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FUZZY REASONING METHODS BY ZADEH AND MAMDANI, AND IMPROVED METHODS

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1. INTRODUCTION

In much of human reasoning, the form of reasoning is approximate rather than exact as in "A red plum is ripe and this plum is more or less red. Then this plum is more or less ripe." Zadeh [1] and Mamdani [2] suggested a method for such a fuzzy reasoning in which antecedents involve fuzzy conditional propositions as an application of fuzzy set theory. Here, we point out that the consequences infered by their methods do not always fit our intuitions, and we suggest an improved method which fits our intuitions under several criteria.

2. FUZZY CONDITIONAL INFERENCE

We shall consider the following form of inference in which a fuzzy conditional proposition is contained.

Ant 1: If x is A then y is B.

Ant 2: x is A'. (1)

Cons : y is B'.

where x and y are the names of objects, and A, A', B, and B' are the labels of fuzzy subsets of universes of discourse U, U, V and V, respectively.

An example of this form of inference is the following.

If a plum is <u>red</u> then the plum is <u>ripe</u>.

This plum is <u>very red</u>.

This plum is <u>very ripe</u>.

From now on, we call this form of inference as "fuzzy conditional inference".

For this form of inference, Zadeh suggested the following method [1]. First, Ant 1 in (1) translates into the following binary fuzzy relation R_m or R_a, that is,

$$R_m = (A \times B) \cup (\neg A \times V) \tag{2}$$

$$R_a = (\neg A \times V) \oplus (U \times B) \tag{3}$$

Mamdani [2] also proposed the following method.

$$R_c = A \times B \tag{4}$$

where $\times$, $\cup$, $\neg$ and $\oplus$ denote cartesian product, union, complement and bounded-sum, respectively. Second, Ant 2 in (1) translates into the unary fuzzy relation (that is, fuzzy set)

$$R = A' \tag{5}$$

Then the consequence B' in Cons of (1) can be obtained by the composition "o" of R and R_m (or R_a, R_c), that is,

$$B'_m = A' \circ ((A \times B) \cup (\neg A \times V)) \tag{6}$$

$$B'_a = A' \circ ((\neg A \times V) \oplus (U \times B)) \tag{7}$$

$$B'_c = A' \circ (A \times B) \tag{8}$$

On the above form of inferences, according to our intuitions it seems that the following relations between A' in Ant 2 and B' in Cons ought to be satisfied.

	A'	B'
Relation I	A	B
Relation II-1	<u>very</u> A	<u>very</u> B
Relation II-2	<u>very</u> A	B
Relation III	<u>more or less</u> A	<u>more or less</u> B
Relation IV-1	<u>not</u> A	unknown
Relation IV-2	<u>not</u> A	<u>not</u> B

Relation II-2 is inconsisted with Relation II-1, but in the Ant 1 if there is not a strong causal relation between "x is A" and "y is B", the satisfaction of Relation II-2 will be permitted. Moreover, Relation IV-1 asserts that when x is not A, any information about y can not be deduced from Ant 1. Thus, this Relation may be thought to be quite natural, since the fuzzy conditional proposition " If x is A then y is B" does not make any assertion when x is not A. The satisfaction of Relation IV-2 is demanded when the fuzzy conditional proposition "If x is A then y is B" means tacitly "If x is A then y is B else y is not B."

3. ZADEH'S AND MAMDANI'S METHODS FOR FUZZY CONDITIONAL INFERENCE

We shall show in this section that Zadeh's methods do not satisfy the relations except Relation IV-1 and that Mamdani's method does not satisfy the relations except Relation I and II-2.

Now, we shall show what will B'_m, B'_a and B'_c be when A' in (6)-(8) is equal to A, very A ($=A^2$)*, more or less A ($=A^{0.5}$) or not A ($=\neg A$), where fuzzy sets A in U and B in V are given as in Fig.1 and Fig.2, respectively.

[1] The case of maximin rule (R_m):

Let A' be A, then B'_m becomes as follows using (6).

$$B'_m = A \circ [(A \times B) \cup (\neg A \times V)]$$

$$= \int_U \mu_A(u)/u \circ \int_{U\times V} (\mu_A(u) \wedge \mu_B(v)) \vee (1 - \mu_A(u))/(u,v)$$

$$= \int_V \bigvee_{u\in U} [\mu_A(u) \wedge ((\mu_A(u) \wedge \mu_B(v)) \vee (1 - \mu_A(u)))]/v \qquad (9)$$

* A^2 and $A^{0.5}$ may be used to approximate the effect of the linguistic modifiers very and more or less, and are defined as

very $A = A^2 = \int \mu_A(u)^2/u$; more or less $A = A^{0.5} = \int \sqrt{\mu_A(u)}/u$

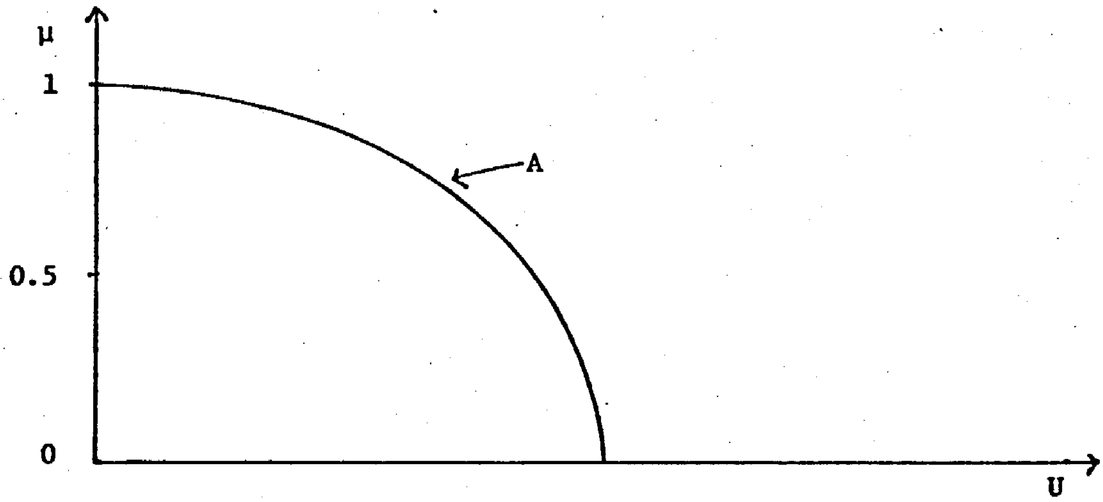

Fig. 1. Membership function $\mu_A(u)$ of fuzzy set A in U

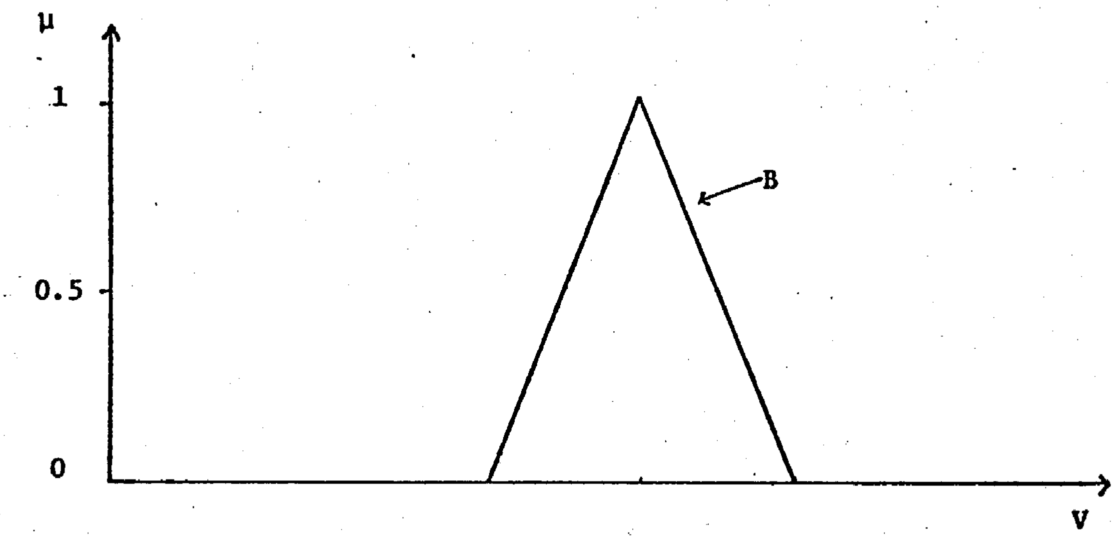

Fig. 2. Membership function $\mu_B(v)$ of fuzzy set B in V

Now, let

$$S_m(\mu_A(u)) = \mu_A(u) \wedge ((\mu_A(u) \wedge \mu_B(v)) \vee (1 - \mu_A(u))) \tag{10}$$

The value $S_m(\mu_A(u))$ with a parameter $\mu_B(v)$ is shown in Fig.3. In Fig.3, if $\mu_B(v)=0.3$, $S_m(\mu_A(u))$ becomes what is shown by ——————, and if $\mu_B(v)=0.8$, $S_m(\mu_A(u))$ becomes what is shown by —·—·—·, and so on. In Fig.1, $\mu_A(u)$ takes all values in the unit interval [0, 1] according to u varying all over U. Thus from Fig.3 we have

$$\bigvee_{u \in U} S_m(\mu_A(u)) = \begin{cases} \mu_B(v) & \text{------ } \mu_B(v) \geq 0.5 \\ 0.5 & \text{------ } \mu_B(v) < 0.5 \end{cases} \tag{11}$$

Therefore, from (9) and (11) we have

$$B'_m = \int_V \bigvee_{u \in U} S_m(\mu_A(u))/v \quad , \tag{12}$$

and the membership function of B'_m is shown in Fig.4.

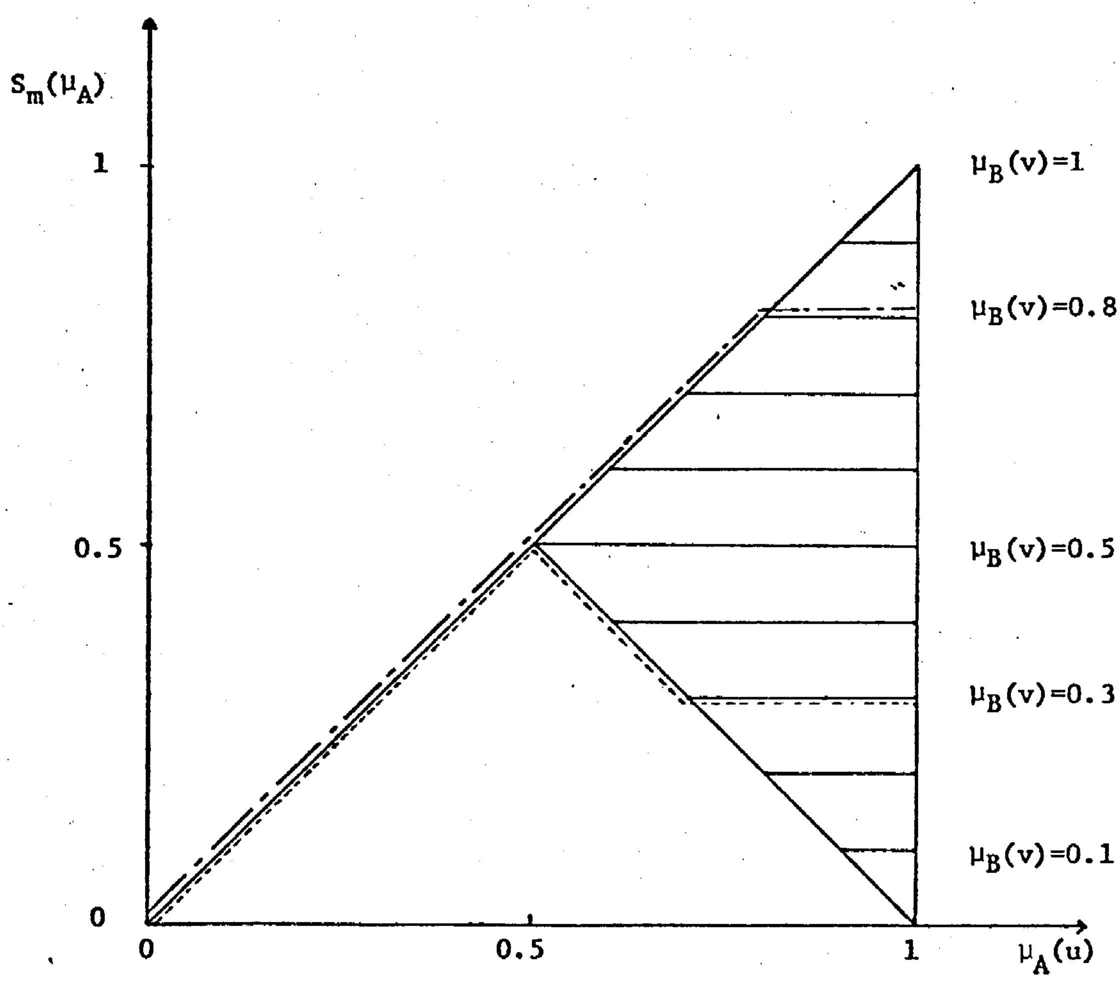

Fig.3. $S_m(\mu_A(u))$ of (10)

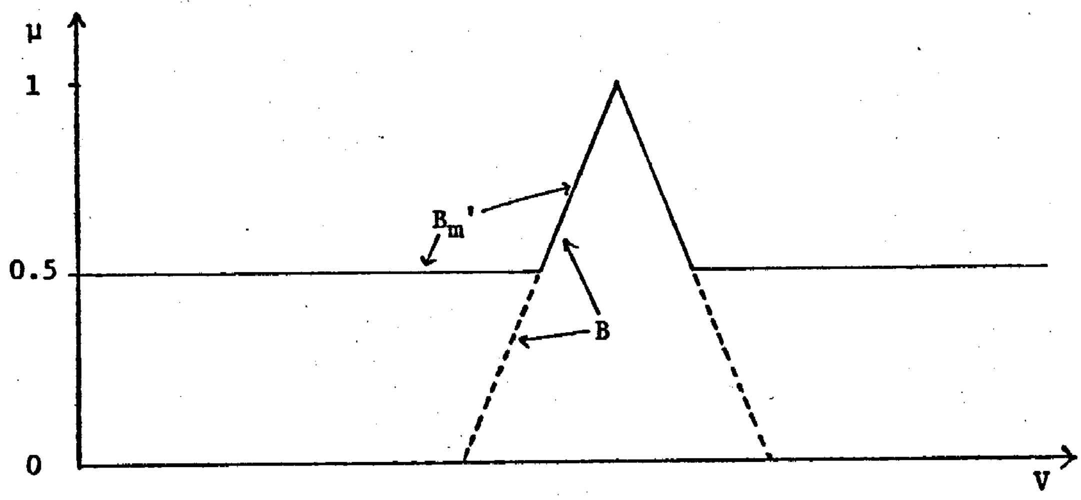

Fig.4. Membership function of B_m' when A' = A

From Fig.4, $B_m' \neq B$ is obtained, and thus it is shown that the Relation I is not satisfied.

Second, suppose A'= <u>very</u> A ($=A^2$), then

$$B_m' = A^2 \circ [(A \times B) \cup (\neg A \times V)]$$

$$= \int_V \mu_A^2(u)/u \circ \int_{U \times V} (\mu_A(u) \wedge \mu_B(v)) \vee (1 - \mu_A(u))/(u,v)$$

$$= \int_V \bigvee_{u \in U} \mu_A^2(u) \wedge [(\mu_A(u) \wedge \mu_B(v)) \vee (1 - \mu_A(u))]/v$$

and the membership function of B_m' is shown in Fig.5. Hence from Fig.5, we can see

$B_m' \neq$ <u>very</u> B

$B_m' \neq B$.

This shows that both Relation II-1 and Relation II-2 are not satisfied.

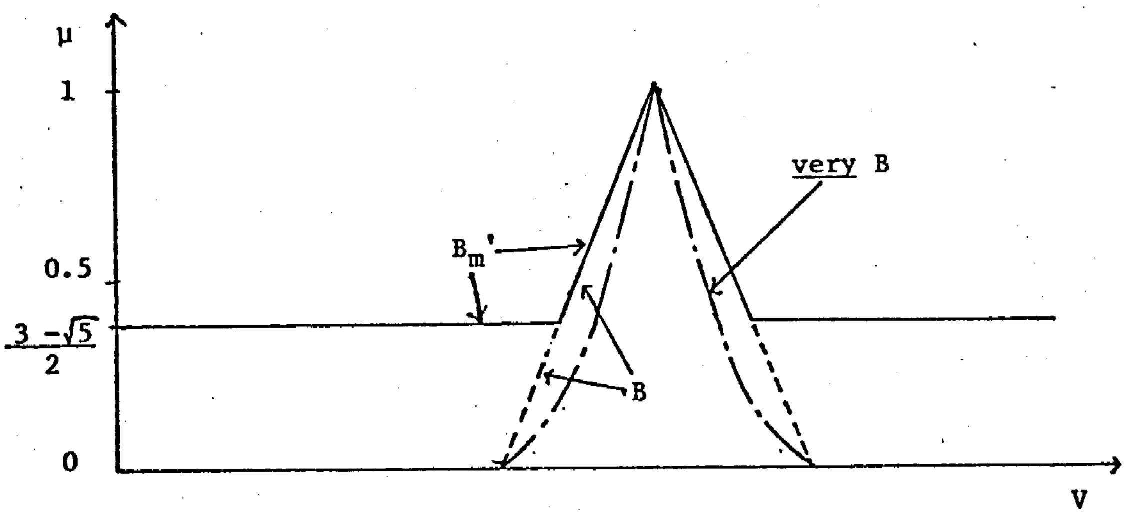

Fig.5. Membership function of B_m' when A' = very A

In a similar way, when A'= more or less A ($=A^{0.5}$), the membership function of B_m' will be as in Fig.6. From Fig.6, $B_m' \neq$ more or less B is obtained and thus Relation III is not satisfied.

Finally we shall show that Relation IV-1 is satisfied when A'= not A. Let A'= not A ($= \neg A$), then

$$B_m' = (\neg A) \circ [(A \times B) \cup (\neg A \times V)]$$

$$= \int_U 1 - \mu_A(u)/u \circ \int_{U \times V} (\mu_A(u) \wedge \mu_B(v)) \vee (1 - \mu_A(u))/(u,v)$$

$$= \int_V \bigvee_{u \in U} (1 - \mu_A(u)) \wedge [(\mu_A(u) \wedge \mu_B(v)) \vee (1 - \mu_A(u))]/v \qquad (13)$$

Now, from Fig.1, there exists $u \in U$ which makes $\mu_A(u)=0$, so that

$$(13) = \int_V 1 \wedge [(0 \wedge \mu_B(v)) \vee 1]/v$$

$$= \int_V 1/v$$

= unknown

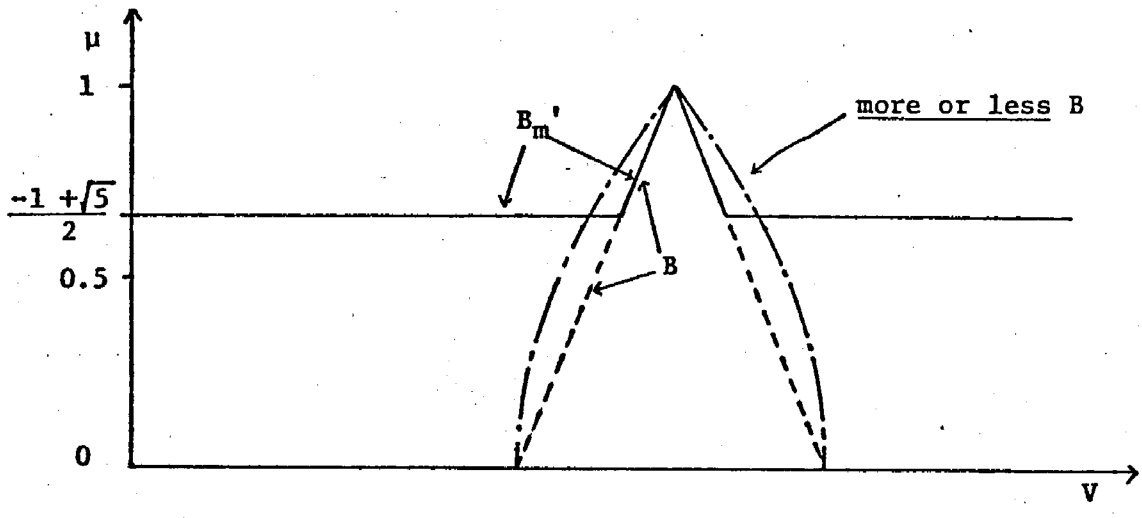

Fig.6. Membership function of B_m' when A' = more or less A

This shows that Relation IV-1 is satisfied.

Since Relation IV-2 is inconsistent with Relation IV-1, it is clear that Relation IV-2 is not satisfied.

[2] The case of arithmetic rule (R_a):

Suppose that $A' = A^{\alpha}$ $(\alpha > 0)$, then the consequence B_a' is obtained as follows:

$$B_a' = A^{\alpha} \circ [(\neg A \times V) \oplus (U \times B)]$$

$$= \int_U \mu_A^{\alpha}(u)/u \circ \int_{U \times V} 1 \wedge (1 - \mu_A(u) + \mu_B(v))/(u,v)$$

$$= \int_V \bigvee_{u \in U} [\mu_A^{\alpha}(u) \wedge (1 \wedge (1 - \mu_A(u) + \mu_B(v)))]/v$$

and the membership function of B_a' at $\alpha = 1$ is shown in Fig.7. From this figure, $B_a' \neq B$ and thus it is found that Relation I is not satisfied.

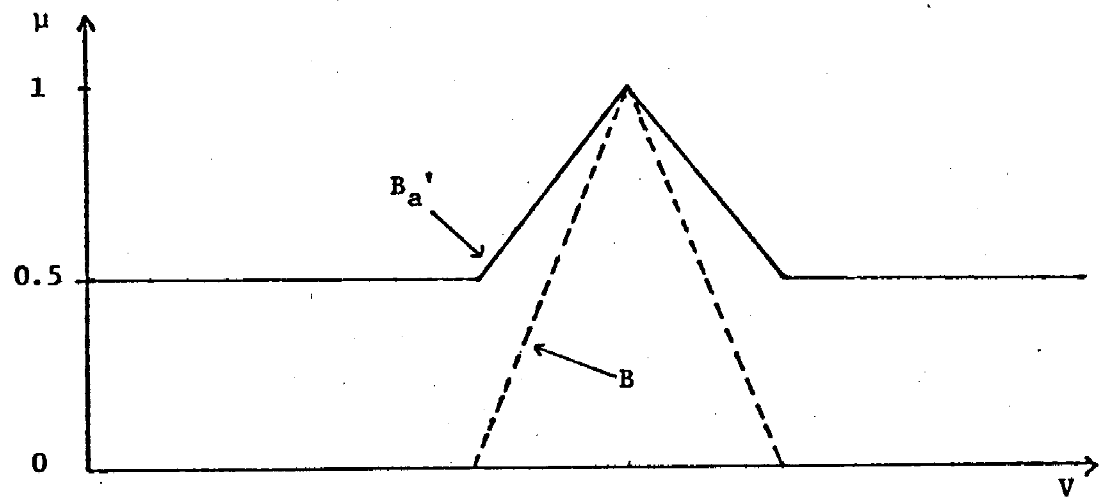

Fig.7. Membership function of B_a' when A' = A

When $\alpha = 2$, that is, A' = very A (= A^2), the membership functions of B_a', very B and B are shown in Fig. 8, which shows that Relation II-1 and II-2 are not saisfied.

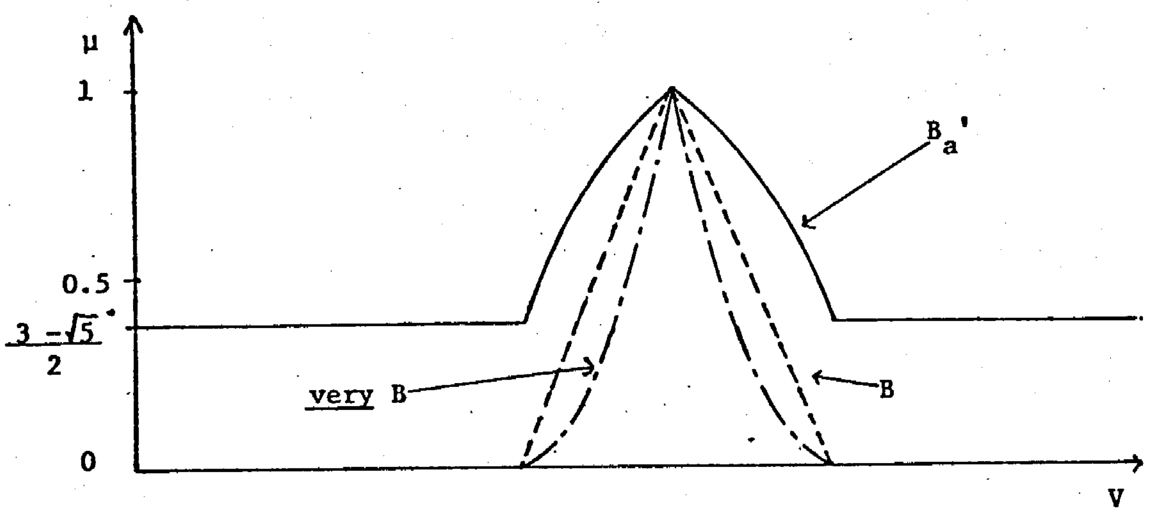

Fig.8. Membership functions of fuzzy sets B_a'(when A' = A), B and very B

When $\alpha = 0.5$, i.e., A' = more or less A (= $A^{0.5}$), the membership function of B_a' is shown as in Fig.9. This shows that Relation III is not satisfied.

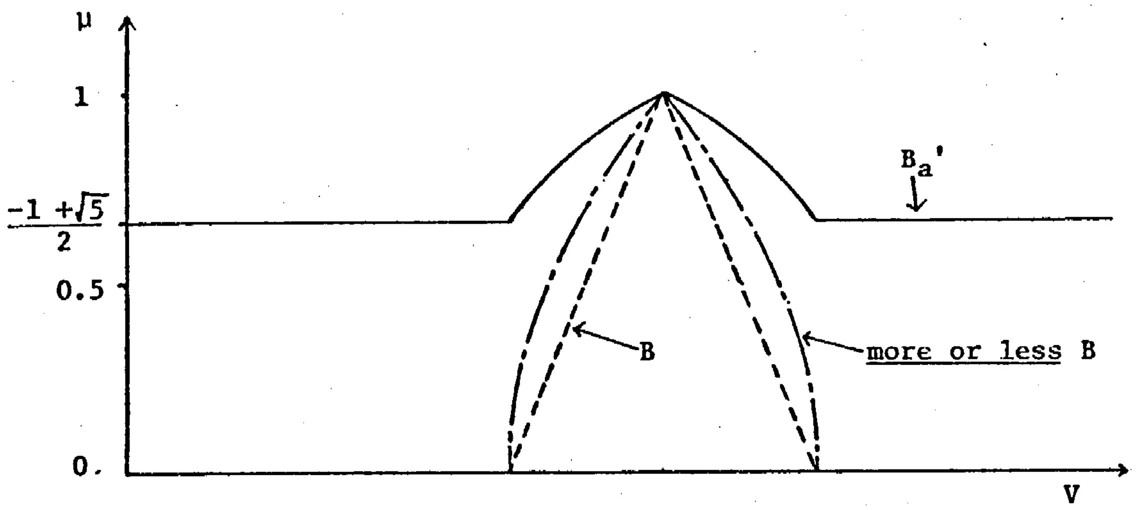

Fig.9. Membership functions of B_a' (when A' = <u>more or less</u> A), B and <u>more or less</u> B

Finally we shall show that Relation IV-1 is satisfied.
Suppose A' = <u>not</u> A (= 7A), then

$$B_a' = (\neg A) \circ [(\neg A \times V) \oplus (U \times B)]$$

$$= \int_V \bigvee_{u \in U} (1 - \mu_A(u)) \wedge [1 \wedge (1 - \mu_A(u) + \mu_B(v))] \,/v$$

$$= \int_V 1 \wedge [1 \wedge (1 + \mu_B(v))] \,/v$$

$$= \int_V 1/v$$

= unknown

This shows that Relation IV-1 is satisfied. Note that this criterion can not be satisfied if $\mu_A(u) > 0$ for all $u \in U$.

[3] The case of mini operation rule (R_c):

First, suppose $A' = A^{\alpha}$, then

$$B'_c = A^{\alpha} \circ (A \times B)$$

$$= \int_U \mu_A^{\alpha}(u)/u \circ \int_{U \times V} \mu_A(u) \wedge \mu_B(v)/(u,v)$$

$$= \int_V \bigvee_{u \in U} \mu_A^{\alpha}(u) \wedge (\mu_A(u) \wedge \mu_B(v))/v \qquad (14)$$

From Fig.1, there exist $u \in U$ which makes $\mu_A(u)=1$, then

$$(14) \quad = \int_V 1 \wedge (1 \wedge \mu_B(v))/v$$

$$= \int_V \mu_B(v)/v$$

$$= B$$

This shows that Relation I and II-2 are satisfied, but Relation II-1 and III are not satisfied.

Second, let A'= <u>not</u> A, then

$$B'_c = (\neg A) \circ (A \times B)$$

$$= \int_U 1 - \mu_A(u)/u \circ \int_{U \times V} \mu_A(u) \wedge \mu_B(v)/(u,v)$$

$$= \int_V \bigvee_{u \in U} [(1 - \mu_A(u)) \wedge \mu_A(u) \wedge \mu_B(v)]/v$$

$$= \begin{cases} \int_V 0.5/v & \text{-------} \quad \mu_B(v) \geqslant 0.5 \\ \int_V \mu_B(v)/v & \text{-------} \quad \mu_B(v) < 0.5 \end{cases}$$

This shows that Relation IV-1 and IV-2 are not satisfied.

It is interesting to note that when A' = unknown (=U), we have

$$B'_C = \int_U 1/u \circ \int_{U\times V} \mu_A(u) \wedge \mu_B(v)/(u,v)$$

$$= \int_V \bigvee_{u\in U} [1 \wedge \mu_A(u) \wedge \mu_B(v)]/v$$

$$= \int_V \mu_B(v)/v$$

This consequence can not be accepted according to our intuitions.

Above discussions show that using the methods (Zadeh's methods and Mamdani's method), almost all criteria stated in Section 2 can not be satisfied and it may be clear that consequences inferred by these methods do not always fit our intuitions.

4. IMPROVED METHODS FOR FUZZY CONDITIONAL INFERENCE

We shall next show the improved methods which satisfy almost these relations.

Let fuzzy subsets A and B in Ant 1 be represented as

$$A = \int_U \mu_A(u)/u \quad , \quad B = \int_V \mu_B(v)/v$$

and suppose that $\mu_A(u)$ and $\mu_B(v)$ satisfy the following conditions.†

(i) $\{ \mu_A(u) \mid u \in U \} \supseteq \{ \mu_B(v) \mid v \in V \}$ (15)

(ii) $\exists u \in U \;\; \mu_A(u)=0 \; ; \; \exists u' \in U \;\; \mu_A(u')=1$ (16)

(iii) $\exists v \in V \;\; \mu_B(v)=0 \; ; \; \exists v' \in V \;\; \mu_B(v')=1$ (17)

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† It is noted that we have discussed the methods by Zadeh and Mamdani under the same conditions.

[4] Method Satisfying the Relation I, II-1, III and IV-1:

If Ant 1 translates into the following fuzzy relation R_s :

$$R_s \hat{=} A \times V \underset{s}{\longrightarrow} U \times B \tag{18}$$

$$\hat{=} \int_{U\times V} [\ \mu_A(u) \underset{s}{\longrightarrow} \mu_B(v)\ ]/(u,v)$$

where
$$\mu_A(u) \underset{s}{\longrightarrow} \mu_B(v) = \begin{cases} 1 \text{ --- } \mu_A(u) \leq \mu_B(v) \\ 0 \text{ --- } \mu_A(u) > \mu_B(v) \end{cases} \tag{19}$$

then the consequence B' is obtained by

$$B' = A' \circ R_s = A' \circ (\ A \times V \underset{s}{\longrightarrow} U \times B\ ) \tag{20}$$

The definition of (19) is based on the implication in S_χ logic system [3].

Using this method, we shall show that Relation I, II-1, III and IV-1 are satisfied under the assumptions of (15)-(17).

As a general case, suppose $A' = A^\alpha$ (α>0), then (20) will be

$$B' = A^\alpha \circ (A \times V \underset{s}{\longrightarrow} U \times B)$$

$$= \int_U \mu_A^\alpha(u)/u \circ \int_{U\times V} \mu_A(u) \underset{s}{\longrightarrow} \mu_B(v)/(u,v)$$

$$= \int_V \bigvee_{u\in U} \mu_A^\alpha(u) \wedge (\mu_A(u) \underset{s}{\longrightarrow} \mu_B(v))/v \tag{21}$$

Here, for each v in V, we can obtain two sets U_1 and U_2 which satisfy the following condition.

$$U_1 \cup U_2 = U, \quad U_1 \cap U_2 = \phi \tag{22}$$

$$\forall u \in U_1 \quad \mu_A(u) \leq \mu_B(v) \tag{23}$$

$$\forall u \in U_2 \quad \mu_A(u) > \mu_B(v) \tag{24}$$

Then

$$(21) = \int_V \bigvee_{u \in U_1} \mu_A^{\alpha}(u)/v \qquad \cdots \text{ from } (23)$$

$$= \int_V \mu_B^{\alpha}(v)/v \qquad \cdots \text{ from } (15) \text{ and } (23)$$

$$= B^{\alpha}$$

This shows that when $\alpha=1$ ($A'=A$), $\alpha=2$ ($A'=A^2$) and $\alpha=0.5$ ($A'=A^{0.5}$), Relation I, II-1 and III are satisfied, respectively.

Next, suppose $A' =$ not A, then, in this case, (20) becomes

$$B' = (\neg A) \circ (A \times V \underset{s}{\longrightarrow} U \times B)$$

$$= \int_U 1 - \mu_A(u)/u \circ \int_{U \times V} \mu_A(u) \underset{s}{\rightarrow} \mu_B(v)/(u,v)$$

$$= \int_V \bigvee_{u \in U} [(1 - \mu_A(u)) \wedge (\mu_A(u) \underset{s}{\rightarrow} \mu_B(v))]/v \qquad (25)$$

From the assumption (16) there exists u in U which makes $\mu_A(u)=0$. Therefore

$$\bigvee_{u \in U} [(1 - \mu_A(u)) \wedge (\mu_A(u) \underset{s}{\rightarrow} \mu_B(v))] = 1$$

Thus,

$$(25) = \int_V 1/v$$

$$= \text{unknown}$$

This shows that Relation IV-1 is satisfied.

Example 1. Let

$$U = V = 0 + 1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9 + 10$$

$$A = \underline{small} = 1/0 + 0.8/1 + 0.6/2 + 0.4/3 + 0.2/4$$

$$B = \underline{middle} = 0.2/2 + 0.4/3 + 0.8/4 + 1/5 + 0.8/6 + 0.4/7 + 0.2/8$$

$$\underline{very\ small} = \underline{small}^2 = 1/0 + 0.64/1 + 0.36/2 + 0.16/3 + 0.04/4$$

$$\underline{very\ middle} = 0.04/2 + 0.16/3 + 0.64/4 + 1/5 + 0.64/6 + 0.16/7 + 0.04/8$$

$$\underline{not\ small} = 0.2/1 + 0.4/2 + 0.6/3 + 0.8/4 + 1/(5 + 6 + \ldots + 10)$$

Then the fuzzy conditional proposition

If x is small then y is middle

translates into

$$R_s = \underline{small} \times V \xrightarrow[s]{} U \times \underline{middle}$$

=

	0	1	2	3	4	5	6	7	8	9	10
0	0	0	0	0	0	1	0	0	0	0	0
1	0	0	0	0	1	1	1	0	0	0	0
2	0	0	0	0	1	1	1	0	0	0	0
3	0	0	0	1	1	1	1	1	0	0	0
4	0	0	1	1	1	1	1	1	1	0	0
5	1	1	1	1	1	1	1	1	1	1	1
6	1	1	1	1	1	1	1	1	1	1	1
7	1	1	1	1	1	1	1	1	1	1	1
8	1	1	1	1	1	1	1	1	1	1	1
9	1	1	1	1	1	1	1	1	1	1	1
10	1	1	1	1	1	1	1	1	1	1	1

1) Let A' = small, then

$$B' = \underline{small} \circ R_S$$

$$= 0.2/2 + 0.4/3 + 0.8/4 + 1/5 + 0.8/6 + 0.4/7 + 0.2/8$$

$$= \underline{middle}$$

2) When A' = very small,

$$B' = (\underline{very\ small}) \circ R_S$$

$$= (\underline{small})^2 \circ R_S$$

$$= 0.04/2 + 0.16/3 + 0.64/4 + 1/5 + 0.64/6 + 0.16/7 + 0.04/8$$

$$= (\underline{middle})^2$$

$$= \underline{very\ middle}$$

3) If A' = not small, then

$$B' = (\underline{not\ small}) \circ R_S$$

$$= 0 + 1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9 + 10$$

$$= V$$

$$\simeq \text{unknown}$$

Stated in English, these inferences may be expressed as follows.

1) If x is small then y is middle.

x is small.

---

y is middle.

2) If x is small then y is middle.

x is very small.

---

y is very middle.

3) If x is small then y is middle.

x is not small.

---

y is unknown.

[5] Method Satisfying the Relation I, II-2, III and IV-1:

If Ant 1 translates into the following fuzzy relation R_g:

$$R_g = A \times V \xrightarrow[g]{} U \times B \tag{26}$$

$$= \int_{U\times V}[\mu_A(u) \xrightarrow[g]{} \mu_B(v)]/(u,v)$$

where
$$\mu_A(u) \xrightarrow[g]{} \mu_B(v) = \begin{cases} 1 & \text{--- } \mu_A(u) \leq \mu_B(v) \\ \mu_B(v) & \text{--- } \mu_A(u) > \mu_B(v) \end{cases} \tag{27}$$

then, under the assumptions of (15)-(17), we can show that this method satisfies the Relation I, II-2, III and IV-1 in a similar way as in Method I. The definition (27) is from Godel's definition of the implication in G_{χ} logic system [3].

[6] Method Satisfying the Relation I, II-1, III and IV-2:

Let Ant 1 translates into the following fuzzy relation R_{sg}:

$$R_{sg} = (A \times V \xrightarrow[s]{} U \times B) \cap (\neg A \times V \xrightarrow[g]{} U \times \neg B) \tag{28}$$

then these relations are satisfied.

[7] Method Satisfying the Relation I, II-2, III and IV-2:

Ant 1 which is translated into the following fuzzy relation R_{gg}:

$$R_{gg} = (A \times V \underset{g}{\longrightarrow} U \times B) \cap (\neg A \times V \underset{g}{\longrightarrow} U \times \neg B) \tag{29}$$

satisfies these relations.

5. SOME PROPERTIES OF R_s AND R_g

In this section, we describe some interesting properties of fuzzy relations R_s defined by (19) and R_g defined by (26). Note that the fuzzy relations R_m and R_a defined by Zadeh do not have these properties and the fuzzy relation R_c defined by Mamdani has only the following Property 1.

Property 1. Let fuzzy conditional propositions P_1, P_2 and P_3 be given as

P_1 = If x is A then y is B

P_2 = If y is B then z is C

P_3 = If x is A then z is C

where A, B and C are fuzzy concepts represented as the following fuzzy sets,

$$A = \int_U \mu_A(u)/u, \quad B = \int_V \mu_B(v)/v, \quad C = \int_W \mu_C(w)/w$$

Let

$$R_s(A, B) = A \times V \underset{s}{\longrightarrow} U \times B$$

$$R_s(B, C) = B \times W \underset{s}{\longrightarrow} V \times C$$

$$R_s(A, C) = A \times W \underset{s}{\longrightarrow} U \times C$$

be fuzzy relations which are translated, respectively, from P_1, P_2 and P_3 using (18) and let

$$R_g(A, B) = A \times V \underset{g}{\longrightarrow} U \times B$$

$$R_g(B, C) = B \times W \underset{g}{\longrightarrow} V \times C$$

$$R_g(A, C) = A \times W \underset{g}{\longrightarrow} U \times C$$

be fuzzy relations translated from P_1, P_2 and P_3, respectively, from (26). Then, under the following conditions, that is,

$$\{\mu_A(u) \mid u \in U\} \supseteq \{\mu_B(v) \mid v \in V\} \supseteq \{\mu_C(w) \mid w \in W\} \tag{30}$$

$$\begin{aligned} &\exists u \in U \quad \mu_A(u) = 0, \qquad &&\exists u' \in U \quad \mu_A(u') = 1 \\ &\exists v \in V \quad \mu_B(v) = 0, \qquad &&\exists v' \in V \quad \mu_B(v') = 1 \\ &\exists w \in W \quad \mu_C(w) = 0, \qquad &&\exists w' \in W \quad \mu_C(w') = 1 \end{aligned} \tag{31}$$

the following equalities are satisfied.

$$R_s(A, C) = R_s(A, B) \circ R_s(B, C) \tag{32}$$

$$R_g(A, C) = R_g(A, B) \circ R_g(B, C) \tag{33}$$

<u>Example 2.</u> Let fuzzy conditional propositions P_1, P_2 and P_3 be

P_1 = If x is A then y is B.

P_2 = If y is B then z is C.

P_3 = If x is A then z is C.

and fuzzy sets A in U, B in V and C in W be given as

$$A = 1/1 + 0.8/2 + 0.6/3 + 0.4/4 + 0.2/5$$

$$B = 0.2/4 + 0.4/5 + 0.8/6 + 1/7$$

$$C = 0.4/2 + 0.8/3 + 1/4 + 0.8/5 + 0.2/6$$

where

$$U = V = W = 1 + 2 + 3 + 4 + 5 + 6 + 7$$

Then, $R_g(A, B)$, $R_g(B, C)$ and $R_g(A, C)$ which are translated from P_1, P_2 and P_3 are obtained as follows:

$$R_g(A, B) = A \times V \underset{g}{\longrightarrow} U \times B$$

$$= \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \end{array} \begin{array}{c} \begin{array}{ccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 \end{array} \\ \begin{bmatrix} 0 & 0 & 0 & 0.2 & 0.4 & 0.8 & 1 \\ 0 & 0 & 0 & 0.2 & 0.4 & 1 & 1 \\ 0 & 0 & 0 & 0.2 & 0.4 & 1 & 1 \\ 0 & 0 & 0 & 0.2 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix} \end{array}$$

$$R_g(B, C) = B \times W \underset{g}{\longrightarrow} V \times C$$

$$= \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \end{array} \begin{array}{c} \begin{array}{ccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 \end{array} \\ \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 & 1 & 0.2 & 0 \\ 0 & 0.4 & 1 & 1 & 1 & 0.2 & 0 \\ 0 & 0.4 & 0.8 & 1 & 0.8 & 0.2 & 0 \end{bmatrix} \end{array}$$

$$R_g(A, C) = A \times W \xrightarrow[g]{} U \times C$$

$$= \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \end{array} \begin{array}{c} \begin{array}{ccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 \end{array} \\ \begin{bmatrix} 0 & 0.4 & 0.8 & 1 & 0.8 & 0.2 & 0 \\ 0 & 0.4 & 1 & 1 & 1 & 0.2 & 0 \\ 0 & 0.4 & 1 & 1 & 1 & 0.2 & 0 \\ 0 & 1 & 1 & 1 & 1 & 0.2 & 0 \\ 0 & 1 & 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix} \end{array}$$

Then, the composition of $R_g(A, B)$ and $R_g(B, C)$ leads to

$$R_g(A, B) \circ R_g(B, C)$$

$$= \begin{bmatrix} 0 & 0 & 0 & 0.2 & 0.4 & 0.8 & 1 \\ 0 & 0 & 0 & 0.2 & 0.4 & 1 & 1 \\ 0 & 0 & 0 & 0.2 & 0.4 & 1 & 1 \\ 0 & 0 & 0 & 0.2 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix} \circ \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 & 1 & 0.2 & 0 \\ 0 & 0.4 & 1 & 1 & 1 & 0.2 & 0 \\ 0 & 0.4 & 0.8 & 1 & 0.8 & 0.2 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0.4 & 0.8 & 1 & 0.8 & 0.2 & 0 \\ 0 & 0.4 & 1 & 1 & 1 & 0.2 & 0 \\ 0 & 0.4 & 1 & 1 & 1 & 0.2 & 0 \\ 0 & 1 & 1 & 1 & 1 & 0.2 & 0 \\ 0 & 1 & 1 & 1 & 1 & 0.2 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix} = R_g(A, C).$$

This shows the satisfaction of the equality (33).

Property 2. For the fuzzy conditional proposition P_1

$$P_1 = \text{If } x \text{ is } A \text{ then } y \text{ is } B$$

and its contradictive proposition P_2, that is,

$$P_2 = \text{If } y \text{ is not } B \text{ then } x \text{ is not } A$$

let $R_s(A, B)$ and $R_s(\neg B, \neg A)$ be fuzzy relations which are translated from P_1 and P_2 using (18). Then the following equality holds.

$$R_s(\neg B, \neg A) = \tilde{R}_s(A, B) \tag{34}$$

where $\tilde{R}_s(A, B)$ denotes the inverse relation of $R_s(A, B)$.

Note that Property 2 does not hold for Rg.

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