Fuzzy Sets and Their Operations

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Among the basic operations which can be performed on fuzzy sets are the operations of union, intersection, complement, algebraic product and algebraic sum. In addition to these operations, new operations called "bounded-sum" and "bounded-difference" were defined by L. A. Zadeh to investigate the fuzzy reasoning which provides a way of dealing with the reasoning problems which are too complex for precise solution. This paper investigates the algebraic properties of fuzzy sets under these nex operations of bounded-sum and bounded-difference and the properties of fuzzy sets in the case where these new operations are combined with the well-known operations of union, intersection, algebraic product and algebraic sum.

1. INTRODUCTION

Among the well-known operations which can be performed on fuzzy sets are the operations of union, intersection, complement, algebraic product and algebraic sum. Much research concerning fuzzy sets and their applications to automata theory, logic, control, game, topology, pattern recognition, integral, linguistics, taxonomy, system, decision making, information retrieval and so on, has been earnestly undertaken by using these operations for fuzzy sets (see the bibliography in Gaines (1977) and Kandel (1978)). For example, union, intersection and complement are found in most of papers relating to fuzzy sets. Algebraic product and algebraic sum are also used in the study of fuzzy events (Zadeh, 1968), fuzzy automata (Santos, 1972), fuzzy logic (Goguen, 1968), fuzzy semantics (Zadeh, 1971) and so on.

In addition to these operations, new operations called "bounded-sum" and

"bounded-difference" are introduced by Zadeh (1975) to investigate the fuzzy reasoning which provides a way of dealing with the reasoning problems which are too complex for precise solution.

This paper investigates the algebraic properties of fuzzy sets under bounded-sum and bounded-difference as well as the properties of fuzzy sets in the case where these new operations are combined with the well-known operations of union, intersection, algebraic product and algebraic sum.

2. Fuzzy Sets and Their Operations

We shall briefly review fuzzy sets and their operations of union, intersection, complement, algebraic product, algebraic sum, bounded-sum, bounded-difference and bounded-product, which is a dual operation for bounded-sum.

Fuzzy Sets: A fuzzy set A in a universe of discourse U is characterized by a membership function μ_A which takes the values in the unit interval $[0, 1]$, i.e.,

$$\mu_A : U \to [0, 1]. \tag{1}$$

The value of μ_A at $u(\in U)$, $\mu_A(u)$, represents the *grade of membership* (*grade*, for short) of u in A and is a point in $[0, 1]$.

The operations of fuzzy sets A and B are listed as follows.

Union:

$$A \cup B \Leftrightarrow \mu_{A \cup B} = \mu_A \vee \mu_B. \tag{2}$$

Intersection:

$$A \cap B \Leftrightarrow \mu_{A \cap B} = \mu_A \wedge \mu_B. \tag{3}$$

Complement:

$$\bar{A} \Leftrightarrow \mu_{\bar{A}} = 1 - \mu_A. \tag{4}$$

Algebraic Product:

$$A \cdot B \Leftrightarrow \mu_{A \cdot B} = \mu_A \mu_B. \tag{5}$$

Algebraic Sum:

$$A \dot{+} B \Leftrightarrow \mu_{A \dot{+} B} = \mu_A + \mu_B - \mu_A \mu_B$$
$$= 1 - (1 - \mu_A)(1 - \mu_B). \tag{6}$$

Bounded-Sum:

$$A \oplus B \Leftrightarrow \mu_{A \oplus B} = 1 \wedge (\mu_A + \mu_B). \tag{7}$$

Bounded-Difference:

$$A \ominus B \Leftrightarrow \mu_{A \ominus B} = 0 \vee (\mu_A - \mu_B). \tag{8}$$

Bounded-Product:

$$A \odot B \Leftrightarrow \mu_{A \odot B} = 0 \vee (\mu_A + \mu_B - 1), \tag{9}$$

where the operations of $\vee$, $\wedge$, $+$, $-$ represent max, min, arithmetic sum, and arithmetic difference, respectively.

3. ALGEBRAIC PROPERTIES OF FUZZY SETS UNDER VARIOUS KINDS OF OPERATIONS

In this section we shall investigate the algebraic properties of fuzzy sets under the operations (2)–(9). We shall first review the well-known properties of fuzzy sets under union (2), intersection (3), and complement (4).

I. *The Case of Union* ($\cup$) *and Intersection* ($\cap$)

Let A, B and C be fuzzy sets in a universe of discourse U, then we have (see Zadeh (1965)):

Idempotent laws:

$$A \cup A = A,$$
$$A \cap A = A. \tag{10}$$

Commutative laws:

$$A \cup B = B \cup A,$$
$$A \cap B = B \cap A. \tag{11}$$

Associative laws:

$$(A \cup B) \cup C = A \cup (B \cup C),$$
$$(A \cap B) \cap C = A \cap (B \cap C). \tag{12}$$

Absorption laws:

$$A \cup (A \cap B) = A,$$
$$A \cap (A \cup B) = A. \tag{13}$$

Distributive laws:

$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C),$$
$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C). \tag{14}$$

Involution law:

$$\bar{\bar{A}} = A. \tag{15}$$

De Morgan's laws:

$$\overline{A \cup B} = \bar{A} \cap \bar{B},$$
$$\overline{A \cap B} = \bar{A} \cup \bar{B}. \tag{16}$$

Identity laws:

$$A \cup \varnothing = A, \qquad A \cup U = U,$$
$$A \cap \varnothing = \varnothing, \qquad A \cap U = A. \tag{17}$$

Complement laws:

$$A \cup \bar{A} \neq U,$$
$$A \cap \bar{A} \neq \varnothing, \tag{18}$$

where $\varnothing$ is an empty fuzzy set defined by $\mu_\varnothing = 0$.

Note. Equations (18) can be expressed more precisely as

$$0.5U \subseteq A \cup \bar{A} \subseteq U,$$
$$\varnothing \subseteq A \cap \bar{A} \subseteq 0.5U, \tag{19}$$

where $\mu_{0.5U} = 0.5\mu_U = 0.5 \times 1 = 0.5$.

THEOREM 1 (Zadeh, 1965). *Fuzzy sets in U form a distributive lattice[1] under $\cup$ and $\cap$, but do not form a Boolean lattice, since $\bar{A}$ is not the complement of A in the lattice sense (see (18, 19)).*

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[1] A set L with two operations $\vee$ and $\wedge$ satisfying idempotent laws, commutative laws, associative laws and absorption laws is said to be a *lattice*. If the lattice L satisfies distributive laws, then L is a *distributive lattice*. If the complement laws $a \vee \bar{a} = 1$ and $a \wedge \bar{a} = 0$ hold, L is a *Boolean lattice*.

THEOREM 2. *Fuzzy sets also form a unitary commutative semiring with zero[2] under the operations $\cup$ and $\cap$.*

Proof. This can be shown by letting $+ = \cup$, $\times = \cap$, $1 = U$ and $0 = \varnothing$ in Footnote 2.

We shall next review the algebraic properties of fuzzy sets under the operations of algebraic product (5) and algebraic product (6) (cf. Kaufmann (1973)).

II. *The Case of Algebraic Sum $(\dotplus)$ and Algebraic Product $(\cdot)$*

Idempotency:

$$A \dotplus A \supseteq A,$$
$$A \cdot A \subseteq A. \tag{20}$$

Commutativity:

$$A \dotplus B = B \dotplus A,$$
$$A \cdot B = B \cdot A. \tag{21}$$

Associativity:

$$(A \dotplus B) \dotplus C = A \dotplus (B \dotplus C),$$
$$(A \cdot B) \cdot C = A \cdot (B \cdot C). \tag{22}$$

Absorption:

$$A \dotplus (A \cdot B) \supseteq A,$$
$$A \cdot (A \dotplus B) \subseteq A. \tag{23}$$

Distributivity:

$$A \dotplus (B \cdot C) \supseteq (A \dotplus B) \cdot (A \dotplus C),$$
$$A \cdot (B \dotplus C) \subseteq (A \cdot B) \dotplus (A \cdot C). \tag{24}$$

De Morgan's laws:

$$\overline{A \dotplus B} = \bar{A} \cdot \bar{B},$$
$$\overline{A \cdot B} = \bar{A} \dotplus \bar{B}. \tag{25}$$

[2] A *semiring* $(R, +, \times)$ is a set R with two operations $+$ and $\times$ of addition and multiplication such that $+$ is associative and commutative, and $\times$ is associative and distributive over $+$, i.e., $a \times (b + c) = (a \times b) + (a \times c)$. A semiring is *unitary* if $\times$ has a unity 1, and is *commutative* if $\times$ is commutative, and is a semiring with *zero* if $+$ has an identity 0 such that $0 \times a = a \times 0 = 0$.

Identities:

$$A \overset{\cdot}{+} \varnothing = A, \qquad A \overset{\cdot}{+} U = U,$$
$$A \cdot \varnothing = \varnothing, \qquad A \cdot U = A. \tag{26}$$

Complementarity:

$$A \overset{\cdot}{+} \bar{A} \neq U,$$
$$A \cdot \bar{A} \neq \varnothing. \tag{27}$$

Remark. Equations (27) can be rewritten more precisely as

$$0.75U \subseteq A \overset{\cdot}{+} \bar{A} \subseteq U,$$
$$\varnothing \subseteq A \cdot \bar{A} \subseteq 0.25U, \tag{28}$$

Thus we can easily obtain the next theorem.

THEOREM 3. *Fuzzy sets in U under algebraic sum $(\overset{\cdot}{+})$ and algebraic product $(\cdot)$ do not constitute such algebraic structures as a lattice and a semiring. Fuzzy sets, however, form a commutative monoid under $\overset{\cdot}{+}$ (or $\cdot$).*[3]

We shall next discuss the absorption property and the distributivity in the case where the operations of algebraic sum, algebraic product, union, and intersection are combined each other.

III. *The Case of Algebraic Sum $(\overset{\cdot}{+})$ and Algebraic Product $(\cdot)$ Combined with Union $(\cup)$ and Intersection $(\cap)$*

Absorption:

$$A \cdot (A \cup B) \subseteq A, \tag{29}$$

$$A \cdot (A \cap B) \subseteq A, \tag{30}$$

$$A \overset{\cdot}{+} (A \cup B) \supseteq A, \tag{31}$$

$$A \overset{\cdot}{+} (A \cap B) \supseteq A, \tag{32}$$

$$A \cup (A \cdot B) = A, \tag{33}$$

$$A \cap (A \cdot B) \subseteq A, \tag{34}$$

$$A \cup (A \overset{\cdot}{+} B) \supseteq A, \tag{35}$$

$$A \cap (A \overset{\cdot}{+} B) = A. \tag{36}$$

[3] A *semigroup* $(S, *)$ is a set S together with an operation $*$ such that $*$ is associative. A *monoid* (or *unitary semigroup*) is a semigroup with identity under $*$. The monoid is called *commutative* if $*$ is commutative.

Distributivity:

$$A \cdot (B \cup C) = (A \cdot B) \cup (A \cdot C), \tag{37}$$

$$A \cdot (B \cap C) = (A \cdot B) \cap (A \cdot C), \tag{38}$$

$$A \dotplus (B \cup C) = (A \dotplus B) \cup (A \dotplus C), \tag{39}$$

$$A \dotplus (B \cap C) = (A \dotplus B) \cap (A \dotplus C), \tag{40}$$

$$A \cup (B \cdot C) \supseteq (A \cup B) \cdot (A \cup C), \tag{41}$$

$$A \cap (B \cdot C) \supseteq (A \cap B) \dotplus (A \cap C), \tag{42}$$

$$A \cup (B \dotplus C) \subseteq (A \cup B) \dotplus (A \cup C), \tag{43}$$

$$A \cap (B \dotplus C) \subseteq (A \cap B) \dotplus (A \cap C). \tag{44}$$

THEOREM 4. *Fuzzy sets form a unitary* $(=U)$ *commutative semiring with zero* $(=\varnothing)$ *under* $\cup$ *(as addition) and algebraic product* $\cdot$ *(as multilication). The duality holds for intersection* $\cap$ *(as addition) and algebraic sum* $\dotplus$ *(as multiplication). Fuzzy sets also form a lattice ordered semigroup*[4] *with zero* $\varnothing$ *and unity* U *under* $\cup$, $\cap$ *and* $\cdot$. *The duality holds for* $\cap$, $\cup$ *and* $\dotplus$.

We shall next discuss the algebraic properties of fuzzy sets under the operations of bounded-sum $\oplus$ (7), bounded-difference $\ominus$ (8) which were defined by Zadeh (1975), and bounded-product $\odot$ (9) which is a new operation dual to bounded-sum. The new operation of bound-product $\odot$ can be expressed by using De Morgan's laws to be shown in

$$A \odot B = \overline{\bar{A} \oplus \bar{B}} = A \ominus \bar{B}. \tag{45}$$

IV. *The Case of Bounded-Sum* $\oplus$, *Bounded-Difference* $\ominus$
     and Bounded-Product $\odot$

Idempotency:

$$A \oplus A \supseteq A, \tag{46}$$

$$A \odot A \subseteq A, \tag{47}$$

$$A \ominus A = \varnothing. \tag{48}$$

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[4] A lattice L which is a semigroup under $*$ and also satisfies the following distributive law is called a *lattice ordered semigroup* and denoted as $L = (L, \vee, \wedge, *)$, where $\vee$ and $\wedge$ are operations of *lub* and *glb* in L, respectively. The distributive law is $x * (y \vee z) = (x * y) \vee (x * z)$. Moreover, $L = (L, \vee, \wedge, *)$ is said to be a *lattice ordered semigroup with unity* I *and zero* 0 if the following are satisfied for any a in L, i.e.,

$$a \vee 0 = a; \quad a * 0 = 0 * a = 0,$$
$$a \vee I = I; \quad a * I = I * a = a.$$

Commutativity:

$$A \oplus B = B \oplus A, \tag{49}$$

$$A \odot B = B \odot A, \tag{50}$$

$$A \ominus B \neq B \ominus A. \tag{51}$$

Associativity:

$$(A \oplus B) \oplus C = A \oplus (B \oplus C), \tag{52}$$

$$(A \odot B) \odot C = A \odot (B \odot C), \tag{53}$$

$$(A \ominus B) \ominus C \subseteq A \ominus (B \ominus C). \tag{54}$$

As a special case of (54), we have

$$A \ominus (A \ominus B) = A \cap B. \tag{55}$$

Absorption:

(i) *Case of $\oplus$ and $\odot$*:

$$A \oplus (A \odot B) \supseteq A, \tag{56}$$

$$A \odot (A \oplus B) \subseteq A. \tag{57}$$

(ii) *Case of $\oplus$ and $\ominus$*:

$$A \oplus (A \ominus B) \supseteq A, \tag{58}$$

$$A \oplus (B \ominus A) = A \cup B, \tag{59}$$

$$A \ominus (A \oplus B) = \varnothing, \tag{60}$$

$$(A \oplus B) \ominus A = \bar{A} \cap B. \tag{61}$$

(iii) *Case of $\odot$ and $\ominus$*:

$$A \odot (A \ominus B) \subseteq A, \tag{62}$$

$$A \odot (B \ominus A) = \varnothing, \tag{63}$$

$$A \ominus (A \odot B) = A \cap \bar{B}, \tag{64}$$

$$(A \odot B) \ominus A = \varnothing. \tag{65}$$

Distributivity:

(i) *Case of $\oplus$ and $\odot$*:

$$A \oplus (B \odot C) \neq (A \oplus B) \odot (A \oplus C), \tag{66}$$

$$A \odot (B \oplus C) \neq (A \odot B) \oplus (A \odot C). \tag{67}$$

(ii)   *Case of* $\oplus$ *and* $\ominus$:

$$A \oplus (B \ominus C) \supseteq (A \oplus B) \ominus (A \oplus C), \qquad (68)$$

$$A \ominus (B \oplus C) \subseteq (A \ominus B) \oplus (A \ominus C), \qquad (69)$$

$$(B \oplus C) \ominus A \neq (B \ominus A) \oplus (C \ominus A). \qquad (70)$$

(iii)   *Case of* $\odot$ *and* $\ominus$:

$$A \odot (B \ominus C) \subseteq (A \odot B) \ominus (A \odot C), \qquad (71)$$

$$A \ominus (B \odot C) \supseteq (A \ominus B) \odot (A \ominus C), \qquad (72)$$

$$(B \odot C) \ominus A \supseteq (B \ominus A) \odot (C \ominus A). \qquad (73)$$

De Morgan's Laws:

$$\overline{A \oplus B} = \bar{A} \odot \bar{B}, \qquad (74)$$

$$\overline{A \odot B} = \bar{A} \oplus \bar{B}, \qquad (75)$$

$$\overline{A \oplus B} = \bar{A} \ominus B. \qquad (76)$$

Furthermore,

$$\overline{A \ominus B} = \bar{A} \oplus B, \qquad (77)$$

$$\bar{A} \ominus \bar{B} = B \ominus A. \qquad (78)$$

Finally,

$$A \odot \bar{B} = A \ominus B, \qquad (79)$$

$$A \ominus \bar{B} = A \odot B. \qquad (80)$$

Identities:

$$A \oplus \varnothing = A, \qquad (81)$$

$$A \oplus U = U, \qquad (82)$$

$$A \odot \varnothing = \varnothing, \qquad (83)$$

$$A \odot U = A. \qquad (84)$$

Moreover,

$$A \ominus \varnothing = A, \qquad (85)$$

$$\varnothing \ominus A = \varnothing, \qquad (86)$$

$$A \ominus U = \varnothing, \qquad (87)$$

$$U \ominus A = \bar{A}. \qquad (88)$$

Complementarity:

$$A \oplus \bar{A} = U, \tag{89}$$

$$A \odot \bar{A} = \varnothing, \tag{90}$$

$$\varnothing \subseteq A \ominus \bar{A} \subseteq U, \tag{91}$$

$$\varnothing \subseteq \bar{A} \ominus A \subseteq U. \tag{92}$$

Remark 1.   From (52) and (53) it is found that the operations $\oplus$ and $\odot$ are associative. Thus can represent $A_1 \oplus A_2 \oplus \cdots \oplus A_n$ and $A_1 \odot A_2 \odot \cdots \odot A_n$ as

$$\mu_{A_1 \oplus A_2 \oplus \cdots \oplus A_n} = 1 \wedge (\mu_{A_1} + \mu_{A_2} + \cdots + \mu_{A_n}),$$

$$\mu_{A_1 \odot A_2 \odot \cdots \odot A_n} = 0 \vee [\mu_{A_1} + \mu_{A_2} + \cdots + \mu_{A_n} - (n-1)].$$

If $A_1 = A_2 = \cdots = A_n (=A)$, then we can have

$$\mu_{A \oplus A \oplus \cdots \oplus A} = 1 \wedge n\mu_A,$$

$$\mu_{A \odot A \odot \cdots \odot A} = 0 \vee (1 - n\mu_{\bar{A}}).$$

Remark 2.   The operations $\cup$, $\cap$, $^-$ over fuzzy sets can be represented by using $\oplus$, $\ominus$, (and $\odot$), that is, by using (59), (55), (61) and (88). Namely,

$$A \cup B = A \oplus (B \ominus A), \tag{93}$$

$$A \cap B = A \ominus (A \ominus B) = (\bar{A} \oplus B) \ominus \bar{A}, \tag{94}$$

$$\bar{A} = U \ominus A. \tag{95}$$

It should be noted that $\oplus$, $\odot$, $\ominus$ are shown not to be represented by $\cup$, $\cap$, and $^-$.

Remark 3.   Fuzzy sets under $\oplus$ and $\odot$ satisfy the complement laws (89)–(90), though they do not satisfy these laws under $\cup$ and $\cap$, and $\cdot$ and $+$. Note that we have $0.5U \subseteq A \cup \bar{A} \subseteq U$; $\varnothing \subseteq A \cap \bar{A} \subseteq 0.5U$ under $\cup$ and $\cap$ from (19), and $0.75U \subseteq A \dotplus \bar{A} \subseteq U$; $\varnothing \subseteq A \cdot \bar{A} \subseteq 0.25U$ under $\dotplus$ and $\cdot$ from (28).

From the above property concerning $\oplus$, $\odot$ and $\ominus$, we can immediately obtain the following theorem.

THEOREM 5.   *Fuzzy sets under $\oplus$ and $\odot$ do not satisfy the absorption and distributive laws and hence do not form such algebraic structures as a lattice and a semiring. The same is true of $\oplus$ and $\ominus$, and of $\odot$ and $\ominus$. Fuzzy sets, however, form a commutative monoid under $\oplus$ (or $\odot$), but do not form such a structure under $\ominus$*

We shall next deal with the absorption and distributive properties for fuzzy sets under the operations of bounded-sum, bounded-difference and bounded-product combined with the operations of union and intersection.

V. *The Case of Bounded-Sum $\oplus$, Bounded-Difference $\ominus$ and Bounded $\odot$ Combined with Union $\cup$ and Intersection $\cap$-*

Absorption:

$$A \cup (A \oplus B) \supseteq A, \tag{96}$$

$$A \cap (A \oplus B) = A, \tag{97}$$

$$A \cup (A \odot B) = A, \tag{98}$$

$$A \cap (A \odot B) \subseteq A, \tag{99}$$

$$A \cup (A \ominus B) = A, \tag{100}$$

$$A \cup (B \ominus A) \neq A, \tag{101}$$

$$A \cap (A \ominus B) \subseteq A, \tag{102}$$

$$A \cap (B \ominus A) \neq A. \tag{103}$$

Moreover,

$$A \oplus (A \cup B) \supseteq A, \tag{104}$$

$$A \oplus (A \cap B) \supseteq A, \tag{105}$$

$$A \odot (A \cup B) \subseteq A, \tag{106}$$

$$A \odot (A \cap B) \subseteq A, \tag{107}$$

$$A \ominus (A \cup B) = \varnothing, \tag{108}$$

$$(A \cup B) \ominus A = B \ominus A \neq A, \tag{109}$$

$$A \ominus (A \cap B) = A \ominus B \subseteq A, \tag{110}$$

$$(A \cap B) \ominus A = \varnothing, \tag{111}$$

Distributivity:

$$A \cup (B \oplus C) \subseteq (A \cup B) \oplus (A \cup C), \tag{112}$$

$$A \cap (B \oplus C) \subseteq (A \cap B) \oplus (A \cap C), \tag{113}$$

$$A \cup (B \odot C) \supseteq (A \cup B) \odot (A \cup C), \tag{114}$$

$$A \cap (B \odot C) \supseteq (A \cap B) \odot (A \cap C), \tag{115}$$

$$A \cup (B \ominus C) \supseteq (A \cup B) \ominus (A \cup C), \tag{116}$$

$$A \cap (B \ominus C) \supseteq (A \cap B) \ominus (A \cap C), \tag{117}$$

and

$$A \oplus (B \cup C) = (A \oplus B) \cup (A \oplus C), \tag{118}$$

$$A \oplus (B \cap C) = (A \oplus B) \cap (A \oplus C), \tag{119}$$

$$A \odot (B \cup C) = (A \odot B) \cup (A \odot C), \tag{120}$$

$$A \odot (B \cap C) = (A \odot B) \cap (A \odot C), \tag{121}$$

$$A \ominus (B \cup C) = (A \ominus B) \cap (A \ominus C), \tag{122}$$

$$A \ominus (B \cap C) = (A \ominus B) \cup (A \ominus C), \tag{123}$$

$$(B \cup C) \ominus A = (B \ominus A) \cup (C \ominus A), \tag{124}$$

$$(B \cap C) \ominus A = (B \ominus A) \cap (C \ominus A), \tag{125}$$

THEOREM 6. *Fuzzy sets satisfy associative laws, commutative laws and distributive law* (120) *under the operations of bounded-product $\odot$ and union $\cup$, and thus they form a unitary (=U) commutative semiring with zero (=Ø) under $\odot$ (as multiplication) and $\cup$ (as addition). Dually, fuzzy sets under the operations of bounded-sum $\oplus$ (as multiplication) and intersection $\cap$ (as addition) form a unitary (=Ø) commutative semiring with zero (=U). Moreover, fuzzy sets under $\oplus$ (as multiplication) and $\cup$ (as addition) form a unitary (=Ø) commutative semiring. The same holds for $\odot$ (as multiplication) and $\cap$ (as addition), where U becomes a unit element for $\cap$. Furthermore, fuzzy sets also form a lattice ordered semigroup with unity U and zero Ø under $\cup$, $\cap$ and $\odot$, where $\odot$ is a semigroup operation. Dually, they form a lattice ordered semigroup with unity Ø and zero U under $\cap$, $\cup$ and $\oplus$.*

As a generalization of V, the following formulas can be obtained.

VI. *Formulas Obtained as a Generalization of* V

$$(A \cup B) \oplus (C \cup D) = (A \oplus C) \cup (A \oplus D) \cup (B \oplus C) \cup (B \oplus D), \tag{126}$$

$$(A \cap B) \oplus (C \cap D) = (A \oplus C) \cap (A \oplus D) \cap (B \oplus C) \cap (B \oplus D), \tag{127}$$

$$(A \cup B) \oplus (A \cap B) = A \oplus B, \tag{128}$$

$$(A \cup B) \odot (C \cup D) = (A \odot C) \cup (A \odot D) \cup (B \odot C) \cup (B \odot D), \tag{129}$$

$$(A \cap B) \odot (C \cap D) = (A \odot C) \cap (A \odot D) \cap (B \odot C) \cap (B \odot D), \tag{130}$$

$$(A \cup B) \odot (A \cap B) = A \odot B, \tag{131}$$

$$(A \cup B) \ominus (C \cap D) = (A \ominus C) \cup (A \ominus D) \cup (B \ominus C) \cup (B \ominus D), \tag{132}$$

$$(A \cap B) \ominus (C \cup D) = (A \ominus C) \cap (A \ominus D) \cap (B \ominus C) \cap (B \ominus D), \tag{133}$$

$$(A \cup B) \ominus (A \cap B) = (A \ominus B) \cup (B \ominus A) = |A - B|. \tag{134}$$

We shall next discuss the absorption and distributive properties under the operations of bounded-sum, bounded-difference and bounded-product combined with the operations of algebraic product and algebraic sum.

VII. *The Case of Bounded-Sum $\oplus$, Bounded-Difference $\ominus$ and Bounded-Product $\oplus$ Combined with Algebraic Product $\cdot$ and Algebraic Sum $\dotplus$*

Absorption:

$$A \cdot (A \oplus B) \subseteq A, \tag{135}$$

$$A \dotplus (A \oplus B) \supseteq A, \tag{136}$$

$$A \cdot (A \odot B) \subseteq A, \tag{137}$$

$$A \dotplus (A \odot B) \supseteq A, \tag{138}$$

$$A \cdot (A \ominus B) \subseteq A, \tag{139}$$

$$A \cdot (B \ominus A) \subseteq A, \tag{140}$$

$$A \dotplus (A \ominus B) \supseteq A, \tag{141}$$

$$A \dotplus (B \ominus A) \supseteq A, \tag{142}$$

and

$$A \oplus (A \cdot B) \supseteq A, \tag{143}$$

$$A \oplus (A \dotplus B) \supseteq A, \tag{144}$$

$$A \odot (A \cdot B) \subseteq A, \tag{145}$$

$$A \odot (A \dotplus B) \subseteq A, \tag{146}$$

$$A \ominus (A \cdot B) = A \cdot \bar{B} \subseteq A, \tag{147}$$

$$(A \cdot B) \ominus A = \varnothing, \tag{148}$$

$$A \ominus (A \dotplus B) = \varnothing, \tag{149}$$

$$(A \dotplus B) \ominus A = \bar{A} \cdot B. \tag{150}$$

Distributivity:

$$A \cdot (B \oplus C) \subseteq (A \cdot B) \oplus (A \cdot C), \tag{151}$$

$$A \dotplus (B \oplus C) \subseteq (A \dotplus B) \oplus (A \dotplus C), \tag{152}$$

$$A \cdot (B \odot C) \supseteq (A \cdot B) \odot (A \cdot C), \tag{153}$$

$$A \dotplus (B \odot C) \supseteq (A \dotplus B) \odot (A \dotplus C), \tag{154}$$

$$A \cdot (B \ominus C) = (A \cdot B) \ominus (A \cdot C), \tag{155}$$

$$A \dotplus (B \ominus C) \supseteq (A \dotplus B) \ominus (A \dotplus C). \tag{156}$$

Furthermore,

$$A \oplus (B \cdot C) \neq (A \oplus B) \cdot (A \oplus C), \tag{157}$$

$$A \oplus (B \dotplus C) \neq (A \oplus B) \dotplus (A \oplus C), \tag{158}$$

$$A \odot (B \cdot C) \neq (A \odot B) \cdot (A \odot C), \tag{159}$$

$$A \odot (B \dotplus C) \neq (A \odot B) \dotplus (A \odot C), \tag{160}$$

$$A \ominus (B \cdot C) \supseteq (A \ominus B) \cdot (A \ominus C), \tag{161}$$

$$(B \cdot C) \ominus A \neq (B \ominus A) \cdot (C \ominus A), \tag{162}$$

$$A \ominus (B \dotplus C) \subseteq (A \ominus B) \dotplus (A \ominus C), \tag{163}$$

$$(B \dotplus C) \ominus A \neq (B \ominus A) \dotplus (C \ominus A). \tag{164}$$

THEOREM 7. *Fuzzy sets under bounded-sum $\oplus$ and algebraic product $\cdot$ do not form such algebraic structures as a lattice and a semiring, since they do not satisfy the distributive laws and the absorption laws. The same is true of $(\oplus, \dotplus)$, $(\odot, \cdot)$, $(\odot, \dotplus)$, $(\ominus, \cdot)$ and $(\ominus, \dotplus)$.*

Remark. Although fuzzy sets do satify the distributive law (155) under $\ominus$ and $\cdot$, they do not satisfy the associative law and the commutative law under $\ominus$ (see (54), (51)) and thus they do not constitute a lattice and a semiring under these operations.

Finally, we shall list the properties of fuzzy sets under containment relation $\subseteq$.

VIII. *Properties of Fuzzy Sets under Containment Relation $\subseteq$*

$$A \odot B \subseteq A \cdot B \subseteq A \cap B, \tag{165}$$

$$A \oplus B \supseteq A \dotplus B \supseteq A \cup B, \tag{166}$$

$$A \ominus B \subseteq A \cap \bar{B}, \tag{167}$$

$$A \subseteq B, C \subseteq D \Rightarrow A \cup C \subseteq B \cup D, \tag{168}$$

$$\Rightarrow A \cap C \subseteq B \cap D, \tag{169}$$

$$\Rightarrow A \cdot C \subseteq B \cdot D, \tag{170}$$

$$\Rightarrow A \dotplus C \subseteq B \dotplus D, \tag{171}$$

$$\Rightarrow A \oplus C \subseteq B \oplus D, \tag{172}$$

$$\Rightarrow A \odot C \subseteq B \odot D, \tag{173}$$

$$A \subseteq B, D \subseteq C \Rightarrow A \ominus C \subseteq B \ominus D, \tag{174}$$

$$A \subseteq B \Leftrightarrow A \ominus B = \varnothing, \tag{175}$$

$$\Leftrightarrow \bar{A} \oplus B = U, \qquad (176)$$

$$\Leftrightarrow A \cup B = B, \qquad (177)$$

$$\Leftrightarrow A \cap B = A. \qquad (178)$$

4. Relationship between Fuzzy Set and Many-Valued Logic

The theory of fuzzy sets may be embedded in a many-valued logic by interpreting the grade of membership $\mu_A(u)$ as representing the truth value of the statement "u is in the fuzzy set A." This section shows that the operations of bounded-sum $\oplus$ and bounded-product $\odot$ for fuzzy sets are corresponding to the operations of "sum" and "product," respectively, in Lukasiewicz's many-valued logic (Izeki, 1963), and then discusses the representation of implication ($\rightarrow$) and equivalence ($\leftrightarrow$) by means of $\oplus$, $\odot$ and $\ominus$.

In many-valued (or continuous) logic (Rescher, 1969; Ginzburg, 1967) and fuzzy logic (Marinos, 1969; Lee and Chang, 1971; Kandel, 1974), the truth value $\vee(P)$ of a statement P takes the value in the unit interval $[0, 1]$, in which $\vee(P) = 1$ indicates the statement P is completely true and $\vee(P) = 0$ indicates P is completely false.

The operations proposed in many-valued logic and fuzzy logic are listed below:

Logical Sum:

$$x \vee y = \max\{x, y\}. \qquad (179)$$

Logical Product:

$$x \wedge y = \min\{x, y\}. \qquad (180)$$

Negation:

$$\bar{x} = 1 - x. \qquad (181)$$

Sum:

$$x \oplus y = 1 \wedge (x + y). \qquad (182)$$

Product:

$$x \odot y = 0 \vee (x + y - 1). \qquad (183)$$

Implication:

$$x \rightarrow y = 1 \wedge (1 - x + y). \qquad (184)$$

Equivalence:

$$x \leftrightarrow y = (1 - x + y) \wedge (1 - y + x). \tag{185}$$

From these definitions it is easily found that the operations of fuzzy sets are closely connected with these operations of many-valued logic and fuzzy logic. Figure 1 gives the pictorial representation of these logical operations and the other logical operations indicated below such as difference $\ominus$, algebraic product $\cdot$, algebraic sum $+$ and other kinds of implications. The logical operations $\ominus$, $\cdot$ and $+$ can be introduced from the operations for fuzzy sets, and two implications can be found in Zadeh (1973) and Goguen (1968), respectively. That is to say,

Difference:

$$x \ominus y = 0 \wedge (x - y). \tag{186}$$

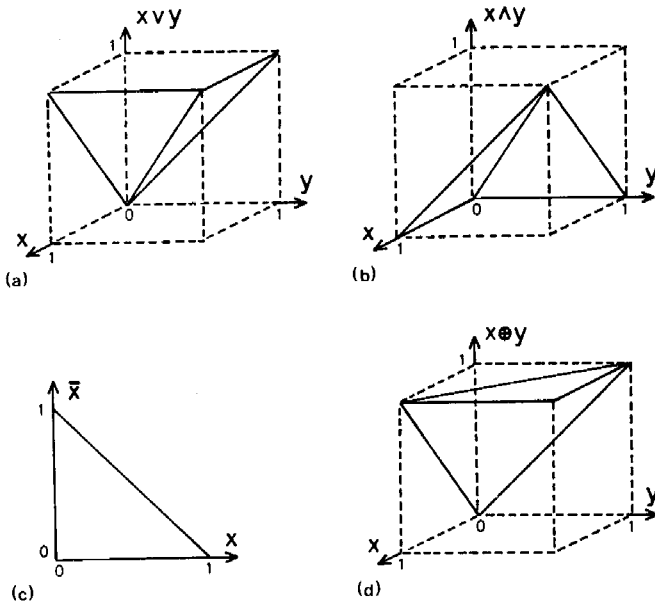*Algebraic Product*:

$$xy = x \cdot y. \tag{187}$$

FIG. 1. Illustration of logical operations. (a) Logical sum $x \vee y = \max\{x, y\}$. (b) Logical product $x \wedge y = \max\{x, y\}$. (c) Negation $\bar{x} = 1 - x$. (d) Sum $x \oplus y = 1 \wedge (x + y)$.

Algebraic Sum:

$$x \dot{+} y = x + y - xy. \tag{188}$$

Implications:

$$x \to y = \bar{x} \vee y, \tag{189}$$

$$x \to y = 1 \cdots x \leqslant y,$$
$$= \frac{y}{x} \cdots x > y. \tag{190}$$

The implication $\to$ of (184) can be defined using the logical operations of sum $\oplus$ (182), product $\odot$ (183) and difference $\ominus$ (186):

$$x \to y = \bar{x} \oplus y \tag{191}$$

$$= \overline{x \odot \bar{y}} \tag{192}$$
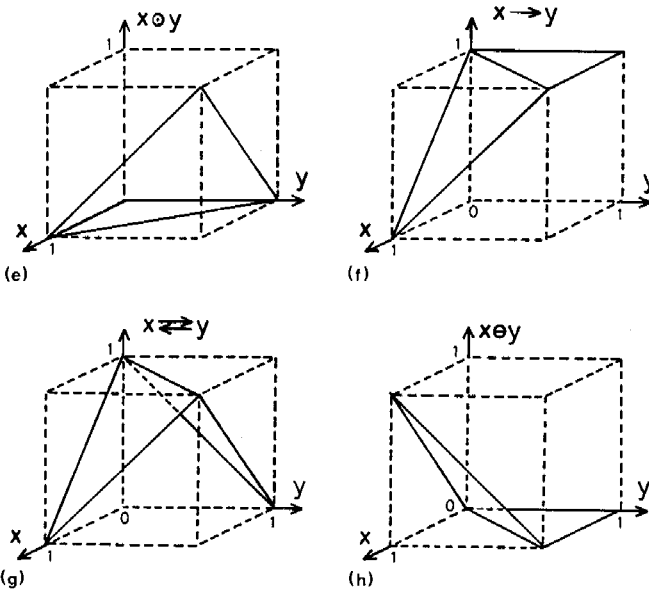
$$= \overline{x \ominus y}. \tag{193}$$

FIG. 1—*Continued.* (e) Product $x \odot y = 0 \vee (x + y - 1)$. (f) Implication $x \to y = 1 \wedge (1 - x + y)$. (g) Equivalence $x \rightleftarrows y = (1 - x + y) \wedge (1 - y + x)$. (h) Difference $x \ominus y = 0 \vee (x - y)$.

Conversely, $\oplus$, $\odot$ and $\ominus$ are also defined by this implication.

$$x \oplus y = \bar{x} \to y, \tag{194}$$

$$x \odot y = \overline{x \to \bar{y}}, \tag{195}$$

$$x \ominus y = \overline{x \to y}. \tag{196}$$

Finally, the equivalence of (185) will be defined as

$$
\begin{aligned}
x \leftrightarrow y &= (x \to y) \wedge (y \to x) \\
&= \overline{(x \ominus y)} \wedge \overline{(y \ominus x)} \\
&= \overline{(x \ominus y) \vee (y \ominus x)} \\
&= \overline{|x - y|} \\
&= 1 - |x - y|, \tag{197}
\end{aligned}
$$

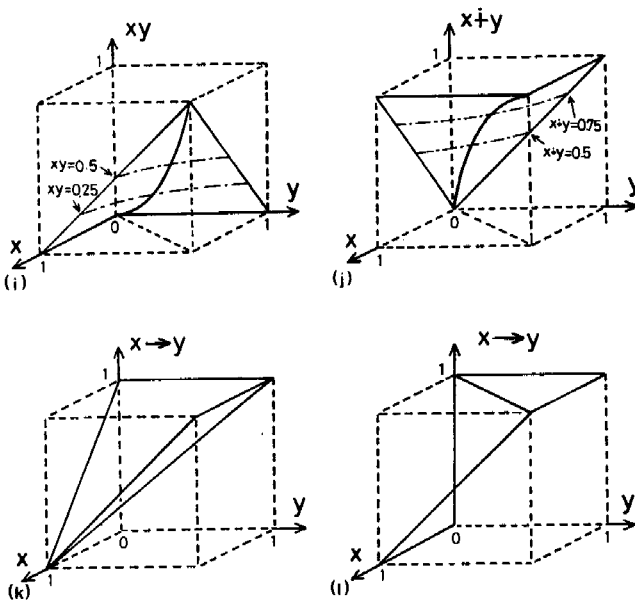where $|z|$ stands for the absolute value of a real number z.

FIG 1—*Continued*. (i) Algebraic product $xy = x \cdot y$. (j) Algebraic sum $x \dotplus y = x + y - xy$. (k) Implication $x \to y = \bar{x} \vee y$. (l) Implication $x \to y = 1 \cdots x \leqslant y; y/x \cdots x > y$.

5. CONCLUSION

We have discussed the algebraic properties of fuzzy sets under the new operations of bounded-sum, bounded-difference and bounded-product and the properties of fuzzy sets under these operations combined with the well-known operations of union, intersection, algebraic product and algebraic sum.

As was indicated in Section 4, the operations over fuzzy sets can be obtained by applying the logical operations of many-valued logic to fuzzy sets. Thus, if we introduce the other kinds of logical operations of many-valued logic to fuzzy sets, we can define various kinds of useful operations for fuzzy sets and, as a result, further fruitful applications of fuzzy sets will be found in a variety of areas.

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