Workshop on Fuzzy Sets
and Management Science,
Brussels, Feb. 19-20, 1981

FUZZY INFERENCE METHODS WITH "IF ... THEN ... ELSE ..."

Masaharu Mizumoto*
Institut für Wirtschaftswissenschaften
RWTH Aachen
Templergraben 64, D-51 Aachen
W. Germany

ABSTRACT

We investigate the properties of fuzzy reasoning methods by Zadeh with a fuzzy conditional proposition "If x is A then y is B else y is C," with A, B and C being fuzzy concepts, and point out that the consequences inferred by his methods do not always fit our intuitions, and suggest new method which fits our intuitions under several criteria.

KEYWORDS

fuzzy set; fuzzy relation; fuzzy conditional proposition; fuzzy reasoning

INTRODUCTION

In much of human reasoning, the form of reasoning is approximate rather than exact as in the statement:

If a demand is large then a price will be high.
The demand of autos is highly large.

The price of autos will be very high.

Zadeh (1975), Mamdani (1977), and Mizumoto et al. (1979a, 1979b, 1979c, 1980) suggested methods for such a reasoning in which the antecedant involves a fuzzy conditional proposition such as "If x is A then y is B," where A and B are fuzzy concepts. In Mizumoto (1979a, 1979b, 1979c, 1980), we investigated the properties of their methods.

As a generalization of such a fuzzy conditional inference containing the proposition "If x is A then y is B," Zadeh (1975) also proposed a fuzzy conditional inference of the form:

Ant 1:	If x is A then y is B else y is C.	
Ant 2:	x is A'.	(1)
Cons:	y is D.	

---

* On leave from Osaka Electro-Communication University, Osaka, Japan till Aug. 1981.

For this form of inference, Zadeh proposed the methods of obtaining the consequence (Cons) from two antecedants (Ant 1 and Ant 2) in (1).

This paper investigates the properties of his methods and points out that the consequences inferred by his methods do not always fit our intuitions, and suggests new method which fits our intuitions under several criteria.

FUZZY CONDITIONAL INFERENCES WITH "IF ... THEN ... ELSE ..."

We shall forcus our attention on the following form of inference in which a fuzzy conditional proposition "If ... then ... else ..." is contained.

Ant 1: If x is A then y is B else y is C.
Ant 2: x is A'. (2)

Cons: y is D.

where x and y are the names of objects, and A, A', B, C and D are fuzzy concepts which are represented by fuzzy sets in universes of discourse U, U, V, V and V, respectively.

An example of such a form of inference is the following:

If a demand is large then a price will be high else a price will be fairly low.
The demand of autos is fairly large.

The price of autos will be more or less high.

The Ant 1 of the form "If x is A then y is B else y is C" in (2) may represent a certain relationship between A and B, C. From this point of view, Zadeh gave the translation rules (Maximin Rule and Arithmetic Rule) for translating the fuzzy conditional proposition "If x is A then y is B else y is C" into a fuzzy relation in U x V.

Let A, B and C be fuzzy sets in U, V and V, respectively, which are written as

$$A = \int_U \mu_A(u)/u \ ; \qquad B = \int_V \mu_B(v)/v \ ; \qquad C = \int_V \mu_C(v)/v \qquad (3)$$

then we have:

(i) Maximin Rule Rm':

$$Rm' = (A \times B) \cup (\neg A \times C) \qquad (4)$$
$$= \int_{U\times V} (\mu_A(u) \wedge \mu_B(v)) \vee ((1-\mu_A(u)) \wedge \mu_C(v)) \ / \ (u,v).$$

(ii) Arithmetic Rule Ra':

$$Ra' = (\neg A \times V \oplus U \times B) \cap (A \times V \oplus U \times C) \qquad (5)$$
$$= \int_{U\times V} 1 \wedge (1-\mu_A(u)+\mu_B(v)) \wedge (\mu_A(u)+\mu_C(v)) \ / \ (u,v).$$

where x, U, ∩, 7, and ⊕ denote cartesian product, union, intersection, complement and bounded-sum for fuzzy sets, respectively.

For the proposition "If x is A then y is B else y is C," it is also possible to define a rule Rb' which is based on the implication in binary logic.

(iii) Fuzzified Binary Rule Rb':

$$Rb' = (\neg A \times V \cup U \times B) \cap (A \times V \cup U \times C) \qquad (6)$$
$$= \int_{U\times V} ((1-\mu_A(u)) \vee \mu_B(v)) \wedge (\mu_A(u) \vee \mu_C(v)) \;/\; (u,v).$$

Furthermore, we shall also introduce new method Rgg' for the proposition "If x is A then y is B else y is C."

(iv) Rule Rgg':

$$Rgg' = (A \times V \underset{g}{\Rightarrow} U \times B) \cap (\neg A \times V \underset{g}{\Rightarrow} U \times C) \qquad (7)$$
$$= \int_{U\times V} (\mu_A(u) \underset{g}{\rightarrow} \mu_B(v)) \wedge (1-\mu_A(u) \underset{g}{\rightarrow} \mu_C(v)) \;/\; (u,v).$$

where

$$\mu_A(u) \underset{g}{\rightarrow} \mu_B(v) = \begin{cases} 1 & \ldots \quad \mu_A(u) \leqq \mu_B(v) \\ \mu_B(v) & \ldots \quad \mu_A(u) > \mu_B(v) \end{cases} \qquad (8)$$

Note that the implication $a \underset{g}{\rightarrow} b$ is based on the implication rule in G_{aleph} logic system by Gödel (Rescher, 1969).

The consequence D in Cons of (2) can be deduced from Ant 1 and Ant 2 using the max-min composition "o" of the fuzzy set A' in U and the fuzzy relation in U x V obtained above. Thus, we can have for each translation rule by the following.

$$Dm = A' \circ Rm' = A' \circ \left[(A \times B) \cup (\neg A \times C)\right] \qquad (9)$$
$$= \int_V \bigvee_u \left\{\mu_{A'}(u) \wedge \left[(\mu_A(u) \wedge \mu_B(v)) \vee ((1-\mu_A(u)) \wedge \mu_C(v))\right]\right\} \;/\; v.$$

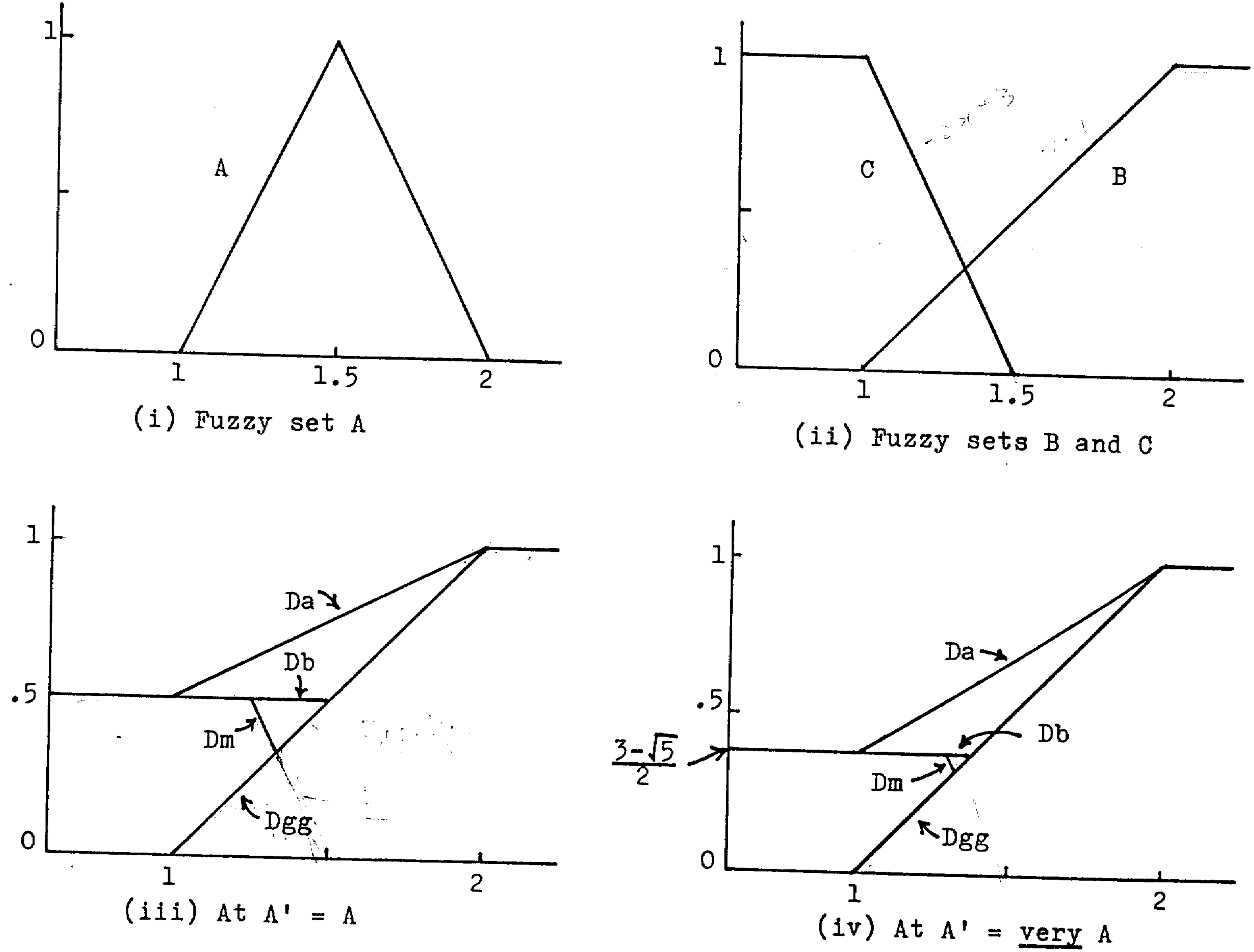
$$Da = A' \circ Ra' = A' \circ \left[(\neg A \times V \oplus U \times B) \cap (A \times V \oplus U \times C)\right] \qquad (10)$$

$$Db = A' \circ Rb' = A' \circ \left[(\neg A \times V \cup U \times B) \cap (A \times V \cup U \times C)\right] \qquad (11)$$

$$Dgg = A' \circ Rgg' = A' \circ \left[(A \times V \underset{g}{\Rightarrow} U \times B) \cap (\neg A \times V \underset{g}{\Rightarrow} U \times C)\right] \qquad (12)$$

Table 1 Inference Results by Each Rule

A' \ D	Dm	Da	Db	Dgg
A	$\mu_B \vee (.5 \wedge \mu_C)$	$\frac{1+\mu_B}{2}$	$\mu_B \vee .5$	μ_B
very A	$\mu_B \vee (\frac{3-\sqrt{5}}{2} \wedge \mu_C)$	$\frac{3+2\mu_B-\sqrt{5+4\mu_B}}{2}$	$\mu_B \vee \frac{3-\sqrt{5}}{2}$	μ_B
more or less A	$\mu_B \vee (\frac{\sqrt{5}-1}{2} \wedge \mu_C)$	$\frac{\sqrt{5+4\mu_B}-1}{2} \wedge \frac{1+\mu_B+\mu_C}{2}$	$\begin{cases} \mu_B & \cdots \mu_B \geq \frac{\sqrt{5}-1}{2} \\ (\mu_B \vee \mu_C \vee .5) \wedge \frac{\sqrt{5}-1}{2} & \cdot\cdot\ \mu_B \leq \frac{\sqrt{5}-1}{2} \end{cases}$	$\begin{cases} \sqrt{\mu_B} & \cdots \mu_B+\mu_C \geq 1 \\ \mu_B \vee (\sqrt{\mu_B} \wedge \mu_C) & \cdot\cdot\ \mu_B+\mu_C<1 \end{cases}$
not A	$\mu_C \vee (.5 \wedge \mu_B)$	$\frac{1+\mu_C}{2}$	$\mu_C \vee .5$	μ_C
not very A	$\mu_C \vee (\frac{\sqrt{5}-1}{2} \wedge \mu_B)$	$\frac{2\mu_C-1+\sqrt{5-4\mu_C}}{2} \wedge \frac{1+\mu_C+\mu_B}{2}$	$\begin{cases} \mu_C & \cdots \mu_C \geq \frac{\sqrt{5}-1}{2} \\ (\mu_C \vee \mu_B \vee .5) \wedge \frac{\sqrt{5}-1}{2} & \cdot\cdot\ \mu_C \leq \frac{\sqrt{5}-1}{2} \end{cases}$	$\begin{cases} 1-(1-\mu_C)^2 & \cdots \mu_B+\mu_C \geq 1 \\ [1-(1-\mu_C)^2 \wedge \mu_B] \vee \mu_C & \cdot\cdot\ \mu_B+\mu_C<1 \end{cases}$
not more or less A	$\mu_C \vee (\frac{3-\sqrt{5}}{2} \wedge \mu_B)$	$\frac{3-\sqrt{5-4\mu_C}}{2}$	$\mu_C \vee \frac{3-\sqrt{5}}{2}$	μ_C



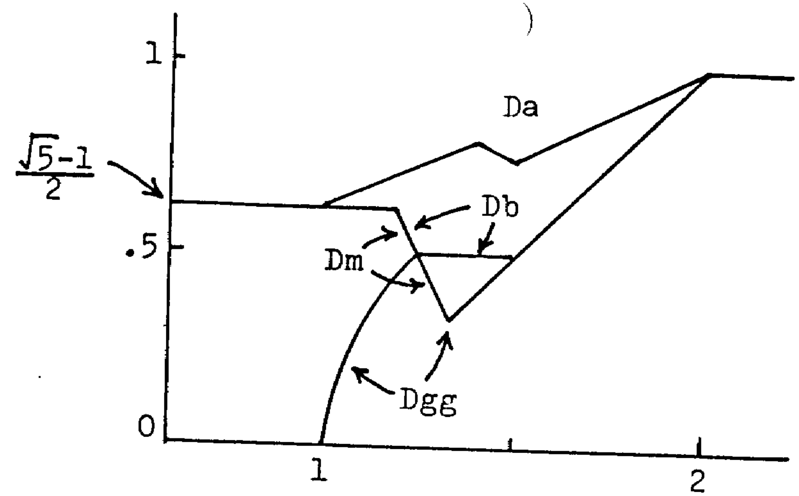
(i) Fuzzy set A

(ii) Fuzzy sets B and C

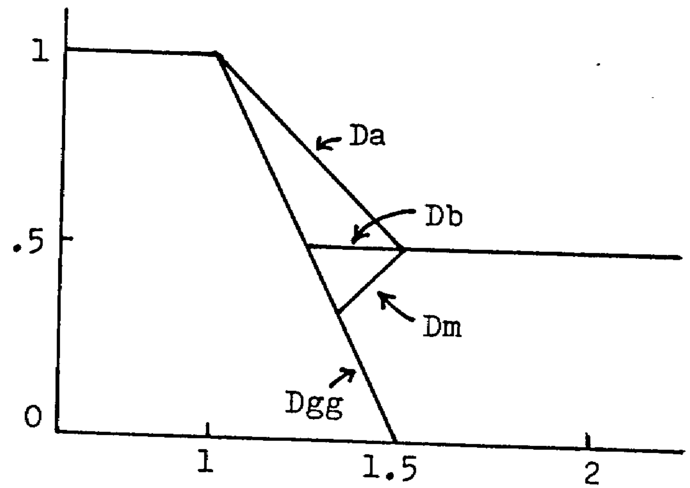
(iii) At A' = A

(iv) At A' = <u>very</u> A

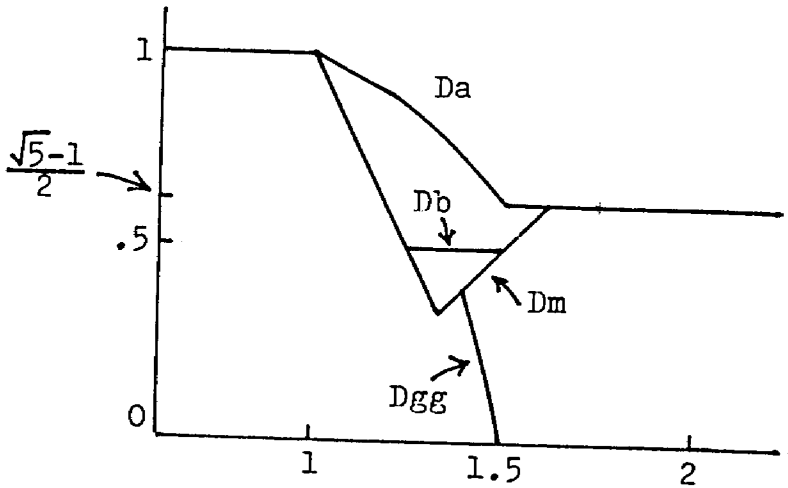
Fig. 1 Fuzzy sets A, B, C, and the consequences Dm, Da, Db, Dgg



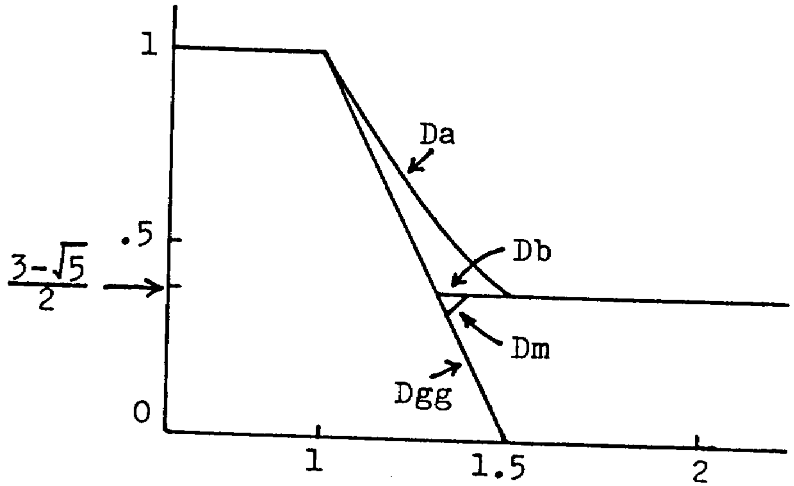
(v) At A' = <u>more or less</u> A



(vi) At A' = <u>not</u> A



(vii) At A' = <u>not very</u> A



(viii) At A' = <u>not more or less</u> A

Fig. 1 (continued)

COMPARISON BETWEEN FUZZY CONDITIONAL INFERENCE METHODS

Using (9)-(12) we shall show what will the consequences Dm, Da, Db and Dgg be when A' is

$$A' = A \tag{13}$$

$$A' = \underline{\text{very}}\ A\ (= A^2) \tag{14}$$

$$A' = \underline{\text{more or less}}\ A\ (= A^{0.5}) \tag{15}$$

$$A' = \underline{\text{not}}\ A\ (= \neg A) \tag{16}$$

$$A' = \underline{\text{not very}}\ A\ (= \neg A^2) \tag{17}$$

$$A' = \underline{\text{not more or less}}\ A\ (= \neg A^{0.5}) \tag{18}$$

which are typical examples for A'. The results are summarized in Table 1. An example is shown in Fig. 1.

According to our intuitions, it seems that the consequence D in (2) should be B and C, respectively, when A' is A and not A under the conditional proposition "If x is A then y is B else y is C." Namely, the following inferences may be a quite natural demand.

Ant 1:	If x is A then y is B else y is C.	
Ant 2:	x is A.	(19)
Cons:	y is B.	

Ant 1:	If x is A then y is B else y is C.	
Ant 2:	x is not A.	(20)
Cons:	y is C.	

From Table 1 it is founded that the consequence Dgg is equal to B (i.e., μ_B) at A' = A, and C (i.e., μ_C) at A' = not A. Namely, the rule Rgg' satisfies the criteria (19) and (20). The other rules Rm', Ra' and Rb' do not satisfy such criteria. It is not yet known, however, what kinds of consequences are most suitable for the consequence D of (2), when A' is equal to very A, more or less A, not very A, and not more or less A. Therefore, we can not conclude that our new method Rgg' is the best one for the fuzzy conditional inference with "If x is A then y is B else y is C," but we may say that Rgg' is a suitable method for the fuzzy conditional inference, since it satisfies the quite natural criteria (19) and (20).

CONCLUSION

In this paper we pointed out that the methods Ra', Rm' and Rb' for the fuzzy conditional inference with "If x is A then y is B else y is C" do not give the consequences which fit our intuitions, and gave an improved method Rgg' which fits our intuitions under quite natural criteria.

The formalization of inference methods for the more complicated form of inference, such as

If x is A_1 then y is B_1 else
If x is A_2 then y is B_2 else
⋮
If x is A_n then y is B_n.
x is A'.

---

y is B'.

would be the future subjects.

REFERENCES

Fukami, S., M. Mizumoto, and K. Tanaka (1980). Some considerations on fuzzy conditional inferences. Fuzzy Sets and Systems, 4, 3, 243-273.

Mamdani, E.H. (1977). Application of fuzzy logic to approximate reasoning using linguistic systems. IEEE Trans. on Comp., c-26, 1182-1191.

Mizumoto, M., S. Fukami, and K. Tanaka (1979a). Fuzzy conditional inferences and fuzzy inferences with fuzzy quantifiers. Proc. of 6th Int. Conf. on Artificial Intelligence (Tokyo, Aug. 20-23), 589-591.

Mizumoto, M., S. Fukami, and K. Tanaka (1979b). Some methods of fuzzy reasonings. In M.M. Gupta, R.K. Ragade, and R.R. Yager (Eds.), Advances in Fuzzy Set Theory and Applications, North-Holland, 117-136.

Mizumoto, M., S. Fukami, and K. Tanaka (1979c). Several methods for fuzzy conditional inferences. Proc. of IEEE Conf. on Decision & Control (Florida, Dec. 12-14), 777-782.

Rescher, N. (1969). Many Valued Logic. McGraw-Hill, New York.

Zadeh, L. A. (1975). Calculus of fuzzy restriction. In L.A. Zadeh, K.S. Fu, K. Tanaka, and M. Shimura (Eds.), Fuzzy Sets and Their Applications to Cognitive and Decision Processes, Academic Press, New York, 1-39.