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FUZZY SETS UNDER VARIOUS OPERATIONS

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ABSTRACT

Among the basic operations which can be performed on fuzzy sets are the operations of union, intersection, complement, algebraic product and algebraic sum. In addition to these operations, new operations called "bounded-sum" and "bounded-difference" were defined by L. A. Zadeh to investigate the fuzzy reasoning which provides a way of dealing with the reasoning problems which are too complex for precise solution.

This paper investigates the algebraic properties of fuzzy sets under these new operations of bounded-sum and bounded-difference and the properties of fuzzy sets in the case where these new operations are combined with the well-known operations of union, intersection, algebraic product and algebraic sum.

INTRODUCTION

Among the well-known operations which can be performed on fuzzy sets are the operations of union, intersection, complement, algebraic product and algebraic sum. A number of researches concerning with fuzzy sets and their applications to automata theory, logic, control, game, topology, pattern recognition, integral, linguistics, taxonomy, system, decision making, information retrieval and so on, have been earnestly investigated by using these operations for fuzzy sets [1, 2]. For example, union, intersection, and complement are found in most of papers relating to fuzzy sets. Algebraic product and algebraic sum are also used in the studies of fuzzy events [3], fuzzy automata [4], fuzzy logic [5], fuzzy semantics [6] and so on.

In addition to these operations, new operations called "bounded-sum" and "bounded-difference" are introduced by L. A. Zadeh [7] to investigate the fuzzy reasoning which provides a way of dealing with the reasoning problems which are too complex for precise solution.

This paper investigates the algebraic properties of fuzzy sets under bounded-sum and bounded-difference as well as the properties of fuzzy sets in the case where these new operations are combined with the well-known operations of union, intersection, algebraic product and algebraic sum.

FUZZY SETS AND THEIR OPERATIONS

We shall briefly review fuzzy sets and their operations of union, intersection, complement, algebraic product, algebraic sum, bounded-sum, bounded-difference, and bounded-product which is a dual operation of bounded -sum.

Fuzzy Sets: A fuzzy set A in a universe of discourse U is characterized by a membership function μ_A which takes the values in the interval [0, 1], i.e.,

$$\mu_A : U \longrightarrow [0, 1] \qquad (1)$$

The value of μ_A at u ($\in$ U), $\mu_A(u)$, represents the grade of membership (grade, for short) of u in A and is a point in [0, 1].

The operations on fuzzy sets A and B are listed as follows.

Union:	$A \cup B$	<==>	$\mu_{A \cup B} = \mu_A \vee \mu_B$	(2)
Intersection:	$A \cap B$	<==>	$\mu_{A \cap B} = \mu_A \wedge \mu_B$	(3)
Complement:	$\bar{A}$	<==>	$\mu_{\bar{A}} = 1 - \mu_A$	(4)
Algebraic Product:	$A \cdot B$	<==>	$\mu_{A \cdot B} = \mu_A \mu_B$	(5)
Algebraic Sum:	$A \dot{+} B$	<==>	$\mu_{A \dot{+} B} = \mu_A + \mu_B - \mu_A \mu_B$	
			$= 1 - (1 - \mu_A)(1 - \mu_B)$	(6)
Bounded-Sum:	$A \oplus B$	<==>	$\mu_{A \oplus B} = 1 \wedge (\mu_A + \mu_B)$	(7)
Bounded-Difference:	$A \ominus B$	<==>	$\mu_{A \ominus B} = 0 \vee (\mu_A - \mu_B)$	(8)
Bounded-Product:	$A \odot B$	<==>	$\mu_{A \odot B} = 0 \vee (\mu_A + \mu_B - 1)$	(9)

where the operations of $\vee$, $\wedge$, +, and - represent max, min, arithmetic sum, and arithmetic difference, respectively.

ALGEBRAIC PROPERTIES OF FUZZY SETS UNDER VARIOUS KINDS OF OPERATIONS

In this section we shall investigate the algebraic properties of fuzzy sets under the operations (2)-(9). We shall first review the well-known properties of fuzzy sets under union (2), intersection (3), complement (4), algebraic product (5), and algebraic sum (6).

Theorem 1 (Zadeh, 1965): *Fuzzy sets in U form a distributive lattice* under $\cup$ and $\cap$, but do not form a Boolean lattice, since $\bar{A}$ is not the complement of A in the lattice sense.*

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* A set L with two operations $\vee$ and $\wedge$ satisfying idempotent laws, commutative laws, associative laws and absorption laws is said to be a lattice. If the lattice L satisfies distributive laws, then L is a distributive lattice. If the complement laws $a \vee \bar{a} = I$ and $a \wedge \bar{a} = O$ holds, L is a Boolean lattice.

<u>Theorem 2</u>: *Fuzzy sets also form a unitary commutative semiring with zero*[*] *under the operations U and ∩.*

<u>Proof</u>. This can be shown by letting + = U, x = ∩, e = Ω, and O = Φ in the footnote, where Ω is a universe of discourse U and Φ is an empty fuzzy set.

<u>Theorem 3</u> (Kaufmann, 1973): *Fuzzy sets under the algebraic sum (∔) and the algebraic product (·) do not constitute such algebraic structures as a lattice and a semiring. Fuzzy sets, however, form a commutative monoid*[**] *under ∔ (or ·).*

<u>Theorem 4</u>: *Fuzzy sets form a unitary (= Ω) commutative semiring with zero (= Φ) under U (as addition) and algebraic product · (as multiplication). The duality holds for intersection ∩ (as addition) and algebraic sum ∔ (as multiplication). Fuzzy sets also form a lattice ordered semigroup*[***] *with zero Φ and unity Ω under U, ∩ and ·. The duality holds for ∪, U and ∔.*

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* A <u>semiring</u> (R,+,x) is a set R with two operations + and x of addition and multiplication such that + is associative and commutative, and x is associative and distributive over +, i.e.,

a x (b + c) = (a x b) + (a x c) and (a + b) x c = (a x c) + (b x c).

A semiring is <u>unitary</u> if x has a unit e, and is <u>commutative</u> if x is commutative, and is a semiring with <u>zero</u> if + has an identity O such that O x a = a x O = O.

** A semigroup (S,·) is a set S together with an operation · such that · is associative. A <u>monoid</u> (or <u>unitary semigroup</u>) is a semigroup with identity under · . The monoid is called <u>commutative</u> if · is commutative.

*** A lattice L which is a semigroup under * and also satisfies the following distributive law is called a <u>lattice ordered semigroup</u> and is denoted as L = (L, v, ∧ , *), where v and ∧ are operations of lub and glb in L, respectively. The distributive law is

x * (y v z) = (x * y) v (x * z) and (x v y) * z = (x * z) v (y * z)

Moreover, L = (L, v, ∧ , *) is said to be a <u>lattice ordered semigroup with unity I and zero O</u> if the followings are satisfied for any x in L, i.e.,

x v O = x, x * O = O * x = O

x v I = I, x * I = I * x = x

We shall next discuss the algebraic properties of fuzzy sets under the operations of bounded-sum ⊕ (7), bounded-difference ⊖ (8) which were defined by Zadeh [7], and bounded-product ⊙ (9) which is a new operation dual to bounded-sum. The new operation of bounded-product ⊙ can be expressed by using De Morgan's laws to be shown in (40) and (41).

$$A \odot B = \overline{\bar{A} \oplus \bar{B}} = A \ominus \bar{B} \qquad (10)$$

(I) The Case of Bounded-Sum ⊕, Bounded-Difference ⊖ and Bounded-Product ⊙:

Idempotency:

$$A \oplus A \supseteq A \qquad (11)$$

$$A \odot A \subseteq A \qquad (12)$$

$$A \ominus A = \phi \qquad (13)$$

Commutativity:

$$A \oplus B = B \oplus A \qquad (14)$$

$$A \odot B = B \odot A \qquad (15)$$

$$A \ominus B \neq B \ominus A \qquad (16)$$

Associativity:

$$(A \oplus B) \oplus C = A \oplus (B \oplus C) \qquad (17)$$

$$(A \odot B) \odot C = A \odot (B \odot C) \qquad (18)$$

$$(A \ominus B) \ominus C \subseteq A \ominus (B \ominus C) \qquad (19)$$

As a special case of (19), we have

$$A \ominus (A \ominus B) = A \cap B \qquad (20)$$

Absorption:

(i) Case of ⊕ and ⊙:

$$A \oplus (A \odot B) \supseteq A \qquad (21)$$

$$A \odot (A \oplus B) \subseteq A \qquad (22)$$

(ii) Case of ⊕ and ⊖:

$$A \oplus (A \ominus B) \supseteq A \qquad (23)$$

$$A \oplus (B \ominus A) = A \cup B \qquad (24)$$

$$A \ominus (A \oplus B) = \phi \qquad (25)$$

$$(A \oplus B) \ominus A = \bar{A} \cap B \qquad (26)$$

(iii) Case of ⊙ and ⊖:

$$A \odot (A \ominus B) \subseteq A \qquad (27)$$

$$A \odot (B \ominus A) = \phi \qquad (28)$$

$$A \ominus (A \odot B) = A \cap \bar{B} \qquad (29)$$

$$(A \odot B) \ominus A = \phi \qquad (30)$$

Distributivity:

(i) Case of ⊕ and ⊙: $A \oplus (B \odot C) \neq (A \oplus B) \odot (A \oplus C)$ (31)

$A \odot (B \oplus C) \neq (A \odot B) \oplus (A \odot C)$ (32)

(ii) Case of ⊕ and ⊖: $A \oplus (B \ominus C) \supseteq (A \oplus B) \ominus (A \oplus C)$ (33)

$A \ominus (B \oplus C) \subseteq (A \ominus B) \oplus (A \ominus C)$ (34)

$(B \oplus C) \ominus A \neq (B \ominus A) \oplus (C \ominus A)$ (35)

(iii) Case of ⊙ and ⊖: $A \odot (B \ominus C) \subsetneq (A \odot B) \ominus (A \odot C)$ (36)

$A \ominus (B \odot C) \supseteq (A \ominus B) \odot (A \ominus C)$ (37)

$(B \odot C) \ominus A \supseteq (B \ominus A) \odot (C \ominus A)$ (38)

De Morgan's Laws: $\overline{A \oplus B} = \bar{A} \odot \bar{B}$ (39)

$\overline{A \odot B} = \bar{A} \oplus \bar{B}$ (40)

Furthermore, $\overline{A \oplus B} = \bar{A} \ominus B$ (41)

$\overline{A \ominus B} = \bar{A} \oplus B$ (42)

$\bar{A} \ominus \bar{B} = B \ominus A$ (43)

Finally, $A \odot \bar{B} = A \ominus B$ (44)

$A \ominus \bar{B} = A \odot B$ (45)

Identities: $A \oplus \Phi = A$ (46)

$A \oplus \Omega = \Omega$ (47)

$A \odot \Phi = \Phi$ (48)

$A \odot \Omega = A$ (49)

Moreover, $A \ominus \Phi = A$ (50)

$\Phi \ominus A = \Phi$ (51)

$A \ominus \Omega = \Phi$ (52)

$\Omega \ominus A = \bar{A}$ (53)

Complementarity: $A \oplus \bar{A} = \Omega$ (54)

$A \odot \bar{A} = \Phi$ (55)

$\Phi \subseteq A \ominus \bar{A} \subseteq \Omega$ (56)

$\Phi \subseteq \bar{A} \ominus A \subseteq \Omega$ (57)

where Φ is an empty set and is defined by $\mu_\Phi = 0$, and Ω is a universe of discourse U and is defined by $\mu_\Omega = 1$.

<u>Remark 1:</u> From (17) and (18) it is found that the operations $\oplus$ and $\odot$ are associative. Thus we can represent $A_1 \oplus A_2 \oplus \ldots \oplus A_n$ and $A_1 \odot A_2 \odot \ldots \odot A_n$ as

$$\mu_{A_1 \oplus A_2 \oplus \ldots \oplus A_n} = 1 \wedge (\mu_{A_1} + \mu_{A_2} + \ldots + \mu_{A_n})$$

$$\mu_{A_1 \odot A_2 \odot \ldots \odot A_n} = 0 \vee [\mu_{A_1} + \mu_{A_2} + \ldots + \mu_{A_n} - (n-1)]$$

If $A_1 = A_2 = \ldots = A_n$ $(= A)$, then we can obtain

$$\mu_{A \oplus A \oplus \ldots \oplus A} = 1 \wedge n\, \mu_A$$

$$\mu_{A \odot A \odot \ldots \odot A} = 0 \vee (1 - n\, \mu_{\bar{A}})$$

<u>Remark 2:</u> The operations $\cup, \cap, \bar{\ }$ over fuzzy sets can be represented by using $\oplus$, $\ominus$, (and $\odot$), that is, by using (24), (20), (26) and (53). Namely,

$$A \cup B = A \oplus (B \ominus A) \tag{58}$$

$$A \cap B = A \ominus (A \ominus B) = (\bar{A} \oplus B) \ominus \bar{A} \tag{59}$$

$$\bar{A} = \Omega \ominus A \tag{60}$$

It should be noted that $\oplus$, $\odot$, and $\ominus$ are easily shown not to be represented by $\cup, \cap$, and $\bar{\ }$.

<u>Remark 3:</u> Fuzzy sets under $\oplus$ and $\odot$ satisfy the complement laws (54)-(55), though they do not satisfy these laws under $\cup$ and $\cap$, and $\cdot$ and $\dotplus$. Note that we have $0.5\Omega \subseteq A \cup \bar{A} \subseteq \Omega$; $\Phi \subseteq A \cap \bar{A} \subseteq 0.5\Omega$ under $\cup$ and $\cap$, and $0.75\Omega \subseteq A \dotplus \bar{A} \subseteq \Omega$; $\Phi \subseteq A \cdot \bar{A} \subseteq 0.25\Omega$ under $\dotplus$ and $\cdot$, where 0.5Ω is defined as $\mu_{0.5\Omega} = 0.5\mu_{\Omega} = 0.5 \times 1 = 0.5$

From the above property concerning $\oplus$, $\odot$ and $\ominus$, we can immediately obtain the following theorem.

<u>Theorem 5:</u> *Fuzzy sets under $\oplus$ and $\odot$ do not satisfy the absorption and distributive laws and hence do not form such algebraic structures as a lattice and a semiring. The same holds for $\oplus$ and $\ominus$, and for $\odot$ and $\ominus$. Fuzzy sets, however, form a commutative monoid under $\oplus$ (or $\odot$), but do not form such a structure under $\ominus$.*

We shall next deal with the absorption and distributive properties for fuzzy sets under the operations of bounded-sum $\oplus$, bounded-difference $\ominus$ and bounded-product $\odot$ combined with the operations of union $\cup$ and intersection $\cap$.

(II) <u>The Case of Bounded-Sum $\oplus$, Bounded-Difference $\ominus$ and Bounded-Product $\odot$ Combined with Union $\cup$ and Intersection $\cap$:</u>

<u>Absorption:</u>

$$A \cup (A \oplus B) \supseteq A \tag{61}$$

$$A \cap (A \oplus B) = A \tag{62}$$

$$A \cup (A \odot B) = A \tag{63}$$

$$A \cap (A \odot B) \subseteq A \tag{64}$$

$$A \cup (A \ominus B) = A \tag{65}$$

$$A \cup (B \ominus A) \neq A \tag{66}$$

$$A \cap (A \ominus B) \subseteq A \tag{67}$$

$$A \cap (B \ominus A) \neq A \tag{68}$$

Moreover,

$$A \oplus (A \cup B) \supseteq A \tag{69}$$

$$A \oplus (A \cap B) \supseteq A \tag{70}$$

$$A \odot (A \cup B) \subseteq A \tag{71}$$

$$A \ominus (A \cap B) \subseteq A \tag{72}$$

$$A \ominus (A \cup B) = \Phi \tag{73}$$

$$(A \cup B) \ominus A = B \ominus A \neq A \tag{74}$$

$$A \ominus (A \cap B) = A \ominus B \subseteq A \tag{75}$$

$$(A \cap B) \ominus A = \Phi \tag{76}$$

<u>Distributivity:</u>

$$A \cup (B \oplus C) \subseteq (A \cup B) \oplus (A \cup C) \tag{77}$$

$$A \cap (B \oplus C) \subseteq (A \cap B) \oplus (A \cap C) \tag{78}$$

$$A \cup (B \odot C) \supseteq (A \cup B) \odot (A \cup C) \tag{79}$$

$$A \cap (B \odot C) \supseteq (A \cap B) \bigcirc (A \cap C) \tag{80}$$

$$A \cup (B \ominus C) \supseteq (A \cup B) \ominus (A \cup C) \tag{81}$$

$$A \cap (B \ominus C) \supseteq (A \cap B) \ominus (A \cap C) \tag{82}$$

and

$$A \oplus (B \cup C) = (A \oplus B) \cup (A \oplus C) \tag{83}$$

$$A \oplus (B \cap C) = (A \oplus B) \cap (A \oplus C) \tag{84}$$

$$A \odot (B \cup C) = (A \odot B) \cup (A \odot C) \tag{85}$$

$$A \odot (B \cap C) = (A \odot B) \cap (A \odot C) \tag{86}$$

$$A \ominus (B \cup C) = (A \ominus B) \cap (A \ominus C) \qquad (87)$$

$$A \ominus (B \cap C) = (A \ominus B) \cup (A \ominus C) \qquad (88)$$

$$(B \cup C) \ominus A = (B \ominus A) \cup (C \ominus A) \qquad (89)$$

$$(B \cap C) \ominus A = (B \ominus A) \cap (C \ominus A) \qquad (90)$$

<u>Theorem 6</u>: *Fuzzy sets satisfy associative laws (18), commutative laws (15) and distributive law (85) under the operations of bounded-product $\odot$ and union $\cup$, and thus they form a unitary (= Ω) commutative semiring with zero (= Φ) under $\odot$ (as multiplication) and $\cup$ (as addition). Dually, fuzzy sets under the operations of bounded-sum $\oplus$ (as multiplication) and intersection $\cap$ (as addition) form a unitary (= Φ) commutative semiring with zero (= Ω). Moreover, fuzzy sets under $\oplus$ (as multiplication) and $\cup$ (as addition) form a unitary (= Φ) commutative semiring. The same holds for $\odot$ (as multiplication) and $\cap$ (as addition), where Ω becomes a unit element for $\cap$. Furthermore, fuzzy sets also form a lattice ordered semigroup with unity Ω and zero Φ under $\cup$, $\cap$ and $\oplus$, where $\oplus$ is a semigroup operation. Dually, they form a lattice ordered semigroup with unity Φ and zero Ω under $\cap$, $\cup$ and $\odot$.*

As a generalization of (II), the following formulas can be easily obtained.

(III) <u>Formulas Obtained as a Generalization of (II)</u>:

$$(A \cup B) \oplus (C \cup D) = (A \oplus C) \cup (A \oplus D) \cup (B \oplus C) \cup (B \oplus D) \qquad (91)$$

$$(A \cap B) \oplus (C \cap D) = (A \oplus C) \cap (A \oplus D) \cap (B \oplus C) \cap (B \oplus D) \qquad (92)$$

$$(A \cup B) \oplus (A \cap B) = A \oplus B \qquad (93)$$

$$(A \cup B) \odot (C \cup D) = (A \odot C) \cup (A \odot D) \cup (B \odot C) \cup (B \odot D) \qquad (94)$$

$$(A \cap B) \odot (C \cap D) = (A \odot C) \cap (A \odot D) \cap (B \odot C) \cap (B \odot D) \qquad (95)$$

$$(A \cup B) \odot (A \cap B) = A \odot B \qquad (96)$$

$$(A \cup B) \ominus (C \cap D) = (A \ominus C) \cup (A \ominus D) \cup (B \ominus C) \cup (B \ominus D) \qquad (97)$$

$$(A \cap B) \ominus (C \cup D) = (A \ominus C) \cap (A \ominus D) \cap (B \ominus C) \cap (B \ominus D) \qquad (98)$$

$$(A \cup B) \ominus (A \cap B) = (A \ominus B) \cup (B \ominus A) = |A - B| \qquad (99)$$

We shall next discuss the absorption and distributive properties under the operations of bounded-sum ⊕, bounded-difference ⊖ and bounded-product ⊙ combined with the operations of algebraic product · and algebraic sum ∔.

(IV) The Case of Bounded-Sum ⊕, Bounded-Difference ⊖ and Bounded-Product ⊕ Combined with Algebraic Product · and Algebraic Sum ∔ :

Absorption:

$$A \cdot (A \oplus B) \subsetneq A \tag{100}$$

$$A \dotplus (A \oplus B) \supseteq A \tag{101}$$

$$A \cdot (A \odot B) \subseteq A \tag{102}$$

$$A \dotplus (A \odot B) \supseteq A \tag{103}$$

$$A \cdot (A \ominus B) \subseteq A \tag{104}$$

$$A \cdot (B \ominus A) \subseteq A \tag{105}$$

$$A \dotplus (A \ominus B) \supseteq A \tag{106}$$

$$A \dotplus (B \ominus A) \supseteq A \tag{107}$$

and

$$A \oplus (A \cdot B) \supsetneq A \tag{108}$$

$$A \oplus (A \dotplus B) \supseteq A \tag{109}$$

$$A \odot (A \cdot B) \subseteq A \tag{110}$$

$$A \odot (A \dotplus B) \subseteq A \tag{111}$$

$$A \ominus (A \cdot B) = A \cdot \bar{B} \subseteq A \tag{112}$$

$$(A \cdot B) \ominus A = \Phi \tag{113}$$

$$A \ominus (A \dotplus B) = \Phi \tag{114}$$

$$(A \dotplus B) \ominus A = \bar{A} \cdot B \tag{115}$$

Distributivity:

$$A \cdot (B \oplus C) \subseteq (A \cdot B) \oplus (A \cdot C) \tag{116}$$

$$A \dotplus (B \oplus C) \subseteq (A \dotplus B) \oplus (A \dotplus C) \tag{117}$$

$$A \cdot (B \odot C) \supseteq (A \cdot B) \odot (A \cdot C) \tag{118}$$

$$A \dotplus (B \odot C) \supseteq (A \dotplus B) \odot (A \dotplus C) \tag{119}$$

$$A \cdot (B \ominus C) = (A \cdot B) \ominus (A \cdot C) \tag{120}$$

$$A \dotplus (B \ominus C) \supseteq (A \dotplus B) \ominus (A \dotplus C) \tag{121}$$

Furthermore,

$$A \oplus (B \cdot C) \neq (A \oplus B) \cdot (A \oplus C) \qquad (122)$$

$$A \oplus (B \dotplus C) = (A \oplus B) \dotplus (A \oplus C) \qquad (123)$$

$$A \odot (B \cdot C) \neq (A \odot B) \cdot (A \odot C) \qquad (124)$$

$$A \odot (B \dotplus C) \neq (A \odot B) \dotplus (A \odot C) \qquad (125)$$

$$A \ominus (B \cdot C) \supseteq (A \ominus B) \cdot (A \ominus C) \qquad (126)$$

$$(B \cdot C) \ominus A \neq (B \ominus A) \cdot (C \ominus A) \qquad (127)$$

$$A \ominus (B \dotplus C) \subseteq (A \ominus B) \dotplus (A \ominus C) \qquad (128)$$

$$(B \dotplus C) \ominus A \neq (B \ominus A) \dotplus (C \ominus A) \qquad (129)$$

From the above property we can easily obtain the following theorem.

<u>Theorem 7:</u> *Fuzzy sets under bounded-sum $\oplus$ and algebraic product $\cdot$ do not form such algebraic structures as a lattice and a semiring, since they do not satisfy the distributive laws and the absorption laws. The same is true of $(\oplus, \dotplus)$, $(\odot, \cdot)$, $(\odot, \dotplus)$, $(\ominus, \cdot)$, and $(\ominus, \dotplus)$.*

<u>Remark:</u> Although fuzzy sets do satisfy the distributive law (120) under $\ominus$ and $\cdot$, they do not satisfy the associative law and the commutative law under $\ominus$ (see (19), (16)) and thus they do not constitute a lattice and a semiring under these operations.

Finally, we shall list the properties of fuzzy sets under containment relation $\subseteq$.

(V) <u>Properties of Fuzzy Sets under Containment Relation $\subseteq$:</u>

$$A \odot B \subseteq A \cdot B \subseteq A \cap B \qquad (130)$$

$$A \oplus B \supseteq A \dotplus B \supseteq A \cup B \qquad (131)$$

$$A \ominus B \subseteq A \cap \bar{B} \qquad (132)$$

$$A \subseteq B,\ C \subseteq D \implies A \cup C \subseteq B \cup D \tag{133}$$

$$\implies A \cap C \subseteq B \cap D \tag{134}$$

$$\implies A \cdot C \subseteq B \cdot D \tag{135}$$

$$\implies A \dotplus C \subseteq B \dotplus D \tag{136}$$

$$\implies A \oplus C \subseteq B \oplus D \tag{137}$$

$$\implies A \odot C \subseteq B \odot D \tag{138}$$

$$A \subseteq B,\ D \subseteq C \implies A \ominus C \subseteq B \ominus D \tag{139}$$

$$A \subseteq B \iff A \ominus B = \Phi \tag{140}$$

$$\iff \bar{A} \oplus B = \Omega \tag{141}$$

$$\iff A \cup B = B \tag{142}$$

$$\iff A \cap B = A \tag{143}$$

CONCLUSION

We have discussed the algebraic properties of fuzzy sets under the new operations of bounded-sum, bounded-difference and bounded-product and the properties of fuzzy sets under these operations combined with the well-known operations of union, intersection, algebraic product and algebraic sum.

If we introduce the other kinds of operations, say, in many-valued logic to fuzzy sets, further fruitful applications of fuzzy sets will be found in a variety of areas.

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