FUZZY CONDITIONAL INFERENCES AND FUZZY INFERENCES WITH FUZZY QUANTIFIERS

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L.A. Zadeh and E.H. Mamdani proposed methods for the fuzzy reasoning in which the antecedent involves a fuzzy conditional inference "If x is A then y is B" with A and B being fuzzy concepts. This paper points out that the consequences inferred by their methods do not always fit our intuitions, and suggests some new methods which fit our intuitions under several criteria such as modus ponens and modus tollens. This paper also contains the discussion of the fuzzy inferences whose antecedents have fuzzy quantifiers such as "most", "some" and "many" using our new methods for fuzzy conditional inferences.

1. FUZZY CONDITIONAL INFERENCES

We shall consider the following form of inference in which a fuzzy conditional proposition is contained.

Ant 1: If x is A then y is B.
Ant 2: x is A'.
__________________________________ (1)
Cons: y is B'.

where x and y are the names of objects, and A, A', B and B' are the labels of fuzzy sets in universes of discourse U, U, V and V, respectively.

For this form of fuzzy conditional inference, several methods are proposed.

Let A and B be fuzzy sets in U and V, respectively, which are written as

$$A = \int_U \mu_A(u)/u \;; \quad B = \int_V \mu_B(v)/v \qquad (2)$$

and let $\times$, $\cup$, $\cap$, $\neg$ and $\oplus$ be cartesian product, union, intersection, complement and bounded-sum for fuzzy sets, respectively.

Then the following fuzzy relations are obtained from a fuzzy conditional statement "If x is A then y is B" in Ant 1 of (1). The fuzzy relations R_m, R_a are proposed by Zadeh [1], R_c is by Mamdani [2], and R_s, R_g, R_{sg} and R_{gg} are new methods proposed here.

$$R_m = (A \times B) \cup (\neg A \times V). \qquad (3)$$

$$R_a = (\neg A \times V) \oplus (U \times B). \qquad (4)$$

$$R_c = A \times B. \qquad (5)$$

$$R_s = A \times V \underset{s}{\longrightarrow} U \times B \qquad (6)$$
$$= \int_{U\times V} [\mu_A(u) \underset{s}{\longrightarrow} \mu_B(v)]/(u,v),$$

where

$$\mu_A(u) \underset{s}{\longrightarrow} \mu_B(v) = \begin{cases} 1 \;\ldots\; \mu_A(u) \le \mu_B(v), \\ 0 \;\ldots\; \mu_A(u) > \mu_B(v). \end{cases}$$

$$R_g = A \times V \underset{g}{\longrightarrow} U \times B \qquad (7)$$
$$= \int_{U\times V} [\mu_A(u) \underset{g}{\longrightarrow} \mu_B(v)]/(u,v),$$

where

$$\mu_A(u) \underset{g}{\longrightarrow} \mu_B(v) = \begin{cases} 1 \;\ldots\; \mu_A(u) \le \mu_B(v), \\ \mu_B(v) \;\ldots\; \mu_A(u) > \mu_B(v). \end{cases}$$

$$R_{sg} = (A \times V \underset{s}{\longrightarrow} U \times B) \cap (\neg A \times V \underset{g}{\longrightarrow} U \times \neg B). \qquad (8)$$

$$R_{gg} = (A \times V \underset{g}{\longrightarrow} U \times B) \cap (\neg A \times V \underset{g}{\longrightarrow} U \times \neg B). \qquad (9)$$

Then the consequence B' in Cons of (1) can be deduced from Ant 1 and Ant 2 using the max-min composition "o" of the fuzzy set A' in U and the fuzzy relation obtained above. Thus, we can have

$$B'_m = A' \circ R_m = A' \circ ((A \times B) \cup (\neg A \times V)),$$

$$B'_a = A' \circ ((\neg A \times V) \oplus (U \times B)),$$

and so on.

Table I Relations between Ant 2 and Cons under Ant 1 in (1), and the satisfaction of the relation under each method

	Ant 2	Cons	R_m	R_a	R_c	R_s	R_g	R_{sg}	R_{gg}
Relation I (modus ponens)	A	B	X	X	O	O	O	O	O
Relation II-1	very A	very B	X	X	X	O	X	O	X
Relation II-2	very A	B	X	X	O	X	O	X	O
Relation III	more or less A	more or less B	X	X	X	O	O	O	O
Relation IV-1	not A	unknown	O	O	X	O	O	X	X
Relation IV-2	not A	not B	X	X	X	X	X	O	O
Relation V (modus tollens)	not B	not A	X	X	X	O	X	O	X

In the above form of fuzzy conditional inference, it seems according to our intuitions that the relations between A' in Ant 2 and B' in Cons in (1) ought to be satisfied as shown in the left part of Table I. Relation II-2 has the result different from that of Relation II-1, but in Ant 1 if there is not a strong casual relation between "x is A" and "y is B", the satisfaction of Relation II-2 will be permitted. Relation IV-1 asserts that when x is not A, any information about y can not be deduced from Ant 1. The satisfaction of Relation IV-2 is demanded when the fuzzy proposition "If x is A then y is B" means tacitly the proposition "If x is A then y is B else y is not B." Relation V corresponds to modus tollens in which the form of this inference is

Ant 1: If x is A then y is B.
Ant 2: y is not B.

Cons: x is not A.

In Table I, it is noted that very A is defined as A^2, more or less A as $A^{0.5}$, not A as $\neg A$, and unknown as V.

The right part of Table I shows the satisfaction (O) or failure (X) of each Relation under the methods given in (3)-(9). It is assumed here that fuzzy sets A and B in (2) satisfy the conditions in the discussion of Relations I-III:

(i) $\{\mu_A(u) \mid u \in U\} \supseteq \{\mu_B(v) \mid v \in V\}$,

(ii) $\exists u \in U \;\; \mu_A(u) = 0; \exists u' \in U \;\; \mu_A(u') = 1$,

(iii) $\exists v \in V \;\; \mu_B(v) = 0; \exists v' \in V \;\; \mu_B(v') = 1$.

But in the discussion of Relation V, we use the condition (i)' instead of (i):

(i)' $\{\mu_A(u) \mid u \in U\} \subseteq \{\mu_B(v) \mid v \in V\}$.

2. FUZZY INFERENCES WITH FUZZY QUANTIFIERS

In this section we shall consider such inferences that the antecedents are quantified by the fuzzy quantifiers such as "most", "a few", etc.

Let us consider a simple form of such inference as

Ant 1: Most Swedes are blond.
Ant 2: Karl is a Swede. (10)
Cons: It is likely that Karl is blond.

In general, this form of inferences may be expressed in symbol as

Ant 1: qX are E.
Ant 2: x' is a member of X. (11)
Cons: It is p that x' is E.

where

q ε Q = most + almost + some + a few + ...
p ε P = likely + very likely + probable + ...

X is a certain set (X = {Swedes} in (10)), and E represents an attribute value of the element of X (E = blond in (10)). q (εQ) is a fuzzy quantifier and can be interpreted as representing a fuzzy rate. p (εP) is a linguistic probability and represents a subjective fuzzy probability (which is denoted by Pr(x' is E)) of the event "x' is E" over X. Since we can assume that p is determined by the fuzzy quantifier q alone, we introduce a function f which is a mapping from a rate space (say, the interval [0,100], with its element interpreted as %) into a probability space [0,1]. Let a fuzzy quantifier q be represented as a fuzzy set in the rate space, then we can get a fuzzy probability f(q) as a fuzzy set in [0,1] by applying the extension principle [1].

Using this function f, we can get the following statement from (11).

$$\frac{\begin{array}{ll}\text{Ant 1'}: & x \in X \longrightarrow Pr(x \text{ is } E) = f(q) \\ \text{Ant 2'}: & x' \in X\end{array}}{\text{Cons'}: \quad Pr(x' \text{ is } E) = f(q)} \qquad (12)$$

Based on this discussion we shall consider the following form of a slightly complicate inference which includes fuzzy quantifier "most", and fuzzy attributes "tall" and "more or less tall".

Ant 1: Most tall men are well-built.
Ant 2: Tom is more or less tall. (13)

Cons: It is likely that Tom is well-built.

In general, this form of inferences may be represented in symbol as

$$\frac{\begin{array}{ll}\text{Ant 1'}: & x \text{ is } A \longrightarrow Pr(x \text{ is } E) = f(q) \\ \text{Ant 2'}: & x' \text{ is } A'\end{array}}{\text{Cons'}: \quad Pr(x' \text{ is } E) = p'} \qquad (14)$$

re A and A' are fuzzy attributes represented by fuzzy sets in a universe of discourse U (In the case of "tall" and "more or less tall", U will be, say, 150cm + 151cm + ... + 200cm). p' is a fuzzy probability which can be obtained from the consequence of the inference.

For this type of inference, it may be thought that the relations in Table II ought to be satisfied between A' and p' under Ant 1' in (14).

Ant 1' of (14) can be read formally as

$$\text{If } x \text{ is } A \text{ then } Pr(x \text{ is } E) \text{ is } f(q). \qquad (15)$$

Thus we can deduce p' of (14) using the methods for fuzzy conditional inferences in Sec. 1.

If Ant 1' of (14), i.e., (15) translates into the fuzzy relation defined in (6), namely

$$R_s(A, f(q)) = A \times V \underset{s}{\longrightarrow} U \times f(q) \qquad (16)$$

where U and V are universes of discourse of A and f(q), respectively, then the consequence Pr(x' is E) of (14) is given by

$$Pr(x' \text{ is } E) = p' = A' \circ R_s(A, f(q)). \qquad (17)$$

Thus it follows from (17) that Relations I°, II°-1, III° and IV° in Table II are satisfied.

Similarly, if (15) translates into the fuzzy relation given by (7), then we find that Relations I°, II°-2, III° and IV° are satisfied.

Example: Let us consider the following antecedents Ant 1 and Ant 2:

Ant 1: Most tall men are well-built.
Ant 2: Tom is very tall. (18)

where tall is a fuzzy set represented by

$$\underline{\text{tall}} = 0.2/160 + 0.4/165 + 0.6/170 + 0.8/175 + 1/180 + 1/185 + 1/190,$$

with the universe of discourse, U, of tall being

$$U = 150 + 155 + 160 + 165 + \dots + 185 + 190.$$

Table II Relations between A' and p'

	A'	p'
Relation I°	A	f(q)
Relation II°-1	very A	very f(q)
Relation II°-2	very A	f(q)
Relation III°	more or less A	more or less f(q)
Relation IV°	not A	unknown

Let the rate space of most be

$$0\% + 10\% + 20\% + 30\% + \dots + 90\% + 100\%,$$

and let the probability space of f(most) be

$$0 + 0.1 + 0.2 + 0.3 + \dots + 0.9 + 1,$$

and f(x) = x ÷ 100 with x in the rate space, then when most is a fuzzy set in the rate space, that is,

$$\underline{\text{most}} = 0.4/70\% + 0.6/80\% + 0.8/90\% + 1/100\%,$$

f(most) will be

$$f(\underline{\text{most}}) = 0.4/0.7 + 0.6/0.8 + 0.8/0.9 + 1/1.$$

The antecedents Ant 1 and 2 can be rewritten from the notation of (14) and (15) as

Ant 1': If x is tall then Pr(x is well-built) is f(most).
Ant 2': Tom is very tall.

Therefore R_s(tall, f(most)) becomes

	0	.1	.2	.3	.4	.5	.6	.7	.8	.9	1
150	1	1	1	1	1	1	1	1	1	1	1
155	1	1	1	1	1	1	1	1	1	1	1
160	0	0	0	0	0	0	0	1	1	1	1
165	0	0	0	0	0	0	0	1	1	1	1
170	0	0	0	0	0	0	0	0	1	1	1
175	0	0	0	0	0	0	0	0	0	1	1
180	0	0	0	0	0	0	0	0	0	0	1
185	0	0	0	0	0	0	0	0	0	0	1
190	0	0	0	0	0	0	0	0	0	0	1

$$Pr(\text{Tom is well-built}) = \underline{\text{very tall}} \circ R_s(\underline{\text{tall}}, f(\underline{\text{most}}))$$
$$= 0.16/0.7 + 0.36/0.8 + 0.64/0.9 + 1/1 = p'.$$

If this fuzzy probability can be approximated by the linguistic probability, say, very likely, then the Cons of Ant 1 and 2 of (18) will be

Cons: It is very likely that Tom is well-built.

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