $B = \int \mu_B(y)/y$ be ordinary fuzzy sets in X and let $*$ be a binary operation on X. Then the operation $*$ can be extended to fuzzy sets A and B by the following relation (the *extension principle* [8]):

$$A * B = \left(\int \mu_A(x)/x\right) * \left(\int \mu_B(y)/y\right) = \int (\mu_A(x) \wedge \mu_B(y))/(x * y)$$

where $\wedge$ denotes min.

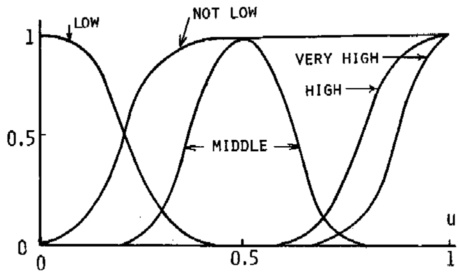

Fig. 1. Example of fuzzy grades.

Let $\mu_A(x)$ and $\mu_B(x)$ be fuzzy grades for fuzzy sets of type 2, A and B, represented as

$$\mu_A(x) = \int f(u)/u, \qquad u \in [0, 1], \tag{4}$$

$$\mu_B(x) = \int g(w)/w, \qquad w \in [0, 1]. \tag{5}$$

Then the operations of algebraic product and algebraic sum for fuzzy grades $\mu_A(x)$ and $\mu_B(x)$ are defined as follows by using the extension principle:

Algebraic product:

$$\begin{aligned} AB \Leftrightarrow \mu_{AB}(x) &= \mu_A(x) \cdot \mu_B(x) \\ &= \left(\int f(u)/u\right) \cdot \left(\int g(w)/w\right) = \int (f(u) \wedge g(w))/uw. \end{aligned} \tag{6}$$

Algebraic sum:

$$\begin{aligned} A \dotplus B \Leftrightarrow \mu_{A \dotplus B}(x) = \mu_A(x) \dotplus \mu_B(x) &= \int (f(u) \wedge g(w))/(u \dotplus w) \\ &= \int (f(u) \wedge g(w))/(u + w - uw). \end{aligned} \tag{7}$$

The complement of a fuzzy set of type 2 A is defined as

Complement:

$$\bar{A} \Leftrightarrow \mu_{\bar{A}}(x) = \neg \mu_A(x) = \int f(u)/(1-u), \tag{8}$$

where $\wedge$ stands for min. We call the operations for fuzzy grades, that is, $\cdot$ as *algebraic product*, $\dotplus$ as *algebraic sum*, and $\neg$ as *negation* hereafter.

Example 2. As a simple example, we shall execute the operation of algebraic product for discrete fuzzy grades $\mu_A(x)$ and $\mu_B(x)$. Let $\mu_A(x)$ and $\mu_B(x)$ be as

$$\mu_A(x) = 0.5/0.2 + 1/0.4 + 0.8/0.6, \tag{9}$$

$$\mu_B(x) = 1/0.2 + 0.9/0.4 + 0.4/0.6. \tag{10}$$

Then from (6) we have $\mu_{AB}(x)$ as follows:

$$\begin{aligned}\mu_A(x) \cdot \mu_B(x) &= \frac{1 \wedge 0.5}{0.2 \times 0.2} + \frac{1 \wedge 1}{0.2 \times 0.4} + \frac{1 \wedge 0.8}{0.2 \times 0.6} \\ &\quad + \frac{0.9 \wedge 0.5}{0.4 \times 0.2} + \frac{0.9 \wedge 1}{0.4 \times 0.4} + \frac{0.9 \wedge 0.8}{0.4 \times 0.6} \\ &\quad + \frac{0.4 \wedge 0.5}{0.6 \times 0.2} + \frac{0.4 \wedge 1}{0.6 \times 0.4} + \frac{0.4 \wedge 0.8}{0.6 \times 0.6} \\ &= 0.5/0.04 + 1/0.08 + 0.8/0.12 \\ &\quad + 0.9/0.16 + 0.8/0.24 + 0.4/0.36.\end{aligned} \tag{11}$$

Example 3. We shall show the example of continuous fuzzy grades. Let $\mu_A(x)$ and $\mu_B(x)$ be continuous fuzzy grades such that

$$\mu_A(x) = \mu_B(x) = \int_0^1 u/u, \tag{12}$$

then we can obtain the algebraic product, algebraic sum and negation of fuzzy grades $\mu_A(x)$ and $\mu_B(x)$ (see Fig. 2):

$$\mu_A(x) \cdot \mu_A(x) = \int_0^1 \sqrt{u}/u, \tag{13}$$

$$\mu_A(x) \dot{+} \mu_A(x) = \int_0^1 1 - \sqrt{1-u}/u, \tag{14}$$

$$\neg \mu_A(x) = \int_0^1 1 - u/u. \tag{15}$$

We shall next define a convex fuzzy grade and a normal fuzzy grade as a special case of fuzzy grades.[3]

Convex fuzzy grades: A fuzzy grade $\mu_A = \int f(u)/u$ is said to be *convex* if for any $u_1, u_2, u_3 \in [0, 1]$ such as $u_1 \leq u_2 \leq u_3$,

$$f(u_2) \geq f(u_1) \wedge f(u_3). \tag{16}$$

Normal fuzzy grades: A fuzzy grade μ_A is *normal* if

$$\bigvee_u f(u) = 1, \tag{17}$$

where $\bigvee = \max$. Otherwise it is *subnormal.*

[3] We shall hereafter abbreviate $\mu_A(x)$ as μ_A for simplicity.

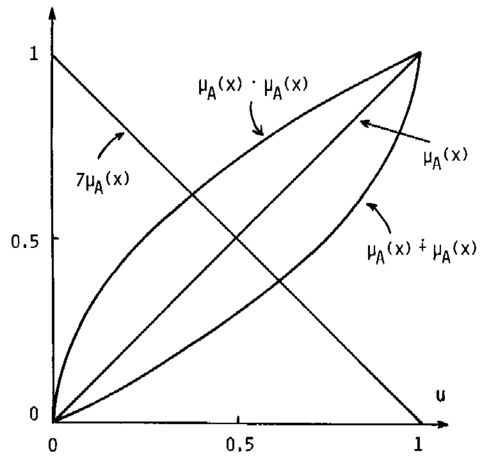

Fig. 2. Negation $\neg\mu_A$, algebraic product $\mu_A \cdot \mu_A$ and algebraic sum $\mu_A \dot{+} \mu_A$ of fuzzy grade μ_A.

A fuzzy grade which is convex and normal is referred to as a *normal convex fuzzy grade.*

Example 4. Fuzzy grades shown in Figs. 1 and 2 are all normal convex fuzzy grades. Fig. 3 indicates that μ_B is subnormal nonconvex and that μ_C is normal nonconvex since the support of μ_C is discrete, that is, μ_C does not satisfy (16).

Level Sets: The *α-level set* of a fuzzy grade $\mu_A = \int f(u)/u$ is a nonfuzzy set denoted as μ_A^α and is defined by

$$\mu_A^\alpha = \{u \mid f(u) \geq \alpha\}, \quad 0 < \alpha \leq 1. \tag{18}$$

It is easy to show that

$$\alpha_1 \leq \alpha_2 \Rightarrow \mu_A^{\alpha_1} \supseteq \mu_A^{\alpha_2}. \tag{19}$$

Let a fuzzy grade μ_A be convex fuzzy grade, then each μ_A^α becomes a convex set (or an *interval*) in [0, 1].

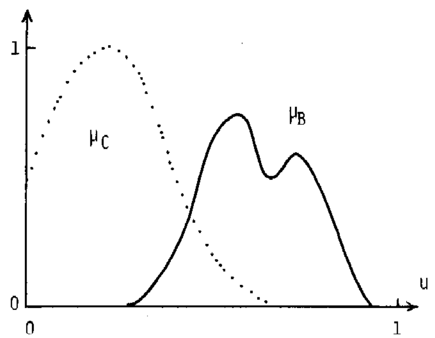

Fig. 3. Example of a subnormal nonconvex fuzzy grade μ_B and normal nonconvex fuzzy grade μ_C.

3. Algebraic properties of fuzzy grades under $\cdot$, $\dot{+}$ and $\neg$

This section discusses the algebraic properties of fuzzy grades under algebraic product ($\cdot$), algebraic sum ($\dot{+}$) and negation ($\neg$). We shall begin with the convexity of fuzzy grades under these operations.

Theorem 1. *If μ_A and μ_B are convex fuzzy grades, $\mu_A \cdot \mu_B$, $\mu_A \dot{+} \mu_B$ and $\neg\mu_A$ are also convex fuzzy grades.*

Proof. In general, let M_1, M_2, N_1, and N_2 be intervals in $[0, 1]$ and let $M_1 \subseteq M_2$ and $N_1 \subseteq N_2$, then we can easily obtain that $M_1 \cdot N_1 \subseteq M_2 \cdot N_2$ and that $M_i \cdot N_i$ $(i = 1, 2)$ are also intervals in $[0, 1]$. (It is noted that let M_1 and N_1 be intervals $[m_1, m_2]$ and $[n_1, n_2]$, respectively, in $[0, 1]$, then $M_1 \cdot N_1$ is $[m_1n_1, m_2n_2]$.) For each $0 < \alpha \leq 1$, the α-level sets μ_A^α and μ_B^α of convex fuzzy grades μ_A and μ_B are intervals in $[0, 1]$. Thus, for any α_1 and α_2 with $0 < \alpha_1 \leq \alpha_2$, the relations $\mu_A^{\alpha_2} \subseteq \mu_A^{\alpha_1}$ and $\mu_B^{\alpha_2} \subseteq \mu_B^{\alpha_1}$ are derived from (19) and hence $\mu_A^{\alpha_2} \cdot \mu_B^{\alpha_2} \subseteq \mu_A^{\alpha_1} \cdot \mu_B^{\alpha_1}$ is obtained, which leads to $(\mu_A \cdot \mu_B)^{\alpha_2} \subseteq (\mu_A \cdot \mu_B)^{\alpha_1}$. Thus, the fuzzy grade $\mu_A \cdot \mu_B$ is shown to be convex.

The convexity of μ_A under negation $\neg$ is proved as follows: The negation of $\mu_A = \int f(u)/u$ is given as $\neg\mu_A = \int f(u)/1-u$, which becomes $\neg\mu_A = \int f(1-u)/u$ when $1-u$ is changed by u. For any real numbers u_1, u_2, u_3 such that $0 \leq u_1 \leq u_2 \leq u_3 \leq 1$, it is obtained that $0 \leq 1-u_3 \leq 1-u_2 \leq 1-u_1 \leq 1$. Thus we can have $f(1-u_2) \geq f(1-u_3) \wedge f(1-u_1)$ by virtue of the convexity of μ_A. Therefore, $\neg\mu_A$ is a convex fuzzy grade.

The convexity of $\mu_A \dot{+} \mu_B$ is proved from the fact that $\mu_A \dot{+} \mu_B$ is given as $\neg(\neg\mu_A \cdot \neg\mu_B)$ (see Theorem 3) and the convexity holds under $\cdot$ and $\neg$.

Remark. It should be noted that for discrete fuzzy grades, the convexity under $\cdot$ and $\dot{+}$ does not hold even if the fuzzy grades are in the shape of 'convex' like μ_C in Fig. 3 (see Example 2).

Theorem 2. *If μ_A and μ_B are normal fuzzy grades, then $\neg\mu_A$, $\mu_A \cdot \mu_B$ and $\mu_A \dot{+} \mu_B$ are also normal fuzzy grades. Furthermore, if μ_A and μ_B are normal convex fuzzy grades, so are $\neg\mu_A$, $\mu_A \cdot \mu_B$ and $\mu_A \dot{+} \mu_B$.*

Next, we shall discuss what laws fuzzy grades satisfy under $\cdot$, $\dot{+}$ and $\neg$.

Theorem 3. *For arbitrary fuzzy grades (including discrete fuzzy grades), the following laws are satisfied under algebraic product ($\cdot$), algebraic sum ($\dot{+}$) and negation ($\neg$):*

Commutative laws: $$\mu_A \cdot \mu_B = \mu_B \cdot \mu_A; \qquad \mu_A \dot{+} \mu_B = \mu_B \dot{+} \mu_A; \tag{20}$$

Associative laws: $$\begin{cases} (\mu_A \cdot \mu_B) \cdot \mu_C = \mu_A \cdot (\mu_B \cdot \mu_C); \\ (\mu_A \dot{+} \mu_B) \dot{+} \mu_C = \mu_A \dot{+} (\mu_B \dot{+} \mu_C); \end{cases} \tag{21}$$

Involution law: $$\neg(\neg\mu_A) = \mu_A; \tag{22}$$

De Morgan's laws: $\begin{cases} \neg(\mu_A \cdot \mu_B) = (\neg\mu_A) \dot{+} (\neg\mu_B); \\ \neg(\mu_A \dot{+} \mu_B) = (\neg\mu_A) \cdot (\neg\mu_B); \end{cases}$ (23)

Part of identity laws:[4] $\mu_A \cdot 1 = \mu_A; \qquad \mu_A \dot{+} 0 = \mu_A.$ (24)

Proof. We shall prove only the De Morgan's law: $\neg(\mu_A \dot{+} \mu_B) = (\neg\mu_A) \cdot (\neg\mu_B)$ of (23). Let $\mu_A = \int f(u)/u$ and $\mu_B = \int g(w)/w$, then it follows from the equality of $u + w - uw$ and $1 - (1-u)(1-w)$ in (7) that

$$\begin{aligned}\neg(\mu_A \dot{+} \mu_B) &= \neg\left(\int f(u) \wedge g(w)/1-(1-u)(1-w)\right) \\ &= \int f(u) \wedge g(w)/(1-u)(1-w) \\ &= \left(\int f(u)/1-u\right) \cdot \left(\int g(w)/1-w\right) \\ &= (\neg\mu_A) \cdot (\neg\mu_B).\end{aligned}$$

Theorem 4. *Normal convex fuzzy grades (needless to say, any fuzzy grades, normal fuzzy grades and convex fuzzy grades) have the following properties. But the identity laws of (29), that is,* $\mu_A \cdot 0 = 0$ *and* $\mu_A \dot{+} 1 = 1$ *can be satisfied by normal fuzzy grades and normal convex fuzzy grades.*

Failure of idempotent laws: $\mu_A \cdot \mu_A \neq \mu_A; \qquad \mu_A \dot{+} \mu_A \neq \mu_A;$ (25)

Failure of absorption laws: $\begin{cases} \mu_A \cdot (\mu_A \dot{+} \mu_B) \neq \mu_A; \\ \mu_A \dot{+} (\mu_A \cdot \mu_B) \neq \mu_A; \end{cases}$ (26)

Failure of distributive laws: $\begin{cases} \mu_A \cdot (\mu_B \dot{+} \mu_C) \neq (\mu_A \cdot \mu_B) \dot{+} (\mu_A \cdot \mu_C); \\ \mu_A \dot{+} (\mu_B \cdot \mu_C) \neq (\mu_A \dot{+} \mu_B) \cdot (\mu_A \dot{+} \mu_C) \end{cases}$ (27)

Failure of complement laws: $\mu_A \cdot (\neg\mu_A) \neq 0, \qquad \mu_A \dot{+} (\neg\mu_A) \neq 1;$ (28)

Failure of identity laws: $\mu_A \cdot 0 \neq 0; \qquad \mu_A \dot{+} 1 \neq 1.$ (29)

Proof. We shall first prove the satisfaction of the identity laws (29) for normal fuzzy grades. Let $\mu_A = \int f(u)/u$ be a normal fuzzy grade, then $\bigvee_u f(u) = 1$ holds from (17). Thus,

$$\mu_A \cdot 0 = \left(\int f(u)/u\right) \cdot 1/0 = \bigvee_u f(u)/0 = 1/0 = 0,$$

which leads to $\mu_A \cdot 0 = 0$. The same holds for $\mu_A \dot{+} 1 = 1$.

Next, we shall give the example of normal convex fuzzy grades which do not satisfy the distributive law: $\mu_A \cdot (\mu_B \dot{+} \mu_C) = (\mu_A \cdot \mu_B) \dot{+} (\mu_A \cdot \mu_C)$ of (27). The failure of the other laws can be proved in the same way.

[4] The other part of identity laws, i.e., $\mu_A \cdot 0 = 0$, $\mu_A \dot{+} 1 = 1$ do not hold in general for arbitrary fuzzy grades (cf. Theorem 4).

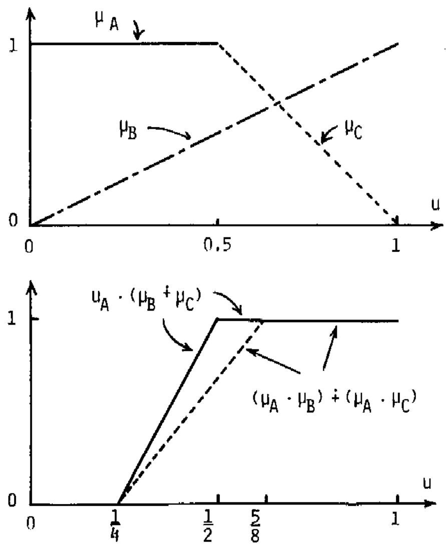

Fig. 4. Illustration of (27).

Let μ_A, μ_B and μ_C be normal convex fuzzy grades such that

$$\mu_A = \int_0^{0.5} 1/u, \qquad \mu_B = \int_0^1 u/u, \qquad \mu_C = \int_{0.5}^1 2(1-u)/u.$$

Then we have (see Fig. 4)

$$\mu_A \cdot (\mu_B \dot{+} \mu_C) = \int_{1/4}^{0.5} 4u - 1/u + \int_{0.5}^1 1/u,$$

$$(\mu_A \cdot \mu_B) \dot{+} (\mu_A \cdot \mu_C) = \int_{1/4}^{5/8} \tfrac{2}{3}(4u-1)/u + \int_{5/8}^1 1/u.$$

From the above theorems, we can immediately obtain the following.

Theorem 5. *Arbitrary fuzzy grades under algebraic product* ($\cdot$) *form a commutative semigroup with identity* 1. *The duality holds for algebraic sum* ($\dot{+}$), *where* 0 *is an identity. The same is true of normal fuzzy grades, convex fuzzy grades and normal convex fuzzy grades.*

Normal convex fuzzy grades (needless to say, any fuzzy grades, normal fuzzy grades, convex fuzzy grades) do not satisfy distributive laws, absorption laws etc. under $\cdot$ *and* $\dot{+}$, *and hence they do not form such algebraic structures as a lattice and a semiring.*

From Theorem 5 and the definitions of (6) and (7), the property concerning with fuzzy sets of type 2 under algebraic product and algebraic sum is derived.

Theorem 6. *Fuzzy sets of type 2 in a set X do not constitute such algebraic structures as a lattice and a semiring under algebraic product and algebraic sum.*

4. Properties of fuzzy grades under algebraic product (·) and algebraic sum ($\dot{+}$) combined with join ($\sqcup$) and meet ($\sqcap$)

This section describes the algebraic properties of fuzzy grades under the operations of algebraic product (·) and algebraic sum ($\dot{+}$) combined with join ($\sqcup$) and meet ($\sqcap$), and shows that normal convex fuzzy grades form a lattice ordered semigroup under join, meet and algebraic product.

At first, we shall briefly review the properties of fuzzy grades under join and meet (cf. [2]).

Join and meet: Join ($\sqcup$) and *meet* ($\sqcap$) of fuzzy grades μ_A and μ_B are defined as follows by using the extension principle:

Join: $$\mu_A \sqcup \mu_B = \int (f(u) \wedge g(w))/(u \vee w), \tag{30}$$

Meet: $$\mu_A \sqcap \mu_B = \int (f(u) \wedge g(w))/(u \wedge w), \tag{31}$$

where $\vee$ and $\wedge$ stand for max and min, respectively.

Property 1 ([2]). Arbitrary fuzzy grades satisfy idempotent laws, commutative laws and associative laws under join ($\sqcup$) and meet ($\sqcap$). Thus, they constitute a partially ordered set.

Property 2 ([2]). Convex fuzzy grades are closed and also satisfy distributive laws under $\sqcup$ and $\sqcap$. Therefore, they form a commutative semiring, but do not form a lattice since they do not satisfy absorption laws.

Property 3 ([2]). Normal convex fuzzy grades are closed and also satisfy absorption laws under $\sqcup$ and $\sqcap$. Thus, they form a distributive lattice under $\sqcup$ and $\sqcap$.

We shall begin with the following theorem.

Theorem 7. *Let μ_A be convex fuzzy grade, and let μ_B and μ_C be arbitrary fuzzy grades, then we obtain the following:*

$$\mu_A \cdot (\mu_B \sqcup \mu_C) = (\mu_A \cdot \mu_B) \sqcup (\mu_A \cdot \mu_C), \tag{32}$$

$$\mu_A \cdot (\mu_B \sqcap \mu_C) = (\mu_A \cdot \mu_B) \sqcap (\mu_A \cdot \mu_C), \tag{33}$$

$$\mu_A \dot{+} (\mu_B \sqcup \mu_C) = (\mu_A \dot{+} \mu_B) \sqcup (\mu_A \dot{+} \mu_C), \tag{34}$$

$$\mu_A \dot{+} (\mu_B \sqcap \mu_C) = (\mu_A \dot{+} \mu_B) \sqcap (\mu_A \dot{+} \mu_C). \tag{35}$$

Proof. We shall only prove (32). The others can be proved in the same way. Since the fuzzy grade μ_A is convex, the α-level set μ_A^α of μ_A is an interval $[a_1, a_2]$ in

[0, 1]. On the other hand, since μ_B and μ_C are arbitrary, each of the α-level sets μ_B^α and μ_C^α can consist of more than one interval. Thus, these α-level sets will be represented as

$$\mu_B^\alpha = \bigcup_{i=1}^{m} [b_{1i}, b_{2i}] \quad \text{and} \quad \mu_C^\alpha = \bigcup_{j=1}^{n} [c_{1j}, c_{2j}].$$

By the way, an interval in [0, 1] can be considered as a special case of fuzzy grade and thus the join of two intervals $[u_1, u_2]$ and $[w_1, w_2]$ can be given as

$$[u_1, u_2] \sqcup [w_1, w_2] = [u_1 \vee w_1, u_2 \vee w_2].$$

Therefore, the α-level set of the left-hand member of (32) will be

$$\begin{aligned}[\mu_A \cdot (\mu_B \sqcup \mu_C)]^\alpha &= \mu_A^\alpha \cdot (\mu_B^\alpha \sqcup \mu_C^\alpha) = [a_1, a_2] \cdot \left\{\bigcup_i [b_{1i}, b_{2i}] \sqcup \bigcup_j [c_{1j}, c_{2j}]\right\} \\ &= [a_1, a_2] \cdot \left\{\bigcup_{i,j} [b_{1i} \vee c_{1j}, b_{2i} \vee c_{2j}]\right\} \\ &= \bigcup_{i,j} [a_1(b_{1i} \vee c_{1j}), a_2(b_{2i} \vee c_{2j})].\end{aligned}$$

On the other hand, the right-hand member of (32) will be

$$\begin{aligned}[(\mu_A \cdot \mu_B) \sqcup (\mu_A \cdot \mu_C)]^\alpha &= (\mu_A^\alpha \cdot \mu_B^\alpha) \sqcup (\mu_A^\alpha \cdot \mu_C^\alpha) \\ &= \left\{[a_1, a_2] \cdot \bigcup_i [b_{1i}, b_{2i}]\right\} \sqcup \left\{[a_1, a_2] \cdot \bigcup_j [c_{1j}, c_{2j}]\right\} \\ &= \left\{\bigcup_i [a_1 b_{1i}, a_2 b_{2i}]\right\} \sqcup \left\{\bigcup_j [a_1 c_{1j}, a_2 c_{2j}]\right\} \\ &= \bigcup_{i,j} [a_1 b_{1i} \vee a_1 c_{1j}, a_2 b_{2i} \vee a_2 c_{2j}] \\ &= \bigcup_{i,j} [a_1(b_{1i} \vee c_{1j}), a_2(b_{2i} \vee c_{2j})] \\ &= [\mu_A \cdot (\mu_B \sqcup \mu_C)]^\alpha.\end{aligned}$$

Thus, we can obtain $\mu_A \cdot (\mu_B \sqcup \mu_C) = (\mu_A \cdot \mu_B) \sqcup (\mu_A \cdot \mu_C)$.

Theorem 8. *Normal convex fuzzy grades form a lattice ordered semigroup*[5] *with zero* 0 *and unity* 1 *under* $\sqcup, \sqcap$ *and* $\cdot$. *The duality holds for* $\sqcap, \sqcup$ *and* $\dot{+}$. *Normal*

[5] A lattice L which is a semigroup under $*$ and also satisfies the following distributive law is called a lattice ordered semigroup and is denoted as $L = (L, \vee, \wedge, *)$, where $\vee$ and $\wedge$ are operations of lub and glb in L, respectively. The distributive law is

$$x * (y \vee z) = (x * y) \vee (x * z); \qquad (x \vee y) * z = (x * z) \vee (y * z).$$

Moreover, $L = (L, \vee, \wedge, *)$ is said to be a *lattice ordered semigroup with unity I and zero* 0 if the following are satisfied for any x in L, i.e.,

$$x \vee 0 = x, \qquad x * 0 = 0 * x = 0,$$

$$x \vee I = I, \qquad x * I = I * x = x.$$

convex fuzzy grades also form a unitary $(=1)$ *commutative semiring*[6] *with zero* $(=0)$ *under* $\sqcup$ *(as addition) and* $\cdot$ *(as multiplication). The duality holds for* $\sqcap$ *and* $\dot{+}$. *Convex fuzzy grades form a unitary* $(=1)$ *commutative semiring under* $\sqcup$ *and* $\cdot$. *The duality holds for* $\sqcap$ *and* $\dot{+}$.

Proof. Normal convex fuzzy grades form a (distributive) lattice under $\sqcup$ and $\sqcap$ (Property 3) and also form a (commutative) semigroup under $\cdot$ (Theorem 5). Moreover, they satisfy the distributive law (32) and have a unity 1 $(=1/1)$ and a zero 0 $(=1/0)$ under $\sqcup$ and $\cdot$. Thus, they form a lattice ordered semigroup with unity and zero under $\sqcup, \sqcap$ and $\cdot$. It follows from Property 1, (21), (32), (20) and Theorem 4 that normal convex fuzzy grades also form a unitary $(=1)$ commutative semiring with zero $(=0)$ under $\sqcup$ (as addition) and $\cdot$ (as multiplication). It is noted that convex fuzzy grades under $\sqcup$ and $\cdot$ form a unitary $(=1)$ commutative semiring *without* zero.

In Theorem 7, it is shown that (32)–(35) hold when μ_A is convex. But, if μ_A is not convex, these identities do not hold even if μ_B and μ_C are convex.

Example 5. We shall show the example which does not satisfy (33) in the case where μ_A is nonconvex and μ_B and μ_C are convex. Let

$$\mu_A = \int_0^{0.5} 1-2u/u + \int_{0.5}^{1} 2u-1/u, \qquad \mu_B = \int_0^{0.5} 2u/u + \int_{0.5}^{1} 1/u,$$

$$\mu_C = \int_0^{0.5} 2u/u + \int_{0.5}^{1} 2(1-u)/u.$$

Then we have (see Fig. 5)

$$\mu_A \cdot (\mu_B \sqcap \mu_C) = \int_0^{3/16} \frac{3-\sqrt{1+16u}}{2}\Big/u + \int_{3/16}^{0.5} \frac{\sqrt{1+16u}-1}{2}\Big/u + \int_{0.5}^{1} 2(1-u)/u,$$

$$(\mu_A \cdot \mu_B) \sqcap (\mu_A \cdot \mu_C) = \int_0^{u_0} 1-2u/u + \int_{u_0}^{0.5} \frac{\sqrt{1+16u}-1}{2}\Big/u + \int_{0.5}^{1} 2(1-u)/u, \qquad u_0 = \frac{5-\sqrt{17}}{4}.$$

Thus it is found that (33) does not hold when μ_A is not convex.

[6] A *semiring* $(R, +, \times)$ is a set R with two operations $+$ and $\times$ of addition and multiplication such that $+$ is associative and commutative, and $\times$ is associative and distributive over $+$, i.e.,

$$a \times (b+c) = (a \times b) + (a \times c); \qquad (a+b) \times c = (a \times c) + (b \times c).$$

A semiring is *unitary* if $\times$ has a unit e, and is *commutative* if $\times$ is commutative, and is a semiring with *zero* if $+$ has an identity 0 such that $0 \times a = a \times 0 = 0$.

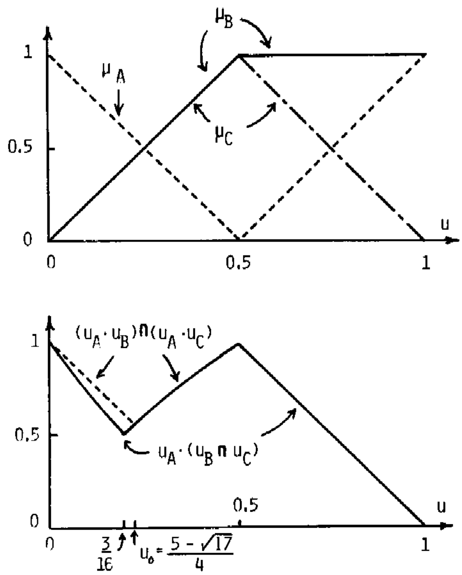

Fig. 5. Illustration of $\mu_A \cdot (\mu_B \sqcap \mu_C) \neq (\mu_A \cdot \mu_B) \sqcap (\mu_A \cdot \mu_C)$ when μ_A is nonconvex.

Theorem 9. *Normal convex fuzzy grades* μ_A, μ_B *and* μ_C *have the following properties. The same holds for arbitrary fuzzy grades.*

$$\mu_A \sqcup (\mu_B \cdot \mu_C) \neq (\mu_A \sqcup \mu_B) \cdot (\mu_A \sqcup \mu_C), \tag{36}$$

$$\mu_A \sqcap (\mu_B \cdot \mu_C) \neq (\mu_A \sqcap \mu_B) \cdot (\mu_A \sqcap \mu_C), \tag{37}$$

$$\mu_A \sqcup (\mu_B \dot{+} \mu_C) \neq (\mu_A \sqcup \mu_B) \dot{+} (\mu_A \sqcup \mu_C), \tag{38}$$

$$\mu_A \sqcap (\mu_B \dot{+} \mu_C) \neq (\mu_A \sqcap \mu_B) \dot{+} (\mu_A \sqcap \mu_C). \tag{39}$$

Theorem 10. *Let* μ_A *and* μ_B *be convex fuzzy grades, then*

$$(\mu_A \sqcup \mu_B) \cdot (\mu_A \sqcap \mu_B) = \mu_A \cdot \mu_B, \tag{40}$$

$$(\mu_A \sqcup \mu_B) \dot{+} (\mu_A \sqcap \mu_B) = \mu_A \dot{+} \mu_B. \tag{41}$$

If μ_A *and/or* μ_B *are nonconvex, the above identities are not satisfied.*

Proof. Let $\mu_A^\alpha = [a_1, a_2]$ and $\mu_B^\alpha = [b_1, b_2]$ be α-level sets of convex fuzzy grades μ_A and μ_B, respectively, then the left-hand member of (40) becomes

$$\begin{aligned}[(\mu_A \sqcup \mu_B) \cdot (\mu_A \sqcap \mu_B)]^\alpha &= ([a_1, a_2] \sqcup [b_1, b_2]) \cdot ([a_1, a_2] \sqcap [b_1, b_2]) \\ &= [a_1 \vee b_1, a_2 \vee b_2] \cdot [a_1 \wedge b_1, a_2 \wedge b_2] = [(a_1 \vee b_1)(a_1 \wedge b_1), (a_2 \vee b_2)(a_2 \wedge b_2)] \\ &= [a_1 b_1, a_2 b_2] = [a_1, a_2] \cdot [b_1, b_2] = \mu_A^\alpha \cdot \mu_B^\alpha = [\mu_A \cdot \mu_B]^\alpha.\end{aligned}$$

Example 6. Let μ_A be nonconvex fuzzy grade and μ_B be convex fuzzy grade, then (40) in Theorem 10 is shown not to be satisfied. Let μ_A be nonconvex fuzzy

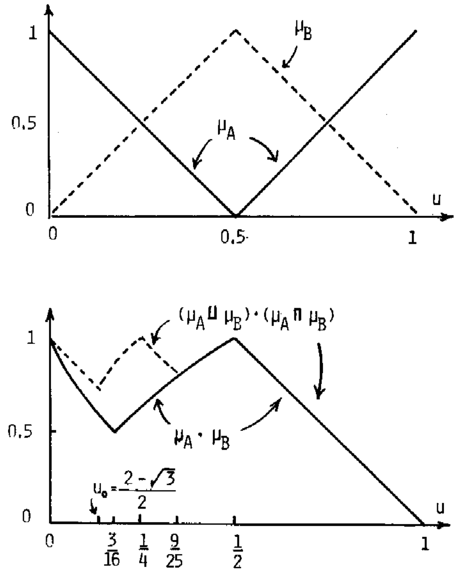

Fig. 6. Illustration of $(\mu_A \sqcup \mu_B)\cdot(\mu_A \sqcap \mu_B) \neq \mu_A \cdot \mu_B$ when μ_A is nonconvex.

grade such as

$$\mu_A = \int_0^{0.5} 1-2u/u + \int_{0.5}^{1} 2u-1/u,$$

and let μ_B be convex fuzzy grade such as

$$\mu_B = \int_0^{0.5} 2u/u + \int_{0.5}^{1} 2(1-u)/u.$$

Then we have (see Fig. 6)

$$(\mu_A \sqcup \mu_B)\cdot(\mu_A \sqcap \mu_B) = \int_0^{u_0} 1-2u/u + \int_{u_0}^{1/4} 2\sqrt{u}/u + \int_{1/4}^{9/25} 2(1-\sqrt{u})/u$$
$$+ \int_{9/25}^{0.5} \frac{-1+\sqrt{1+16u}}{2} \Big/ u + \int_{0.5}^{1} 2(1-u)/u,$$
$$u_0 = 1 - \frac{\sqrt{3}}{2},$$

$$\mu_A \cdot \mu_B = \int_0^{3/16} \frac{3-\sqrt{1+16u}}{2} \Big/ u + \int_{3/16}^{0.5} \frac{-1+\sqrt{1+16u}}{2} \Big/ u + \int_{0.5}^{1} 2(1-u)/u.$$

Thus it has been shown that (40) does not hold when μ_A and/or μ_B are nonconvex.

5. Conclusion

The operations of algebraic product and algebraic sum on ordinary fuzzy sets are used in the studies of fuzzy events [7], fuzzy automata [5], fuzzy logic [1] and so on. Thus, these operations performed on fuzzy sets of type 2 will find a number of applications in the studies of fuzzy sets.

The algebraic properties of fuzzy sets of type 2 under bounded-sum and bounded-difference combined with union, intersection, algebraic product and algebraic sum will be presented in subsequent papers.

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